Note
On odd path extendable graphs *

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Abstract

A connected graph G is said to be odd path extendable if for any odd path P of G, the graph $G - V(P)$ contains a perfect matching. In this paper, we at first time introduce the concept of odd path extendable graphs. Some simple necessary and sufficient conditions for a graph to be odd path extendable are given. In particular we show that if a graph is odd path extendable, it is hamiltonian.

Keywords Matching, odd path extendable, n-extendable

1 Introduction

All graphs considered in this paper are simple connected graphs. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. For $S \subset V(G)$, $G[S]$ denotes the subgraph of G induced by S. For a path P in G, we denote by $V(P)$ the set of vertices of P. The number of edges of a path is its length, a path is even (odd) if its length is even (odd). A graph G is bicritical if the deletion of any two distinct vertices of G results in a graph with a perfect matching. Let G be a graph with perfect matchings. An edge of

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G is said to be allowed if it lies in a perfect matching of G. A graph G is said to be elementary if its allowed edges form a connected subgraph of G. An elementary graph G is said to be 1-extendable (or matching covered) if all of its edges are allowed. Other terminologies and notations not defined here can be found in [1] and [2].

1-extendable graphs play an important role in matching theory. Lovász and Plummer [2, Chapter 5] showed that in a certain sense any elementary graphs could be constructed using only 1-extendable bipartite graphs and bicritical graphs as "building blocks". It turns out that 1-extendable bipartite graphs have so-called "ear constructions": A graph G is 1-extendable bipartite if and only if it can be represented as $G = x + P_1 + ... + P_r$, where x is an edge and each P_i (the so-called "ear") is a path of odd length joining two vertices in different color classes of $x + P_1 + ... + P_{i-1}$ and having no other vertex in common with $x + P_1 + ... + P_{i-1}$ [2, Chapter 4]. By this fact and the definition of 1-extendable graph G, we know that for a 1-extendable bipartite graph G, it always contains an odd path P (the length of P might be one) such that $G - V(P)$ has a perfect matching. This observation motives us to introduce a new class of graphs called odd path extendable graphs.

Definition. Let G be a connected graph. If for any odd path P of G, the graph $G - V(P)$ contains a perfect matching, then G is said to be odd path extendable.

It is obvious that every odd path extendable graph must have an even number of vertices. We can also easily see that the complete graphs K_{2n}, complete bipartite graphs $K_{n,n}$ and cycles C_{2n} (where $n \geq 2$) are odd path extendable.

The relations between odd path extendability and some existing notions of matching extendability is as following. Clearly, if a graph is odd path extendable then it is 1-extendable, but the reverse does not hold. For example, the graph $K_{n,n} - e$ (here $n \geq 3$, e is an edge in $K_{n,n}$) is 1-extendable, but $K_{n,n} - e$ is not odd path extendable. Recall that a connected graph G is said to be *n-extendable* if G has n independent edges and any n independent edges are contained in a perfect matching of G, where $1 \leq n \leq \frac{|V(G)|-2}{2}$. The concept of n-extendable graphs was introduced by Plummer [3] in 1980. Since then, there has been extensive research on this topic. For detailed results on n-extendable graphs, see two surveys [4, 5]. Obviously, if a graph G is n-extendable for all n, then it is odd path extendable. Thus the odd path extendability of graphs can be thought of as a variant of n-extendability. An induced matching M in a graph G is a matching where no two edges of M are joined by an edge of G. A connected graph G is called an *induced-matching extendable* (simply IM-extendable) graph if any induced matching M is included in a perfect matching of G. This concept was first proposed by Yuan [8]. Many results on IM-extendability can be found in [6-8]. An odd path extendable graph may not be an IM-extendable graph, for example, a cycle C_{2n} is odd path extendable, but C_{2n} is not IM-extendable. Conversely, an IM-extendable graph may not be an odd path extendable graph, for example, the cube $C_4 \times K_2$ is IM-extendable, but it is easily checked that $C_4 \times K_2$ is not odd path extendable.

In this paper, we introduce the notion of odd path extendable graphs and give some simple necessary and sufficient conditions for a graph to be odd path extendable. In particular, we show that if a graph is odd path extendable, it is hamiltonian.

2 Characterization and some structural properties of odd path extendable graphs

In this section, we will give a simple characterization of odd path extendable graphs based on Tutte's 1-factor Theorem and study some structural properties of such graphs.

Theorem A (Tutte's 1-factor Theorem). A graph G has a perfect matching if and only if $c_o(G-S) \leq |S|$ for all $S \subseteq V(G)$.

Theorem 1. A graph G is odd path extendable if and only if

$$c_o(G-S) \leq |S| - (k+1)$$

for every $S \subset V(G)$ such that $G[S]$ contains an odd path of length k.

Proof. The necessity is trivial by Theorem A.

Conversely, let P be an arbitrary odd path in G. Assume that the length of P is k. Define $G' = G - V(P)$. To prove that G' has a perfect matching, it suffices to show that $c_o(G' - S') \leq |S'|$ for any $S' \subset V(G')$ by Theorem A. In fact, letting $S = S' \cup V(P)$, note that $G[S]$ contains an odd path P of length k, we have $c_o(G' - S') = c_o(G - S) \leq |S| - (k+1) = |S'|$ by the hypothesis. Thus G is an odd path extendable graph. □

It is well known that a 1-extendable graph is 2-connected [2]. As mentioned above, an odd path extendable graph is 1-extendable, hence we have the following.

Theorem 2. If a graph G is odd path extendable, then G is 2-connected.

The following is an obvious observation from the definition of the odd path extendable graphs.

Lemma 3. Let G be an odd path extendable graph and let P be an odd path of G. Then $G - V(P)$ contains no odd component.

Based on this observation, we will obtain some structural properties of odd path extendable graphs. Call a cycle is even (odd) if its length is even (odd).

Theorem 4. Let G be an odd path extendable non-bipartite graph and C an odd cycle in G. Let $A = V(G) \setminus V(C)$. Then $G[A]$ is connected.

Proof. Let $C = x_1x_2...x_tx_1$ be an arbitrarily odd cycle of G. Set $A = V(G) \setminus V(C)$. Clearly, $A \neq \emptyset$.

Suppose that $G[A]$ is disconnected. Let D_1, D_2,...,D_k be the components of $G - V(C)$, where $k \geq 2$. Since $|V(G)|$ is even and $|V(C)|$ is odd, by parity, one of D_1, D_2,...,D_k must be an odd component. Without loss of generality, we assume that D_1 is an odd component. Since G is connected, there exists an edge from D_2 to C. Denote such an edge by xx_j, where $x \in V(D_2)$, $x_j \in \{x_1, x_2, ..., x_t\}$. But then $P = xx_jx_{j+1}...x_tx_1...x_{j-1}$ is an odd path and $G - V(P)$ contains the odd component D_1. By Lemma 3, we get a contradiction. □

Now we will show that if a graph is odd path extendable, it is hamiltonian.

Theorem 5. Let G be an odd path extendable graph, then G has a Hamilton cycle.

Proof. First we have the following claim.

Claim. G has a Hamilton path.

Suppose that G has no Hamilton path. We shall derive a contradiction.

Let $P = v_0v_1...v_m$ be a longest path in G of length m ($m < |V(G)|-1$). Then $G - V(P)$ contains some components.

For a component G' of $G - V(P)$, define *the associated path* $P_{G'}$ with respect to G' as follows: $P_{G'} = v_iv_{i+1}...v_j$ is a subpath of $P = v_0v_1...v_m$, where i is the minimal subscript for which $v_i \in V(P)$ is adjacent (in G) to a vertex of G', and j is the maximal subscript for which $v_j \in V(P)$ is adjacent (in G) to a vertex of G'. Since G is 2-connected, $v_i \neq v_j$. Furthermore, by the maximality of P, $v_i \neq v_0$, $v_j \neq v_m$ (see Fig. 1).

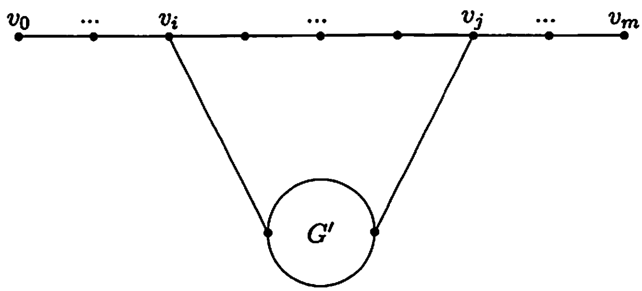

Fig. 1.

Case 1. m is even.

Since $|V(G)|$ is even, by parity, $G - V(P)$ would contain an odd component, say G_1. Let $P_{G_1} = v_iv_{i+1}...v_j$ be the associated path with respect to G_1.

It is clear that P_{G_1} is an even path, otherwise P_{G_1} is an odd path, $G - V(P_{G_1})$ contains an odd component G_1 and by Lemma 3 we have a contradiction. But then the subpath $P'_{G_1} = v_{i-1}v_i...v_j$ of P is an odd path and $G - V(P'_{G_1})$ contains an odd component G_1, a contradiction.

Case 2. m is odd.

In this case, $G - V(P)$ may contain even or odd components. We consider two cases.

Case 2.1. $G - V(P)$ contains an even component.

Let G_2 be an even component of $G - V(P)$ and let $P_{G_2} = v_iv_{i+1}...v_j$ be the associated path with respect to G_2. Suppose v_i is adjacent to a vertex $x \in V(G_2)$. If P_{G_2} is an odd path, then $P'_{G_2} = xv_iv_{i+1}...v_jv_{j+1}$ is also an odd path. Clearly, $G - V(P'_{G_2})$ would contain an odd component, a contradiction.

If P_{G_2} is an even path, then $P''_{G_2} = xv_iv_{i+1}...v_j$ is an odd path, and $G - V(P''_{G_2})$ would contain an odd component, again a contradiction.

Case 2.2. $G - V(P)$ contains an odd component.

Let G_3 be an odd component of $G - V(P)$ and let $P_{G_3} = v_iv_{i+1}...v_j$ be the associated path with respect to G_3.

It is clear that P_{G_3} is an even path, otherwise P_{G_3} is an odd path, $G - V(P_{G_3})$ contains an odd component G_3, a contradiction. But then the subpath $P'_{G_3} = v_{i-1}v_i...v_j$ of P is an odd path and $G - V(P'_{G_3})$ contains an odd component G_3, a contradiction.

The claim thus holds.

Let $P_h = v_1v_2...v_n$ be a Hamiltonian path in G, where $n = |V(G)|$. Then v_1 is adjacent to v_n, otherwise the subpath $P'_h = v_2...v_{n-1}$ of P_h is an odd path and $G - V(P'_h)$ contains two isolated vertices v_1 and v_n, a contradiction. Now $C = P_h + v_1v_n$ is a Hamiltonian cycle in G. □

The class of odd path extendable graphs seems to be quite restricted. We close the paper with the following problem.

- Determine all odd path extendable graphs.

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