On construction of infinite families of k-tight optimal double-loop networks *

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Abstract

A double-loop network(DLN) $G(N;r,s)$ is a digraph with the vertex set $V=\{0,1,\ldots,N-1\}$ and the edge set $E=\{v\rightarrow v+r \pmod N$ and $v\rightarrow v+s \pmod N)|v\in V\}$. Let $D(N;r,s)$ be the diameter of $G(N;r,s)$ and let us define $D(N)=\min\{D(N;r,s)|1\leq r<s<N$ and $\gcd(N,r,s)=1\}$, $D_1(N)=\min\{D(N;1;s)|1<s<N\}$ and $lb(N)=\lceil\sqrt{3N}\,\rceil-2$. It is known that $lb(N)$ is a sharp lower bound for both $D(N)$ and $D_1(N)$. A given DLN $G(N;r,s)$ is called k-tight if $D(N;r,s)=lb(N)+k(k\geq 0)$. A k-tight DLN $G(N;r,s)$ is called optimal if $D(N)=lb(N)+k(k\geq 0)$, and a k-tight DLN $G(N;1,s)$ is called restricted optimal if $D_1(N)=lb(N)+k(k\geq 0)$. Coppersmith proved that there exists an infinite family of N for which the minimum diameter $D(N)\geq\sqrt{3N}+c(logN)^{1/4}$, where c is a constant.

In this paper, we first propose some new approaches to construct infinite families of k-tight double-loop networks (not necessarily restricted optimal) starting from almost all k-tight restricted optimal double-loop networks $G(N;1,s)$. Secondly we prove by Chinese Remainder Theorem that infinite families containing no k-tight($0\leq k\leq m$) optimal double-loop networks $G(N;r,s)$ can be constructed for any integer $m\geq 0$.

Keywords: *Double-loop network, k-tight optimal, k-tight restricted optimal, L-shaped tile, infinite family*

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1 Introduction

Double-loop digraphs $G = G(N; r, s)$, with $1 \leq r < s < N$ and $\gcd(N, r, s) = 1$, have the vertex set $V = \{0, 1, \ldots, N-1\}$ and the adjacencies are defined by $v \rightarrow v + r(\mod N)$ and $v \rightarrow v + s(\mod N)$ for $v \in V$. These kinds of digraphs have been widely studied as architecture for local area networks, known as double-loop networks (DLN). For surveys about these networks, see[3,7].

From the metric point of view, the minimization of the diameter of G corresponds to a faster transmission of messages in the network. The diameter of G is denoted by $D(N; r, s)$. As G is vertex symmetric, its diameter can be computed from the expression $\max\{d(0; i) | i \in V\}$, where $d(u; v)$ is the distance from u to v in G. For a fixed integer $N > 0$, the optimal value of the diameter is denoted by

$$D(N) = \min\{D(N; r, s) | 1 \leq r < s < N \text{ and } \gcd(N, r, s) = 1\}.$$

Several works studied the minimization of the diameter (for a fixed N) with $r = 1$. Let us denote

$$D_1(N) = \min\{D(N; 1; s) | 1 < s < N\}.$$

Since the work of Wong and Coppersmith [10], a sharp lower bound is known for $D_1(N)$:

$$D_1(N) \geq \lceil \sqrt{3N} \rceil - 2 = lb(N).$$

Fiol et al. in [8] proved that $lb(N)$ is also a sharp lower bound for $D(N)$. A given DLN $G(N; r, s)$ is called k-tight if $D(N; r, s) = lb(N) + k (k \geq 0)$. A k-tight DLN with N nodes is called optimal if $D(N) = lb(N) + k (k \geq 0)$, where integer N is called k-tight optimal. A k-tight DLN $G(N; 1, s)$ is called restricted optimal if $D_1(N) = lb(N) + k (k \geq 0)$. A k-tight restricted optimal DLN $G(N; 1, s)$ is also optimal if $D_1(N) = D(N)$. The 0-tight DLN are known as tight ones and they are also optimal.

Although the identity $D(N) = D_1(N)$ holds for most values of N, there are also another infinite set of integers with $D(N) < D_1(N)$. These other integral values of N are called non-unit step integers or nus integers in [2]. Thus, for most restricted optimal k-tight DLN $G(N; 1, s)$, it is also k-tight optimal.

The metrical properties of $G(N; r, s)$ are fully contained in its related L-shaped tile $L(N; l, h, x, y)$ with $N = lh - xy$. In Figure 1, we illustrate generic dimensions of an L-shaped tile.

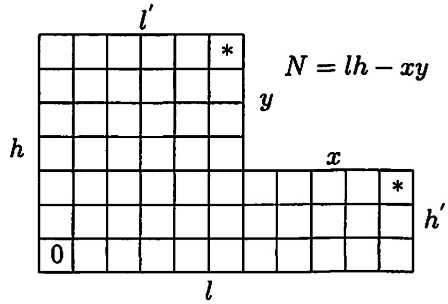

Figure 1: Generic dimensions of an L-shaped tile

Let $D(L) = D(L(N;l,h,x,y)) = \max\{l+h-x-2, l+h-y-2\}$. For obvious reasons, the value $D(L)$ is called the diameter of the tile L. It is known that an L-shaped tile $L(N;l,h,x,y)$ can be assigned to a $G(N;r,s)$ without any confusion [2,8]. It is said that the tile $L(N;l,h,x,y)$ can be realized by $G(N;r,s)$. However, we can not find double-loop network $G(N;r,s)$ from some L-shaped tiles. When an L-shaped tile $L(N;l,h,x,y)$ has diameter $lb(N)+k$, we say it is k-tight.

Coppersmith in a private communication to D.F.Hsu (quoted in [3,5,7]) proved that there exists an infinite number of N for which the minimum diameter $D(N) \geq \sqrt{3N} + c(logN)^{1/4}$, where c is a constant. Xu and Liu [11] gave an infinite family of 4-tight optimal double-loop networks. It is known that finding infinite families of k-tight optimal DLN is a difficult task as the value k increases.

The remaining of this paper will be organized as follows. Some lemmas, which will be used throughout this paper, are introduced in Section 2. In section 3, we propose some new approaches to construct infinite families of k-tight double-loop networks (not necessarily optimal) starting from almost all k-tight optimal double-loop networks $G(N;1,s)$. In section 4, we prove by Chinese Remainder Theorem that infinite families containing no k-tight($0 \leq k \leq m$) optimal double-loop networks $G(N;r,s)$ can be constructed for any integer $m \geq 0$, and infinite families of k-tight($k > m$) restricted optimal double-loop networks $G(N;1,s)$ can be constructed for any integer $m \geq 0$. Finally, section 5 presents an example to illustrate our main approaches.

2 Preliminary

The following Lemma 1, 2, 3 and 4 can be found in [6 or 8 or 9].

Lemma 1[6, 9]. Let t be a nonnegative integer. We define $I_1(t) = [3t^2+1, 3t^2+2t]$, $I_2(t) = [3t^2+2t+1, 3t^2+4t+1]$ and $I_3(t) = [3t^2+4t+2, 3(t+1)^2]$. Then we have $[4, 3T^2+6T+3] = \bigcup_{t=1}^{T}\bigcup_{i=1}^{3} I_i(t)$, where $T > 1$, and $lb(N) = 3t+i-2$ if $N \in I_i(t)$ for $i = 1, 2, 3$.

Lemma 2[8, 9]. Let $L(N; l, h, x, y)$ be an L-shaped tile, $N = lh - xy$. Then,

(a) There exists a $G(N; 1, s)$ realizing the L-shaped tile iff $l > y$ and $h \geq x$ or $l \geq y$ and $h > x$, and $\gcd(h, y) = 1$, where $s \equiv \alpha l - \beta(l-x)(\mod N)$ for some integral values α and β satisfying $\alpha y + \beta(h-y) = 1$.

(b) There exists a $G(N; r, s)$ realizing the L-shaped tile iff $l > y$ and $h \geq x$ or $l \geq y$ and $h > x$, and $\gcd(l, h, x, y) = 1$, where $r \equiv \alpha h + \beta y(\mod N)$, $s \equiv \alpha x + \beta l(\mod N)$ for some integral values α and β satisfying $\gcd(N, r, s) = 1$.

Lemma 3[9]. Let $L(N; l, h, x, y)$ be an L-shaped tile, $N = lh - xy$. Then
(a) If $L(N; l, h, x, y)$ is realizable, then $|y-x| < \sqrt{N}$;
(b) If $x > 0$ and $|y-x| < \sqrt{N}$, then
$D(L(N; l, h, x, y)) \geq \sqrt{3N - \frac{3}{4}(y-x)^2} + \frac{1}{2}|y-x| - 2$;
(c) Let $f(z) = \sqrt{3N - \frac{3}{4}z^2} + \frac{1}{2}z$. Then $f(z)$ is strictly increasing when $0 \leq z \leq \sqrt{N}$.

Lemma 4[9]. Let $N(t) = 3t^2 + At + B \in I_i(t)$ and L be the L-shaped tile $L(N(t); l, h, x, y)$, where A and B are integral values; $l = 2t + a$, $h = 2t + b$, $z = |y - x|$, a, b, x, y are all integral polynomials of variable t, and $j = i + k (k \geq 0)$. Then L is k-tight iff the following identity holds

$$(a+b-j)(a+b-j+z) - ab + (A+z-2j)t + B = 0. \qquad (1)$$

The following Lemma 5 is the generalization of Theorem 2 in [11], and can be found in [12].

Lemma 5[12]. Let $H(z, j) = (2j-z)^2 - 3[j(j-z) + (A+z-2j)t + B]$, and the identity (1) be an equation of a and b. A necessary condition for the equation (1) to have integral solution is that $4H(z, j) = s^2 + 3m^2$, where s and m are integers.

It is easy to show that the following Lemma 6 is equivalent to Theorem 1 in [11]. Lemma 6 can be found in [12].

Lemma 6[12]. Let n, s and m be integers, $n = s^2 + 3m^2$. If n has a prime factor p, here $p \equiv 2(\mod 3)$, then there exists an even integer q, such that n is divisible by p^q, but not divisible by p^{q+1}.

Lemma 7[12]. Let $N = N(t) = 3t^2 + At + B \in I_i(t)$ and L-shaped tile $L(N; l, h, x, y)$ be k-tight($k \geq 0$) and realizable. Let $z = |y - x|$. Then the following hold

Case 1. If $A = 0$ or $A = 2$(if $i = 2$) or $A = 4$(if $i = 3$), and $3N - \frac{3}{4}(2k+3)^2 > (3t + \frac{A-1}{2})^2$, then $0 \leq z \leq 2k + 2$.

Case 2. If $A = 1$ or $A = 3$ or $A = 5$, and $3N - \frac{3}{4}(2k+2)^2 > (3t + \frac{A-1}{2})^2$, then $0 \leq z \leq 2k + 1$.

Case 3. If $A = 2$(if $i = 1$) or $A = 4$(if $i = 2$) or $A = 6$, and $3N - \frac{3}{4}(2k+1)^2 > (3t + \frac{A-1}{2})^2$, then $0 \leq z \leq 2k$.

Lemma 8. There exists an infinite number of prime p, where $p \neq 2$ and $p \equiv 2(\mod 3)$.

Proof. We prove it by contradiction.

Suppose there are only primes: $p_1, p_2, \ldots, p_m$, such that $p_i \neq 2$ and $p_i \equiv 2(\mod 3)$ for $1 \leq i \leq m$.

Let $p = 3p_1p_2 \cdots p_m + 2$. It is easy to know that p has a prime factor q, such that $q \notin \{2, p_1, p_2, \ldots, p_m\}$ and $q \equiv 2(\mod 3)$, which is a contradiction.

We have this lemma. □

3 Infinite families of k-tight double-loop networks

Here we must note that the conditions of Lemma 7 are satisfied by almost all k-tight optimal and realizable L-shaped tile $L(N(t); l, h, x, y)$. If an L-shaped tile $L(N(t); l, h, x, y)$ does not satisfy the condition in Case 3. That is,

$$3N(t) - \frac{3}{4}(2k+1)^2 \leq (3t + \frac{A-1}{2})^2$$

Hence,

$$3(t + B) \leq \frac{3}{4}(2k+1)^2 + (\frac{A-1}{2})^2.$$

We may let $A_1 = A - 1$ and $B_1 = B + t$. Then this is Case 2, and if the following holds,

$$3(t + B_1) > \frac{3}{4}(2k + 2)^2 + (\frac{A_1 - 1}{2})^2.$$

This is equivalent to the condition in Case 2.

Otherwise, we may let $A_2 = A_1 - 1 = A - 2$ and $B_2 = B_1 + t = B + 2t$. Then this is Case 1, and if the following holds,

$$3(t + B_2) > \frac{3}{4}(2k + 3)^2 + (\frac{A_2 - 1}{2})^2.$$

This is equivalent to the condition in Case 1, to which there is no counter example up to now.

Theorem 1 has the same idea of procreation given in [1].

Theorem 1. Let $N(t) = 3t^2 + At + B \in I_i(t)$, where $1 \leq i \leq 3$. If L-shaped tile $L(N(t_0); l_0, h_0, x_0, y_0)$ satisfies the conditions of Lemma 7 and can be realized by a double-loop network $G(N(t_0); 1, s_0)$, where $l_0 = 2t_0 + a_0, h_0 = 2t_0 + b_0, x_0 = t_0 + a_0 + b_0 - j, z = y_0 - x_0 \geq 0$(or $y_0 = t_0 + a_0 + b_0 - j, z = x_0 - y_0 \geq 0$), $j = i + k$, a_0 and b_0 are integers, then an infinite family of k-tight double-loop networks $G(N; 1, s)$ (not necessarily restricted optimal) can be constructed starting from $L(N(t_0); l_0, h_0, x_0, y_0)$.

Proof. From Lemma 7, $2j - A - z \geq 0$. Let $q = h_0 - 2y_0$. We only prove the case of $q > 0$ and $y_0 - x_0 \geq 0$. The others are similar.

Case 1. $2j - A - z = 0$. The equation (1) is equivalent to the following, $(a + b - j)(a + b - j + z) - ab + B = 0$.

Let $t = qf + t_0, l = 2qf + l_0, h = 2qf + h_0, x = qf + x_0, y = qf + y_0, h' = h - y, l' = l - x$, where f is a nonnegative integer.

From Lemma 2(a), $\gcd(y_0, h_0 - y_0) = 1$, that is $\gcd(y_0, h_0 - 2y_0) = 1$, thus,

$$\gcd(y, h) = \gcd(y, h - 2y) = \gcd(qf + y_0, h_0 - 2y_0) = \gcd(y_0, q) = 1.$$

Suppose $\alpha_0 y_0 + \beta_0(h_0 - 2y_0) = 1$, then $\alpha_0(y - f(h' - y)) + \beta_0(h' - y) = 1$, that is $(\alpha_0 f + \alpha_0 - \beta_0)y + (-\alpha_0 f + \beta_0)h' = 1$

Let $s(f) \equiv (\alpha_0 f + \alpha_0 - \beta_0)l + (\alpha_0 f - \beta_0)l' (\mod N(t))$.

Let $t_1 = qf$. Then,

$$\begin{aligned} lh - xy - N(t_0) &= (2t_1 + l_0)(2t_1 + h_0) - (t_1 + x_0)(t_1 + y_0) - N(t_0) \\ &= 3t_1^2 + t_1(2l_0 + 2h_0 - x_0 - y_0) \\ &= 3t_1^2 + t_1(6t_0 + 2j - z) \\ &= 3t_1^2 + t_1(6t_0 + A) \\ &= N(t) - N(t_0) \end{aligned}$$

Thus, $lh - xy = N(t)$, and L-shaped tile $L(N(t); l, h, x, y)$ can be realized by $G(N(t); 1, s(f))$.

From Lemma 4, L-shaped tile $L(N(t); l, h, x, y)$ is k-tight, but not necessarily optimal.

Case 2. $2j - A - z > 0$.

Let $r = 2j - A - z, a = rqf + a_0, b = -2rqf + b_0$, where f is a nonnegative integer. By equation (1), we have $t = 3rq^2f^2 + qf(-3b_0 + 2j - z) + t_0$.

Let $l = 2t + a, h = 2t + b, x = t + a + b - j, y = t + a + b - j + z, h' = h - y, l' = l - x,$ Note that

$y = t + a + b - j + z = 3rq^2f^2 + qf(-3b_0 + 2j - z - r) + y_0 = 3rq^2f^2 + qf(-3b_0 + A) + y_0$,

and from Lemma 2, we have,

$\gcd(y, h) = \gcd(y, h - 2y) = \gcd(3rq^2f^2 + qf(-3b_0 + A) + y_0, h_0 - 2y_0) = \gcd(y_0, q) = 1$.

Suppose $\alpha_0 y_0 + \beta_0(h_0 - 2y_0) = 1$, and note that

$$y = 3rq^2f^2 + qf(-3b_0 + A) + y_0 = f(h' - y)(3rqf - 3b_0 + A) + y_0,$$

thus,

$$\alpha_0(y - f(h' - y)(3rqf - 3b_0 + A)) + \beta_0(h' - y) = 1,$$

That is

$$(\alpha_0 f(3rqf - 3b_0 + A) + \alpha_0 - \beta_0)y + (-\alpha_0 f(3rqf - 3b_0 + A) + \beta_0)h' = 1.$$

Let $s(f) \equiv (\alpha_0 f(3rqf - 3b_0 + A) + \alpha_0 - \beta_0)l + (\alpha_0 f(3rqf - 3b_0 + A) - \beta_0)l' (\mod N(t))$.

Let $t_1 = 3rq^2f^2 + qf(-3b_0 + 2j - z)$. Then,

$$\begin{aligned}
& lh - xy - N(t_0) \\
= \;& (2t_1 + rqf + l_0)(2t_1 - 2rqf + h_0) - (t_1 - rqf + x_0)(t_1 - rqf + y_0) - N(t_0) \\
= \;& 3t_1^2 + t_1(2l_0 + 2h_0 - x_0 - y_0) - 3(rqf)^2 + rqf(h_0 - 2l_0 + x_0 + y_0) \\
= \;& 3t_1^2 + t_1(6t_0 + 2j - z) - 3(rqf)^2 + rqf(3b_0 - 2j + z) \\
= \;& 3t_1^2 + t_1(6t_0 + A) + rt_1 - 3(rqf)^2 + rqf(3b_0 - 2j + z) \\
= \;& N(t) - N(t_0) + 3(rqf)^2 + rqf(-3b_0 + 2j - z) - 3(rqf)^2 + rqf(3b_0 - 2j + z) \\
= \;& N(t) - N(t_0)
\end{aligned}$$

Thus, $lh - xy = N(t)$, and L-shaped tile $L(N(t); l, h, x, y)$ can be realized by $G(N(t); 1, s(f))$.

From Lemma 4, L-shaped tile $L(N(t); l, h, x, y)$ is k-tight, but not necessarily optimal.

We have this theorem. □

4 Infinite families containing no k-tight $(0 \leq k \leq m)$ optimal double-loop networks $G(N; r, s)$ for any integer $m \geq 0$

Theorem 2 is our main result, which will be used to prove Theorem 3.
Theorem 2. Given $N(t) = 3t^2 + At + B \in I_i(t), 1 \leq i \leq 3$, infinite families of $\{N(t); t = de + c\}$,where d, c are integers, $e \geq 0$ is an integral variable, can be constructed for any integer $m \geq 0$, so that there does not exist any k-tight$(0 \leq k \leq m)$ optimal double-loop network $G(N; r, s)$ with $N(de + c)$ nodes.

Proof. Let $N(t) = 3t^2 + At + B \in I_i(t), i = \frac{A+1}{2}, 1 \leq i \leq 3$, where $t > t_0$, t_0 is a constant, $A = 1$ or $A = 3$ or $A = 5$. In fact, we can get the same results in the case $i = 1 + \frac{A}{2}$ and $A = 0$ or $A = 2$ or $A = 4$.

When t_0 is large enough, we may guarantee that $B > \frac{1}{4}(2m + 1)^2 + (\frac{A}{2})^2/3$ and $N(t) = 3t^2 + At + B \in I_i(t)$. Then,

$$3B - \tfrac{3}{4}(2m + 1)^2 > (\tfrac{A}{2})^2. \text{ That is, } 3N(t) - \tfrac{3}{4}(2m + 1)^2 > (3t + \tfrac{A}{2})^2.$$

For any $k(0 \leq k \leq m))$, $3N(t) - \frac{3}{4}(2k + 1)^2 > (3t + \frac{A}{2})^2$. From Lemma 7, we know that there does not exist any k-tight L-shaped tile $L(N(t); l, h, x, y)$ for $z \geq (2k + 1)$. For $0 \leq z \leq 2k, j = i + k$, if we can guarantee that $4H(z, j)$ has a prime factor p with an odd power, where $p \equiv 2($ mod 3), then by Lemma 6, 5, and 4, we know that there does not exist any k-tight L-shaped tile.

Now our only task is to prove that $4H(z, j)$ has a prime factor p with an odd power, where $p \equiv 2($ mod 3), $0 \leq z \leq 2k, j = i + k, 0 \leq k \leq m$. Since $H(z, j) = (2j - z)^2 - 3[j(j - z) + (A + z - 2j)t + B], A + z - 2j \leq 2i - 1 + 2k - 2(i + k) = -1$, hence $4H(z, j)$ are all polynomials of order 1. Let us denote these polynomials by: $a_i t + b_i, 1 \leq i \leq d$, where $d = (m + 1)^2$.

From Lemma 8, let us denote all primes by: $p_1, p_2, \ldots, p_i, \ldots$, where $p_i \equiv 2($ mod 3) for $i \geq 1$.

Without loss of generality, we assume that $\gcd(a_i, p_i) = 1$ for $1 \leq i \leq d$. Suppose $\alpha_i a_i + \beta_i p_i^2 = 1$, then we have $\alpha'_i a_i + \beta'_i p_i^2 = p_i - b_i$, that is, $\alpha'_i a_i + b_i = -\beta'_i p_i^2 + p_i$. Thus there exists c_i, such that $a_i t + b_i \equiv p_i($ mod $p_i^2)$ for any $t = p_i^2 e + c_i, e \geq 0$.

Since p_i^2 mutually prime to each other, by Chinese Remainder Theorem, we know that there exists a solution to the following congruences.

$$\begin{cases} t \equiv c_1(\mod p_1^2) \\ t \equiv c_2(\mod p_2^2) \\ \cdots\cdots \\ t \equiv c_d(\mod p_d^2) \end{cases}$$

Suppose the solution is $t = p_1^2 p_2^2 \cdots p_d^2 e + c, e \geq 0$ and $c > t_0$. If $0 \leq z \leq 2k, j = i + k, 0 \leq k \leq m, t = p_1^2 p_2^2 \cdots p_d^2 e + c$, then $4H(z, j)$ has a prime factor p with an odd power, where $p \equiv 2(\mod 3)$.

Therefore, there does not exist any k-tight$(0 \leq k \leq m)$ optimal double-loop network $G(N(t); r, s)$, where $t = p_1^2 p_2^2 \cdots p_d^2 e + c(e \geq 0)$.

We have this theorem. □

Based on Theorem 1 and Theorem 2, it is easy to prove the following Theorem 3.

Theorem 3. For any integer $m \geq 0$, there exists integer $k(k > m)$, such that infinite families of k-tight restricted optimal double-loop networks $G(N; 1, s)$ can be constructed.

Proof. From Theorem 2, we have $N(t) = 3t^2 + At + B \in I_i(t), i = \frac{A+1}{2}, 1 \leq i \leq 3$, where $t = p_1^2 p_2^2 \cdots p_d^2 e + c(e \geq 0), A = 1$ or $A = 3$ or $A = 5$, there does not exist any k-tight$(0 \leq k \leq m)$ double-loop network $G(N(t); r, s)$.

Suppose that there exists a $(m+k_1)$-tight restricted optimal double-loop network $G(N(t_1); 1, s_1)$;

There exists a $(m + k_2)$-tight restricted optimal double-loop network $G(N(t_2); 1, s_2)$ and $k_2 < k_1$;

There exists a $(m + k_3)$-tight restricted optimal double-loop network $G(N(t_3); 1, s_3)$ and $k_3 < k_2$;

$\cdots\cdots$

Since $k_i \geq 1$, this sequence is finite. Hence we may assume that there does not exist any k-tight$(0 \leq k \leq m + n)$ restricted optimal double-loop network with $N(t)$ nodes, where $t = p_1^2 p_2^2 \cdots p_d^2 e + t_0(e \geq 0)$ and $n \geq 0$, but there does exist a $(m + n + 1)$-tight restricted optimal double-loop network $G(N(t_0); 1, s(t_0))$, and $3N(t) - \frac{3}{4}(2(m + n + 1) + 2)^2 > (3t + \frac{A-1}{2})^2$ for $e \geq 0$.

From Theorem 1, an infinite family of $(m+n+1)$-tight restricted optimal double-loop networks can be constructed starting from $(m+n+1)$-tight restricted optimal double-loop network $G(N(t_0); 1, s(t_0))$, which has an L-shaped tile $L(N(t_0); l_0, h_0, x_0, y_0)$.

In the case of $q = h_0 - 2y_0 > 0$, and $y_0 - x_0 \geq 0, 2j - A - z > 0$. The others are similar. Let $N(t) = 3t^2 + At + B, t = 3rq^2f^2 + qf(-3b_0 + 2j - z) + t_0, f = p_1^2 p_2^2 \cdots p_d^2 e (e \geq 0)$. Then there exists a $(m+n+1)$-tight restricted optimal double-loop network $G(N(t); 1, s)$.

We have this theorem. □

5 An example

To illustrate the above theorems, we present an example.

Example 1. Let $N(t) = 3t^2 + 3t + 17 \in I_2(t), m = 2$. Now we derive an infinite family of 3-tight optimal double-loop networks $G(N(t); 1, s)$.

Since $B = 17 > \frac{1}{4}(2m+1)^2 + (\frac{A}{2})^2/3 = \frac{1}{4}(2 \cdot 2 + 1)^2 + (\frac{3}{2})^2/3$, from Theorem 2, we now only need to consider $H(z, j) = (2j - z)^2 - 3[j(j - z) + (3 + z - 2j)t + 17]$, for $0 \leq z \leq 2k, j = 2 + k, 0 \leq k \leq 2$.

It is easy to show that,
$H(0, 2) = 3t - 47$, when $t \equiv 3 \pmod 4$, has a factor 2 with power 1;
$H(2, 3) = 3t - 44$, when $t \equiv 3 \pmod{25}$, has a factor 5 with power 1;
$H(1, 3) = 6t - 44$, when $t \equiv 3 \pmod 4$, has a factor 2 with power 1;
$H(0, 3) = 9t - 42$, when $t \equiv 3 \pmod{25}$, has a factor 5 with power 1;
$H(4, 4) = 3t - 35$, when $t \equiv 3 \pmod 4$, has a factor 2 with power 1;
$H(3, 4) = 6t - 38$, when $t \equiv 3 \pmod{25}$, has a factor 5 with power 1;
$H(2, 4) = 9t - 39$, when $t \equiv 19 \pmod{121}$, has a factor 11 with power 1;
$H(1, 4) = 12t - 38$, when $t \equiv 3 \pmod 4$, has a factor 2 with power 1;
$H(0, 4) = 15t - 35$, when $t \equiv 3 \pmod{25}$, has a factor 5 with power 1.

Thus we focus on the following congruences,

$$\begin{cases} t \equiv 3 \pmod 4 \\ t \equiv 3 \pmod{25} \\ t \equiv 19 \pmod{121} \end{cases}$$

There is a solution, $t = 100 \cdot 121 \cdot e + 503$, where $e \geq 0$ is an integer.

From Theorem 2, $\{N(t) = 3t^2 + 3t + 17; t = 100 \cdot 121 \cdot e + 503, e \geq 0\}$ is an infinite family, in which there does not exist any k-tight$(0 \leq k \leq 2)$ optimal double-loop network $G(N(t); r, s)$.

For $t = 503$, $N(t) = 3t^2 + 3t + 17 = 760553$, $D(760553; 1, 156540) = lb(N) + 3$, and it is checked by computer that $D(760553) = lb(N) + 3$, thus $G(760553; 1, 156540)$ is a 3-tight optimal double-loop network.

From $G(760553; 1, 156540)$, we get 3-tight L-shaped tile $L(760553 : 993, 996, 475, 481)$. Further, we know that, $a_0 = -13, b_0 = -10, j = 2+3 = 5, z = 6, q = h_0 - 2y_0 = 34, 2j - A - z = 1 \neq 0$.

From Theorem 1, let $a = 34f - 13, b = -68f - 10, t = 3(34)^2 f^2 + 34f(34) + 503$. Since $7(481) - 99(34) = 1$. Note that $y = f(h' - y)(3rqf - 3b_0 + A) + y_0$, thus,

$$7\{y - f(h' - y)[3(34)f + 33]\} - 99(h' - y) = 1.$$

That is,

$$\{7f[3(34)f + 33] + 106\}y + \{-7f[3(34)f + 33] - 99\}h' = 1.$$

Let, $s(f) \equiv \{7f[3(34)f + 33] + 106\}(2t + a) + \{7f[3(34)f + 33] + 99\}(t - b + j)(\mod N(t))$.

From Theorem 1, $\{G(3t^2 + 3t + 17; 1, s(f)) | t = 3(34)^2 f^2 + 34f(34) + 503, f = 50 \cdot 121 \cdot e, e \geq 0\}$ is an infinite family of 3-tight optimal double-loop networks.

Let $e = 0$. Then $f = 0, t = 503, s(0) = 106 \times 993 + 99 \times 518 = 156540$. Thus, $G(N(503); 1, s(0))$ is a 3-tight optimal double-loop network.

References

[1] F. Aguiló, M.A. Fiol, An efficient algorithm to find optimal double loop networks, *Discrete Mathematics*,138(1995) 15-29.

[2] F. Aguiló, E. Simó, M. Zaragozá, Optimal double-loop networks with non-unit steps, *The Electronic Journal of Combinatorics*, 10 (2003), ♯R2.

[3] J.-C. Bermond, F. Comellas, D.F. Hsu, Distributed loop computer networks: a survey, *J. Parallel Distribut. Comput.*, 24 (1995) 2-10.

[4] C.F. Chan, C. Chen and Z.X. Hong, A simple algorithm to find the steps of double-loop networks, *Discrete Applied Mathematics*, Vol.121 (2002) 61-72.

[5] P. Erdős, D.F. Hsu, Distributed loop networks with minimum transmission delay, *Theoret. Comput. Sci.*, 100 (1992) 223-241.

[6] P.Esqué, F. Aguiló and M.A. Fiol, Double commutative-step digraphs with minimum diameters, *Discrete Mathematics*,114 (1993)147-157.

[7] F.K. Hwang, A complementary survey on double-loop networks, *Theoret. Comput. Sci.*, 263(2001) 211-229.

[8] M.A. Fiol, J.L.A. Yebra, I. Alegre and M. Valero, A discrete optimization problem in local networks and data alignment, *IEEE Trans. Comput.*,C-36 (1987) 702-713

[9] Li Q., Xu J. and Zhang Z., The infinite families of optimal double loop networks, *Discrete Applied Mathematics*, Vol.46(1993)179-183.

[10] C.K. Wong and D. Coppersmith, A combinatorial problem related to multimode memory organizations, *J. Assoc. Comput. Mach.*, 21 (1974) 392-402.

[11] Xu J. and Liu Q., An infinite family of 4-tight optimal double loop networks, *Science in China, Series A*, Vol. A-46 num. 1 (2003) 139-143.

[12] Zhou J. and Xu X., On infinite families of optimal double-loop networks with non-unit steps, *Ars Combinatoria*, accepted.