# On the Graceful Conjecture of Permutation Graphs of Paths *

Zhihe Liang†
Department of Mathematics, Hebei Normal University
Shijiazhuang 050016, P. R. China

**Abstract.** S.M.Lee proposed the conjecture: for any $n > 1$ and any permutation $f$ in $S(n)$, the permutation graph $P(P_n, f)$ is graceful. For any integer $n > 1$, we discuss gracefulness of the permutation graphs $P(P_n, f)$ when $f$=(123), $(n-2, n-1, n)$, $(i, i+1)$, $1 \leq i \leq n-1$, (12)(34)...$(2m-1, 2m)$, $1 \leq m \leq n/2$, and give some general results.

**Keywords:** permutation graph; graceful; graceful conjecture of permutation graphs of paths

## 1 Introduction

Let $G$=$(V, E)$ be a finite simple $(p, q)$ graph. The graph $G$ is called graceful if there exists an one-to-one mapping $g$ from $V$ to $\{0, 1, 2, ..., q\}$ such that the induced edge labels $\{g^*(uv) = |g(u) - g(v)| \mid uv \in E\}$ is equal to $\{1, 2, ..., q\}$. The mapping $g$ is said to be a graceful labelling of $G$, and $g^*$ is said to be a edge labelling of $G$. The concept of graceful graph is due to Rosa [1]. In 1967, he introduced the notion of $\beta$-valuation, which Golomb [2] subsequently called graceful labelling. In general, it is hard to decide whether a given graph is graceful. Even if a graph is known to be graceful, it may still be difficult to find a graceful labelling. Research has focused on specific classes of trees, bipartite graphs, cycles, cycle with a pendant edge attached to at each vertex and wheel, cycles with chords, grid graphs, one

*This work was supported by National Natural Science Foundation of China.
†E-mail address: zhiheliang@163.com.cn (Z. Liang)

point union of two cycles, one-point union of complete graphs, unicyclic graphs, etc (see [4]). The theory of graceful graphs has attracted many mathematicians mainly because of its aesthetic aspect, as well as its wide range of applications in such areas as radar pulse codes, x-ary crystallography, circuit design, missile guidance, radio astronomy, sonar ranging, and broadcast frequency assignments (see [4,5,6]). Recently, instead of studying special classes of graceful graphs, attention has been turned to their construction. In this paper, we will consider the construction of graceful graphs via permutation graphs.

Chartrand and Harary introduced the concept of permutation graphs. For a graph $G$ with $n$ vertices labelled $1, 2, ..., n$, where $n \geq 3$, and a permutation $f$ in $S(n)$, the symmetric group on the symbols $\{1, 2, ..., n\}$, the $f$-permutation graph on $G$, denoted $P(G, f)$, consists of two disjoint copies of $G$, say $G_1$ and $G_2$, along with the $n$ edges obtained by joining $u_i$ in $G_1$ with $v_{f(i)}$ in $G_2$, $i$=1,2,...,$n$. It is obvious that the graph $P(G, (1))$ is isomorphic to $G \times K_2$. If $G$ is a cycle $C_n$, then the (1)-permutation graph of $C_n$, $P(C_n, (1))$, is a prism graph. Let $P_n$ denote the path with $n$ vertices. In 1983, Lee proposed the graceful conjecture of permutation graphs of paths: for any $n > 1$ and any permutation $f$ in $S(n)$, the permutation graph $P(P_n, f)$ is graceful. Gallian restated this conjecture in [4]. It and Ringel-Kotzig graceful tree conjecture that postulates the existence of a graceful labelling of any tree have received a lot of attention. The two conjectures are important. Gracefulness of $P(G, f)$ have been considered in the following literature.

(1) In [9], $G$ is hypercube and $f$ is an identity map.

(2) In [4, 10], $G$ is a star $K_{1,n}$ and $f$ is an identity map.

(3) In [11, 12], $G$ is a cycle and $f$ is an identity map.

For the graceful permutation graphs of paths, known results are as follows.

**Lemma 1.1**[7,13] Let $n > 1$ be a positive integer and $f$ a permutation in $S(n)$. Then permutation graph of path $P(P_n, f)$ is graceful

(1) for any $n < 5$ and any permutation $f$ in $S(n)$;

(2) when $f$=(1) and (12);

(3) when $n$=$2k$ and $f$=$(12)(34)...(k,k+1)...(2k-1,2k)$;

(4) when $n$=$2k+1$ and $f$=$(23)(45)...(k,k+1)...(2k,2k+1)$;

(5) when $n$=$2k$ and $f$=$(34)(5,6)...(2k-1,2k)$;

(6) when $n$=$2k+1$ and $f$=$(45)(6,7)...(2k,2k+1)$;

(7) $f$=(12345), (2345), (234), (123456) and (23)(45).

Let $n > 1$ be a positive integer and $f$ a permutation in $S(n)$. In this article, we prove that $P(P_n, f)$ is graceful when

(1) $f$=$(n-2, n-1, n)$;

(2) $n - 5m + 11 \equiv 1 \pmod 3$, $f$=$(2m-1, 2m)$ for $1 \le m \le (n+7)/2$, or $f$=$(2m, 2m+1)$ for $m \ge 1$;

(3) $n - 5m + 11 \equiv 2 \pmod 3$, $f$=$(2m, 2m+1)$ for $m < (n+6)/5$, or $f$=$(2m-1, 2m)$ for $m \ge 1$;

(4) $n - 5m + 11 \equiv 0 \pmod 3$ and $m < (n+8)/5$, $f$=$(2m-1, 2m)$, or $(2m, 2m+1)$;

(5) $m \equiv 0 \pmod 3$ and $n < (10m+6)/3$ in conditions 1 and 2 (see section 3), $f$=$(12)(34)...(2m-1,2m)$;

(6) $m \le 3$ and $n > 1$ in conditions 1 and 2 (see section 3), $f$=$(12)(34)...(2m-1,2m)$;

(7) $m \not\equiv 0 \pmod 3$ and $n > 1$ in conditions 1 and 2 (see section 3), $f$=$(12)(34)...(2m-1,2m)$ for $1 \le m \le \lfloor n/2 \rfloor$;

(8) under conditions 3 and 4 (see section 3), $f$=$(12)(34)...(2m-1,2m)$ for $1 \le m \le \lfloor n/2 \rfloor$.

In addition, if the function $g(x)$ has value $n+1-x$ if $f(x) \ne x$, or $x$ if $f(x) = x$, then $P(P_n, f^{-1})$, $P(P_n, gf)$ and $P(P_n, (gf)^{-1})$ are graceful.

Let $Z$ be the ring of integers and let $a, b \in Z$. The following notations are used frequently.

$[a,b]=\{x \mid x \in Z,\ a \le x \le b\}$,

$[a,b]_k=\{x \in Z \mid a \le x \le b, x \equiv a(mod\ k)\}$,

$\lfloor x \rfloor=\max\{y \mid y \le x, y \in Z\}$ for any real number $x$,

$f(S)=\{f(x) |\ x \in S\}$ where $S$ is a set and $f$ is a function,

$(a_1, a_2, ..., a_n)^{-1}=(a_n, a_{n-1}, ..., a_1)$ if $(a_1, a_2, ..., a_n)$ is a sequence.

# 2 Preliminary results

**Lemma 2.1** Let $(a_1, a_2, \cdots, a_{n+1})$ be a sequence, and $a_{2i}=a+(i-1)d_1$, $a_{2i-1}=b+(i-1)d_2$, $c_{2i-1}=|a_{2i}-a_{2i-1}|$, $c_{2i}=|a_{2i+1}-a_{2i}|$. If $d_1=-d_2$, then $(c_1, c_2, \cdots, c_n)$ is an arithmetic progression with common difference $d$, where $d=-d_2$ when $a_{2i} > a_{2i-1}$ for any $i \in [1, \lfloor (n+1)/2 \rfloor]$, $d=d_2$ when $a_{2i} < a_{2i-1}$ for any $i \in [1, \lfloor (n+1)/2 \rfloor]$.

**Proof** Since $a_{2i}-a_{2i-1}=(a-b)+(i-1)(d_1-d_2)$ and $a_{2i+1}-a_{2i}=d_2-[(a-b)+(i-1)(d_1-d_2)]$, the theorem is true. □

Let $(a_1, a_2, ..., a_m)$ and $(b_1, b_2, ..., b_n)$ be two strictly monotone sequences, we call them complementary if their elements form a continual integers section $[a, a+m+n-1]$, where $a=\min\{a_i, b_j | i \in [1,m], j \in [1,n]\}$. This section is said to be their generating set. For instance, the two sequences $(2,5,6,8,9,13)$ and $(12,11,10,7,4,3)$ are complementary. Let $A=[n, n+3m-1]$ be a strictly monotone increasing sequence, and $B=\{a-3i | i \in [0, m-1]\}$ be a strictly monotone decreasing sequence. The strictly monotone increasing sequence $\bigcup_{i=0}^{m-1}\{|n+6i-a|, |n+6i-a+1|, |n+6i-a+2|\}$ is denoted by $A \rhd B$.

**Lemma 2.2** For strictly monotone increasing sequences $U=[u, u+3v-1]$, $Y=\bigcup_{i=0}^{y-1}\{z+6i, z+6i+1, z+6i+2\}$, and strictly monotone decreasing sequence $W=\{w-3i \mid i \in [0, v-1]\}$,

(1) if $w > u+6v-4$, then $Y$ and $U \rhd W$ are complementary if and only if $(w-u-6v+4-z)sgn(y-v)=3$ for $|y-v|=1$, or $|w-u-6v+4-z|=3$ for $y=v$;

(2) if $w < u$, then $Y$ and $U \rhd W$ are complementary if and only if $(u-w-z)sgn(y-v)=3$ for $|y-v|=1$, or $|u-w-z|=3$ for $y=v$.

**Proof** (1) When $w > u+6v-4$, $U \rhd W=\bigcup_{i=0}^{v-1}\{w-u-6v+4+6i, w-u-6v+5+6i, w-u-6v+6+6i\}$ is a strictly monotone increasing sequence.

Since $Y$ and $U \rhd W$ are complementary, $|y-v| \le 1$. If $y=v+1$, then $z=(w-u-6v+4)-3$, i.e. $(w-u-6v+4-z)sgn(y-v)=3$. If $y=v-1$, then $(w-u-6v+4-z)sgn(y-v)=3$. If $y=v$, by definition of complementary sets, we have $|w-u-6v+4-z|=3$.

Conversely, when $(w-u-6v+4-z)sgn(y-v)=3$ for $|y-v|=1$, or $|w-u-6v+4-z|=3$ for $y=v$, since $Y\bigcup(U \rhd W)$ is a continual integers section, $Y$ and $U \rhd W$ are complementary.

Similarly, we can show (2). □

**Lemma 2.3** (1) For strictly monotone increasing sequence $Y=\bigcup_{i=0}^{y-1}\{z+6i, z+6i+1, z+6i+2\}$, there exist strictly monotone increasing sequence $X=[x, x+3y-1]$ and strictly monotone decreasing sequence $Z=[s-3i \mid i \in [0, y-1]\}$, such that $Y=X \rhd Z$.

(2) Let $U=[u, u+3v-1]$ be a strictly monotone increasing sequences, $W=\{w-3i \mid i \in [0, v-1]\}$ a strictly monotone decreasing sequence. Then there exist strictly monotone increasing sequence $X=[x, x+3y-1]$ and strictly monotone decreasing sequence $Z=\{z-3i \mid i \in [0, y-1]\}$, such that $U \rhd W$ and $X \rhd Z$ are complementary.

**Proof** (1) When $s < x$, we take $z=x-s$. When $s > x+6y-4$, take

$z=s-x-6y+4$.

(2) By Lemma 2.2 and (1), we can obtain (2). □

Let $P_n$ be a path with $n$ vertices, $f$ a permutation in $S(n)$, $V(G_j)=\{v_{j1}, v_{j2}, ..., v_{jn}\}$, $j = 1,2$. Then the vertex set of the permutation graph $P(P_n, f)$ is $V(G_1)\bigcup V(G_2)$, and its edge set is $E(G_1)\bigcup E(G_2)\bigcup\{v_{1,j}v_{2,f(j)} \mid j \in [1,n]\}$. The permutation graph $P(P_n, f)$ has $2n$ vertices and $3n-2$ edges. Let $g$ be a function and $f=(a_1, a_2, ..., a_s)$ be a permutation, define $gf=g(f)=(g(a_1), g(a_2), ..., g(a_s))$.

**Lemma 2.4** Let $n > 1$ be a positive integer, $f$ a permutation in $S(n)$. Define the function $g$ as follows:

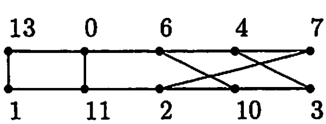
$$g(x) = \begin{cases} n+1-x, & if\ f(x) \neq x \\ x, & if\ f(x) = x \end{cases} \qquad (*)$$

then $P(P_n, f^{-1})$, $P(P_n, gf)$ and $P(P_n, (gf)^{-1})$ are graceful when permutation graph $P(P_n, f)$ is graceful.

**Proof** Since $P(P_n, f^{-1}) \cong P(P_n, f)$, $P(P_n, gf) \cong P(P_n, f)$ and $P(P_n, (gf)^{-1}) \cong P(P_n, f)$, this theorem is true. □

Therefore, if show the graceful conjecture of permutation graphs of paths, we construct at most $(n!+2)/2$ graceful permutation graphs of paths.

**Example 1** If $n=5$, $f=(3,4,5)$ and $g$ is defined as (*), then $f^{-1}=(5,4,3)$, $gf=(3,2,1)$ and $(gf)^{-1}=(1,2,3)$. The permutation graph $P(P_5, f)$ is graceful(see below figure). Therefore $P(P_5, f^{-1})$, $P(P_5, gf)$ and $P(P_5, (gf)^{-1})$ are graceful by Lemma 2.4.



# 3 Main results

**Theorem 3.1** For any $n > 1$ and $f$=$(n-2, n-1, n)$ in $S(n)$, the permutation graph $P(P_n, f)$ is graceful.

**Proof** It clearly suffices to prove this for $n \geq 5$ by Lemma 1.1 (1). When $n$=5 see Example 1. Let the vertex set and the edge set of graph $P(P_n, f)$ be $V$ and $E$ respectively.

Case 1: when $6 \leq n \leq 13$, let

$$g(v_{1,j}) = \begin{cases} 3n-2-(j-1)/2, & j \in [1, n-3]_2 \\ (j-2)/2, & j \in [2, n-3]_2 \end{cases}$$

$$g(v_{2,j}) = \begin{cases} n-4+(j-1)/2, & j \in [1, n-3]_2 \\ 2n+2-j/2, & j \in [2, n-3]_2 \end{cases}$$

and the 6-tuple $(g(v_{1,n-2}), g(v_{1,n-1}), g(v_{1,n}), g(v_{2,n-2}), g(v_{2,n-1}), g(v_{2,n}))$ be $(6,14,8,4,9,7)$ when $n$=6, $(8,12,6,5,13,10)$ when $n$=7, $(13,14,18,15,7,12)$ when $n$=8, $(11,9,13,10,16,8)$ when $n$=9, $(16,24,18,13,14,17)$ when $n$=10, $(5,14,17,12,11,18)$ when $n$=11, $(21,13,17,15,14,19)$ when $n$=12, $(7,16,18,22,15,21)$ when $n$=13.

Case 2: when $n \geq 14$ and $n$ is even, let

$$g(v_{1,j}) = \begin{cases} (j-1)/2, & j \in [1, n-3]_2 \\ 3n-1-j/2, & j \in [2, n-4]_2 \\ (n+6)/2, & j = n-2 \\ n/2, & j = n-1 \\ (3n+4)/2, & j = n \end{cases}$$

$$g(v_{2,j}) = \begin{cases} 2n-(j-1)/2, & j \in [1, n-5]_2 \\ n-2+j/2, & j \in [2, n-4]_2 \\ 5n/2, & j = n-3 \\ (n-2)/2, & j = n-2 \\ (n+10)/2, & j = n-1 \\ (n+2)/2, & j = n \end{cases}$$

Then $g(V)$= $[0, (n+2)/2] \bigcup \{(n+6)/2, (n+10)/2\} \bigcup [n-1, (3n-8)/2] \bigcup \{(3n+4)/2\} \bigcup [(3n+6)/2, 2n] \bigcup \{5n/2\} \bigcup [(5n+2)/2, 3n-2] \subseteq [0, 3n-2]$.

Since $|g(V)|$=$2n$, $g$ is an injection. Let

$A$=$\{g^*(v_{1,j}v_{1,j+1})|\ j \in [1, n-1]\}$=$[2n+3, 3n-2] \bigcup \{5, 3, n+2\}$,

$B$=$\{g^*(v_{2,j}v_{2,j+1})|\ j \in [1, n-1]\}$=$[7, n+1] \bigcup \{n+4, 2n+1, 6, 4\}$,

$C$=$\{g^*(v_{1,n-2}v_{2,n-1}), g^*(v_{1,n-1}v_{2,n}), g^*(v_{1,n}v_{2,n-2})\}$=$\{2, 1, n+3\}$,

$D$=$\{g^*(v_{1,j}v_{2,j})|\ j \in [1, n-3]\}$=$[n+5, 2n] \bigcup \{2n+2\}$.

Then $g^*(E)$=$A \bigcup B \bigcup C \bigcup D$=$[1, 3n-2]$. This implies that the graph $P(P_n, f)$ is graceful.

Case 3: when $n \geq 15$ and $n$ is odd, let

$$g(v_{1,j}) = \begin{cases} (j-1)/2, & j \in [1, n-4]_2 \\ 3n-1-j/2, & j \in [2, n-3]_2 \\ (5n-9)/2, & j = n-2 \\ (5n-3)/2, & j = n-1 \\ (3n-7)/2, & j = n \end{cases}$$

$$g(v_{2,j}) = \begin{cases} 2n-(j-1)/2, & j \in [1, n-4]_2 \\ n-2+j/2, & j \in [2, n-5]_2 \\ (n-3)/2, & j = n-3 \\ (5n-1)/2, & j = n-2 \\ (5n-13)/2, & j = n-1 \\ (5n-5)/2, & j = n \end{cases}$$

Then $g(V)$=$[0, (n-3)/2] \bigcup [n-1, (3n-7)/2] \bigcup [(3n+5)/2, 2n] \bigcup \{(5n-13)/2, (5n-9)/2\} \bigcup [(5n-5)/2, 3n-2] \subseteq [0, 3n-2]$.

Since $|g(V)|$=$2n$, $g$ is an injection. Let

$A$=$\{g^*(v_{1,j}v_{1,j+1})|\ j \in [1, n-1]\}$=$[2n+3, 3n-2] \bigcup \{5, 3, n+2\}$,

$B$=$\{g^*(v_{2,j}v_{2,j+1})|\ j \in [1, n-1]\}$=$[7, n+1] \bigcup \{n+4, 2n+1, 6, 4\}$,

$C$=$\{g^*(v_{1,n-2}v_{2,n-1}), g^*(v_{1,n-1}v_{2,n}), g^*(v_{1,n}v_{2,n-2})\}$=$\{2, 1, n+3\}$,

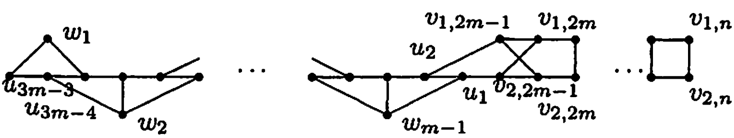
$D$=$\{g^*(v_{1,j}v_{2,j})|\ j \in [1, n-3]\}$=$[n+5, 2n] \bigcup \{2n+2\}$.

Then $g^*(E)$=$A \bigcup B \bigcup C \bigcup D$=$[1, 3n-2]$. This implies that the graph $P(P_n, f)$ is graceful. □

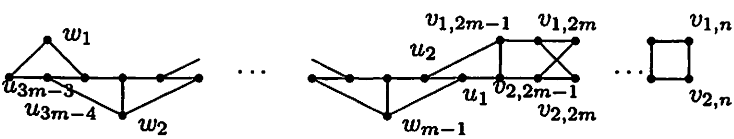
Let $s$ be odd and $t$ be even. In what follows, the symbols $S$ and $T$ denote the sequences $(g(u_1), g(u_3), ..., g(u_s))$ and $(g(u_2), g(u_4), ..., g(u_t))$, respectively.

**Theorem 3.2** Let $n > 1$ be an integer and $1 \leq m \leq \lfloor n/2 \rfloor$. Then
(1) when $n - 5m + 11 \equiv 1 \pmod 3$, the permutation graph $P(P_n, (2m-1, 2m))$ for any integer $m < (n+7)/5$ and $P(P_n, (2m, 2m+1))$ for any integer $m \geq 1$ are graceful; when $n - 5m + 11 \equiv 2 \pmod 3$, the permutation graph $P(P_n, (2m, 2m+1))$ for any integer $m < (n+6)/5$ and $P(P_n, (2m-1, 2m))$ for any integer $m \geq 1$ are graceful;
(2)when $n - 5m + 11 \equiv 0 \pmod 3$ and $m < (n+8)/5$, the permutation graph $P(P_n, (2m-1, 2m))$ and $P(P_n, (2m, 2m+1))$ are graceful.

**Proof** Let $P(P_n, f)$=$(V, E)$. Because of Lemma 2.4, we only need to discuss $m$ in $[1, \lfloor n/4 \rfloor]$. When $m$=1, see Lemma 1.1. When $m > 1$, the permutation graph $P(P_n, (2m-1, 2m))$ is isomorphic to the following graph.




The permutation graph $P(P_n, (2m, 2m+1))$ is isomorphic to the following graph.




The graceful labelling $g$ is given by

| $g(v_{1,j})$ | range of $j$ | set of vertex labels |
|---|---|---|
| (j+1)/2-m | $[2m-1, n]_2$ | $[0, \lfloor(n+1)/2\rfloor-m]$ |
| 3n-2+m-j/2 | $[2m, n]_2$ | $[3n-2+m-\lfloor n/2\rfloor, 3n-2]$ |

| $g(v_{2,j})$ | range of $j$ | set of vertex labels |
|---|---|---|
| n-m+1-(j-1)/2 | $[2m-1, n]_2$ | $[n-m+1-\lfloor(n-1)/2\rfloor, n-2m+2]$ |
| 2n+m-3+j/2 | $[2m, n]_2$ | $[2n+2m-3, 2n+m-3+\lfloor n/2\rfloor]$ |

| $g(u_j)$ | range of $j$ | set of vertex labels |
|---|---|---|
| 2n+2m-4-(j-1)/2 | $[1,3(m-1)]_2$ | $[2n+2m-2-\lfloor 3m/2\rfloor, 2n+2m-4]$ |
| n-2m+2+j/2 | $[2, 3(m-1)]_2$ | $[n-2m+3, n-2m+1+\lfloor(3m-1)/2\rfloor]$. |

When $f$=$(2m-1,2m)$, let

$$g(w_1)=\begin{cases} n-3+5m/2, & m\ is\ even\\ 2n+2-(5m+1)/2, & m\ is\ odd\end{cases}$$

When $f$=$(2m,2m+1)$, let

$$g(w_1)=\begin{cases} n-2+5m/2, & m\ is\ even\\ 2n+1-(5m+1)/2, & m\ is\ odd\end{cases}$$

Let $A$=$\{v_{1,j},v_{2,j}\mid j\in[2m-1,n]\}$, $B$=$\{uv\mid u\ or\ v\in A, uv\in E(P(P_n,f))\}$, $T$=$\{w_1,w_2,\cdots,w_{m-1}\}$ and $D$=$\{uv\mid u\in T, uv\in E(P(P_n,f))\}$.

By the definition of $g$, we have

$g(V\setminus T)$=$\{g(u_i),g(v_{1,j}),g(v_{2,j})\mid i\in[1,3(m-1)], j\in[2m-1,n]\}$

$=[0,\lfloor(n+1)/2\rfloor-m]\bigcup[n-m+1-\lfloor(n-1)/2\rfloor,n-m+1+\lfloor(m-1)/2\rfloor]$

$\bigcup[2n+m-2-\lfloor m/2\rfloor,2n+m-3+\lfloor n/2\rfloor]\bigcup[3n-2+m-\lfloor n/2\rfloor,3n-2]$.

$A_1$=$\{g^*(v_{1,j}v_{1,j+1})\mid j\in[2m-1,n-1]\}$=$[2n+2m-2,3n-2]$,

$B_1$= $\{g^*(v_{2,j}v_{2,j+1})\mid j\in[2m-1,n-1]\}$=$[n+4m-5,2n+2m-5]$.

When $f$=$(2m-1,2m)$, we have

$C_1$=$\{g^*(v_{1,2m-1}v_{2,2m}),g^*(v_{1,2m}v_{2,2m-1}),g^*(v_{1,2m-1}u_2),g^*(v_{2,2m-1}u_1)\}$

$=\{2n+2m-3,2n+2m-4,n-2m+3,n+4m-6\}$,

$D_1$=$\{g^*(v_{1,j}v_{2,j})\mid j\in[2m+1,n]\}$=$[1,n-2m]$.

Therefore, the set of edge labels in $B$ is

$g^*(B)$=$A_1\bigcup B_1\bigcup C_1\bigcup D_1$= $[1,n-2m]\bigcup\{n-2m+3\}\bigcup[n+4m-6,3n-2]$.

When $f$=$(2m,2m+1)$, we have

$C_1$=$\{g^*(v_{1,2m-1}v_{2,2m-1}),g^*(v_{1,2m+1}v_{2,2m}),g^*(v_{1,2m}v_{2,2m+1}),g^*(v_{1,2m-1}u_2),$
$g^*(v_{2,2m-1}u_1)\}$

$=\{n-2m+2,2n+2m-3,2n+2m-4,n-2m+3,n+4m-6\}$,

$D_1$=$\{g^*(v_{1,j}v_{2,j})\mid j\in[2m+2,n]\}$=$[1,n-2m-1]$.

Therefore, the set of edge labels in $B$ is $g^*(B)$=$A_1\bigcup B_1\bigcup C_1\bigcup D_1$

$=[1,n-2m-1]\bigcup\{n-2m+2,n-2m+3\}\bigcup[n+4m-6,3n-2]$.

The edge labels set of path $P$=$u_1u_2...u_{3(m-1)}$ is $[n+m-2,n+4m-7]$.

For part (1), we firstly verify $g^*$ is a bijection.

When $n-5m+11 \not\equiv 0 \pmod 3$ and $m$ is odd, let
$W=(g(w_2),g(w_4),...,g(w_{m-1}))=(n+m-1,n+m+2,...,n+(5m-11)/2)^{-1}$, $Z=(g(w_3),g(w_5),...,g(w_{m-2}))^{-1}=(2n-(5m-11)/2,2n-(5m-11)/2+3,...,2n-m-2)^{-1}$, $U=S^{-1}$, $s=3m-4$, $X=T$, $t=3m-7$. By Lemma 2.2 and 2.3 we obtain
$g^*(D)=\{n-2m+1,n-2m+2\}\bigcup[n-2m+4,n+m-3]$ when $f=(2m-1,2m)$,
$g^*(D)=\{n-2m,n-2m+1\}\bigcup[n-2m+4,n+m-3]$ when $f=(2m,2m+1)$.

Therefore, $g^*$ is a bijection from $E$ onto $[1,3n-2]$.

When $n-5m+11 \not\equiv 0 \pmod 3$ and $m$ is even, let
$(g(w_2),g(w_4),...,g(w_{m-2}))=(2n-5m/2+4,2n-5m/2+7,...,2n-m-2)$,
$(g(w_3),g(w_5),...,g(w_{m-1}))=(n+5m/2-7,n+5m/2-10,...,n+m-1)$.

In the same way, we obtain
$g^*(D)=\{n-2m+1,n-2m+2\}\bigcup[n-2m+4,n+m-3]$ when $f=(2m-1,2m)$,
$g^*(D)=\{n-2m,n-2m+1\}\bigcup[n-2m+4,n+m-3]$ when $f=(2m,2m+1)$.

Since $g^*(B)\cup g^*(E(P))\cup g^*(D)=[1,3n-2]$, $g^*$ is a bijection from $E$ onto $[1,3n-2]$.

In the following, we verify $g$ is an injection. For the permutation graph $P(P_n,(2m-1,2m))$,

(i) when $n-5m+11\equiv 1 \pmod 3$ and odd number $m<(n+7)/5$, we have $g(w_2)-g(w_1)=4>0$, $g(w_1)-g(w_3)=n+7-5m>0$, $g(w_2)-g(w_3)=n-5m+11$. Therefore the members in $g(T)$ are pairwise distinct. Since $g(T)\cap g(V\setminus T)$ is an empty set and $|g(V)|=2n$, $g$ is an injection.

(ii) When $n-5m+11\equiv 1 \pmod 3$ and even number $m<(n+7)/5$, we have $g(w_2)-g(w_1)=n+7-5m>0$, $g(w_1)-g(w_3)=4>0$, $g(w_2)-g(w_3)=n-5m+11$. Therefore the members in $g(T)$ are pairwise distinct. Since $g(T)\cap g(V\setminus T)$ is an empty set and $|g(V)|=2n$, $g$ is an injection.

(iii) When $n-5m+11\equiv 2 \pmod 3$, since $|g(V)|=2n$, $g$ is an injection.

For the permutation graph $P(P_n,(2m,2m+1))$,

(i) when $n-5m+11\equiv 2 \pmod 3$ and odd number $m<(n+6)/5$, we have $g(w_2)-g(w_1)=-5<0$, $g(w_1)-g(w_3)=n+6-5m>0$, $g(w_2)-$

$g(w_3)=n-5m+11$. Therefore the members in $g(T)$ are pairwise distinct. Since $g(T)\cap g(V\setminus T)$ is an empty set and $|g(V)|=2n$, $g$ is an injection.
(ii) When $n-5m+11\equiv 2 \pmod 3$ and even number $m<(n+6)/5$, we have $g(w_2)-g(w_1)=n+6-5m>0$, $g(w_1)-g(w_3)=5>0$, $g(w_2)-g(w_3)=n-5m+11$. Therefore the members in $g(T)$ are pairwise distinct. Since $g(T)\cap g(V\setminus T)$ is an empty set and $|g(V)|=2n$, $g$ is an injection.
(iii) When $n-5m+11\equiv 1 \pmod 3$, since $|g(V)|=2n$, $g$ is an injection.

Therefore, $g$ is a graceful labelling of $P(P_n,f)$ for $f=(2m-1,2m)$ and $(2m,2m+1)$.

For part (2), when $n-5m+11\equiv 0 \pmod 3$, $m$ is even, and $m<(n+8)/5$

| $g(w_j)$ | range of $j$ | set of vertex labels |
|---|---|---|
| $2n+m/2-5$ | $\{2\}$ | $\{2n+m/2-5\}$ |
| $n-(3j-5m-1)/2$ | $[3,m-1]_2$ | $[n+m+2,n-4+5m/2]_3$ |
| $2n+(3j-5m-4)/2$ | $[4,m-2]_2$ | $[2n+4-5m/2,2n-m-5]_3$ |

When $n-5m+11\equiv 0 \pmod 3$, $m$ is odd, and $m<(n+8)/5$

| $g(w_j)$ | range of $j$ | set of vertex labels |
|---|---|---|
| $n+3-(m-1)/2$ | $\{2\}$ | $\{n+3-(m-1)/2\}$ |
| $n-(3j-5m-1)/2$ | $[4,m-1]_2$ | $[n+m+2,n+(5m-11)/2]_3$ |
| $2n-2+(3j-5m)/2$ | $[3,m-2]_2$ | $[2n-(5m-5)/2,2n-m-5]_3$ |

The result is obvious when $m\le 4$. When $m>4$, for $P(P_n,(2m-1,2m))$,
(i) when $m$ is even, we have $g(w_2)\ne g(w_1)$, $g(w_4)-g(w_3)=n+8-5m>0$, $g(w_2)>g(w_{m-2})$, $g(w_1)>g(w_3)$, $g(w_4)-g(w_1)=n-5m+7\not\equiv 0 \pmod 3$. Therefore the members in $g(T)$ are pairwise distinct. Since $g(T)\cap g(V\setminus T)$ is an empty set and $|g(V)|=2n$, $g$ is an injection.
(ii) When $m$ is odd, we have $g(w_2)\ne g(w_1)$, $g(w_3)>g(w_2)$, $g(w_1)<g(w_3)$, $g(w_3)-g(w_4)=n-5m+8>0$, $g(w_1)-g(w_4)=n-5m+7\not\equiv 0 \pmod 3$. Therefore the members in $g(T)$ are pairwise distinct. Since $g(T)\cap g(V\setminus T)$ is an empty set and $|g(V)|=2n$, $g$ is an injection.

Similarly, we can show that the function $g$ is an injection for $P(P_n,(2m,2m+1))$.

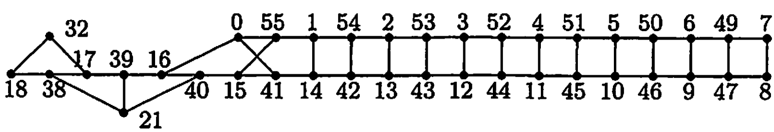
Also

$g^*(D)=\{n-2m+1, n-2m+2\} \bigcup [n-2m+4, n+m-3]$ when $f=(2m-1, 2m)$,

$g^*(D)=\{n-2m, n-2m+1\} \bigcup [n-2m+4, n+m-3]$ when $f=(2m, 2m+1)$.

Since $g^*(E)=g^*(B) \cup g^*(E(P)) \cup g^*(D)=[1, 3n-2]$, $g^*$ is a bijection from $E$ onto $[1, 3n-2]$. Therefore, $g$ is a graceful labelling of $P(P_n, f)$ for $f=(2m-1, 2m)$ and $(2m, 2m+1)$. □

**Example 2** When $n=19$, $m=3$, there is $n-5m+11 \equiv 0 \pmod 3$. The graceful labeling of graph $P(P_n, (2m-1, 2m))$ is as follows:




For any $n$ and $m$, the combinations of their values can be partitioned into four cases:

- condition 1: $n \equiv 0 \pmod 4$ and $m \equiv 0 \pmod 2$, or $n \equiv 2 \pmod 4$ and $m \equiv 1 \pmod 2$,
- condition 2: $n \equiv 0 \pmod 4$ and $m \equiv 1 \pmod 2$, or $n \equiv 2 \pmod 4$ and $m \equiv 0 \pmod 2$,
- condition 3: $n \equiv 1 \pmod 4$ and $m \equiv 0 \pmod 2$, or $n \equiv 3 \pmod 4$ and $m \equiv 1 \pmod 2$,
- condition 4: $n \equiv 3 \pmod 4$ and $m \equiv 0 \pmod 2$, or $n \equiv 1 \pmod 4$ and $m \equiv 1 \pmod 2$.

**Theorem 3.3** If $n > 1$ is any integer and $f=(12)(34)...(2m-1, 2m)$, $1 \leq m \leq \lfloor n/2 \rfloor$ is a permutation in $S(n)$, then

(1) when $m \equiv 0 \pmod 3$ and $n < (10m+6)/3$ in conditions 1 and 2, $P(P_n, f)$ is graceful;

(2) when $m \leq 3$ and $n > 1$ in conditions 1 and 2, $P(P_n, f)$ is graceful;

(3) when $m \not\equiv 0 \pmod 3$ and $n > 1$ in conditions 1 and 2, $P(P_n, f)$ is

graceful;

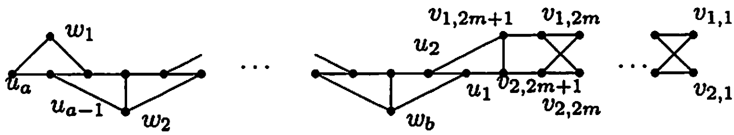
(4) under conditions 3 and 4, $P(P_n, f)$ is graceful.

**Proof** Let the permutation graph $P(P_n, f)$=$(V, E)$. Let

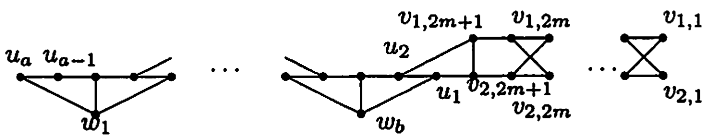
$$b = \begin{cases} (n-2m-1)/2, & n \text{ is odd} \\ (n-2m-2)/2, & n \text{ is even} \end{cases}$$

$$a = \begin{cases} 3b, & n \text{ is odd} \\ 3b+2, & n \text{ is even} \end{cases}$$

When $n$ is odd, the permutation graph $P(P_n, f)$ is isomorphic to the following graph.




When $n$ is even, the permutation graph $P(P_n, f)$ is isomorphic to the following graph.




We define the function $g$ as follows:

| $g(v_{1j})$ | range of $j$ | set of vertex labels |
|---|---|---|
| $m-(j-1)/2$ | $[1, 2m+1]_2$ | $[0, m]$ |
| $3n-m-2+j/2$ | $[2, 2m]_2$ | $[3n-m-1, 3n-2]$ |

| $g(v_{2j})$ | range of $j$ | set of vertex labels |
|---|---|---|
| $3n-m-2-(j-1)/2$ | $[1, 2m+1]_2$ | $[3n-2m-2, 3n-m-2]$ |
| $m+1+j/2$ | $[2, 2m]_2$ | $[m+2, 2m+1]$ |

| $g(u_j)$ | range of $j$ | set of vertex labels |
|---|---|---|
| $2m+2+(j-1)/2$ | $[1, a]_2$ | $[2m+2, 2m+2+\lfloor (a-1)/2 \rfloor]$ |
| $3n-2m-2-j/2$ | $[2, a]_2$ | $[3n-2m-2-\lfloor a/2 \rfloor, 3n-2m-3]$ |

Let $A$={$v_{1j}, v_{2j} \mid j \in [1, 2m+1]$}, $B$={$uv \mid u$ *or* $v \in A, uv \in E$}. Then the set of edge labels in $B$ is $[1, 2m] \bigcup [3n-4m-4, 3n-2]$. The set of edge labels in path $P$=$u_1u_2...u_a$ is $[g^*(u_a u_{a-1}), 3n-4m-5]$, where

$$g^*(u_a u_{a-1}) = 3n - 4m - 3 - a = \begin{cases} (3n-2m-3)/2, & n \text{ is odd} \\ (3n-2m-4)/2, & n \text{ is even} \end{cases}$$

We distinguish eight cases to define the labelling:

Case 1: $m \not\equiv 0 (\text{mod } 3)$ in condition 1

| $g(w_j)$ | range of $j$ | set of vertex labels |
|---|---|---|
| 4m+1+(a+3j)/2 | [1,b]$_2$ | [4m+(a+5)/2,4m+⌊(a+3b+2)/2 ⌋]$_3$ |
| 3n-4m-(a+3j+3)/2 | [2,b]$_2$ | [3n-4m-⌊(a+3b+3)/2⌋,3n-4m-(a+9)/2]$_3$ |

The condition 1 is equivalent to $a$ is odd and $n$ even.

Case 2: $5m \not\equiv 2 (\text{mod } 3)$ in condition 4

| $g(w_j)$ | range of $j$ | set of vertex labels |
|---|---|---|
| 4m+(a+3j+2)/2 | [1,b]$_2$ | [4m+(a+5)/2,4m+⌊(a+3b+2)/2⌋]$_3$ |
| 3n-4m-(a+3j+3)/2 | [2,b]$_2$ | [3n-4m-⌊(a+3b+3)/2⌋,3n-4m-(a+9)/2]$_3$ |

The condition 4 is equivalent to $a$ is odd and $n$ odd.

Case 3: $m \not\equiv 0$ (mod 3) in condition 2

| $g(w_j)$ | range of $j$ | set of vertex labels |
|---|---|---|
| 3n-4m-(a+3j+3)/2 | [1,b]$_2$ | [3n-4m-⌊(a+3b+3)/2⌋,3n-4m-(a+6)/2]$_3$ |
| 4m+(a+3j+2)/2 | [2,b]$_2$ | [4m+(a+8)/2,4m+⌊(a+3b+2)/2⌋]$_3$ |

The condition 2 is equivalent to $a$ is even and $n$ even.

Case 4: $5m \not\equiv 2 (\text{mod } 3)$ in condition 3

| $g(w_j)$ | range of $j$ | set of vertex labels |
|---|---|---|
| 3n-4m-(a+3j+3)/2 | [1,b]$_2$ | [3n-4m-⌊(a+3b+3)/2⌋,3n-4m-(a+6)/2]$_3$ |
| 4m+(a+3j+2)/2 | [2,b]$_2$ | [4m+(a+8)/2,4m+⌊(a+3b+2)/2 ⌋]$_3$ |

The condition 3 is equivalent to $a$ is even and $n$ odd.

Case 5: $m \leq 3$ and $n > 1$, or $m \equiv 0$ (mod 3) and $n < (10m+6)/3$ in condition 1

| $g(w_j)$ | range of $j$ | set of vertex labels |
|---|---|---|
| 3n-2m-(a+9)/2 | {1} | {3n-2m-(a+9)/2 } |
| 4m+(a+3j-4)/2 | [3,b]$_2$ | [4m+(a+5)/2,4m+⌊(a+3b-4)/2 ⌋]$_3$ |
| 3n-4m-(a+3j-3)/2 | [2,b]$_2$ | [3n-4m-⌊(a+3b-3)/2⌋,3n-4m-(a+3)/2]$_3$ |

Case 6: $m \not\equiv 0 \pmod 3$ in condition 4

| $g(w_j)$ | range of $j$ | set of vertex labels |
|---|---|---|
| 3n-2m-(a+7)/2 | {1} | {3n-2m-(a+7)/2 } |
| 4m+(a+3j-2)/2 | $[3,b]_2$ | $[4m+(a+7)/2, 4m+\lfloor(a+3b-2)/2\rfloor]_3$ |
| 3n-4m-(a+3j-1)/2 | $[2,b]_2$ | $[3n-4m-\lfloor(a+3b-1)/2\rfloor, 3n-4m-(a+5)/2]_3$ |

Case 7: $m \leq 3$ and $n > 1$, or $m \equiv 0 \pmod 3$ and $n < (10m+6)/3$ in condition 2

| $g(w_j)$ | range of $j$ | set of vertex labels |
|---|---|---|
| 2m+(a+8)/2 | {1} | {2m+(a+8)/2 } |
| 4m+(a+3j-4)/2 | $[2,b]_2$ | $[4m+(a+2)/2, 4m+\lfloor(a+3b-4)/2\rfloor]_3$ |
| 3n-4m-(a+3j-3)/2 | $[3,b]_2$ | $[3n-4m-\lfloor(a+3b-3)/2\rfloor, 3n-4m-(a+6)/2]_3$ |

Case 8: $m \not\equiv 0 \pmod 3$ in condition 3

| $g(w_j)$ | range of $j$ | set of vertex labels |
|---|---|---|
| 2m+(a+6)/2 | {1} | {2m+(a+6)/2 } |
| 4m+(a+3j-2)/2 | $[2,b]_2$ | $[4m+(a+4)/2, 4m+\lfloor(a+3b-2)/2\rfloor]_3$ |
| 3n-4m-(a+3j-1)/2 | $[3,b]_2$ | $[3n-4m-\lfloor(a+3b-1)/2\rfloor, 3n-4m-(a+8)/2]_3$ |

Let $T$=$\{w_j \mid j \in [1,b]\}$, $D$=$\{uv \mid u \in T, uv \in E\}$. From Lemma 2.2 and 2.3, we determine the set of edge labels of $D$ to be $g^*(D)$=$[2m+1, 3n-4m-4-a]$. Since

$g^*(E)$=$g^*(B) \cup g^*(E(P)) \cup g^*(D)$=$[1, 3n-2]$,

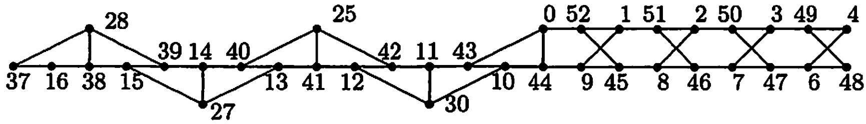
$g^*$ is a bijection from $E$ onto $[1, 3n-2]$. It is easy to see that the set of vertex labels of $T$ is contained in $[2m+\lfloor(a+5)/2\rfloor, 3n-2m-\lfloor(a+6)/2\rfloor]$ and the members in $T$ are pairwise distinct. From the values of $g(w_j)$, we have

$g(V \setminus T)$=$\{g(v_{1i}), g(v_{2i}), g(u_j) \mid i \in [1, 2m+1], j \in [1, a]\}$

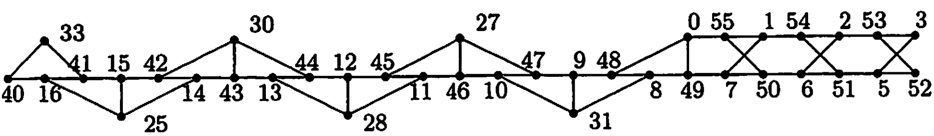
$=[0, m] \cup [m+2, 2m+2+\lfloor(a-1)/2\rfloor] \cup [3n-2m-2-\lfloor a/2\rfloor, 3n-2]$.

Since $g(T) \cap g(V \setminus T)$ is an empty set and $|g(V)|$=$2n$, $g$ is an injection. Therefore, this theorem is true. □

**Example 3** For Case 3 in Theorem 3.3, when $n$=18, $m$=4, we have $b$=4, $a$=14. The graceful labeling of graph $P(P_{18}, (1,2)(3,4)(5,6)(7,8))$ is as follows:




**Example 4** For Case 4 in Theorem 3.3, when $n$=19, $m$=3, we have $b$=6, $a$=18. The graceful labeling of graph $P(P_{19}, (1,2)(3,4)(5,6))$ is as follows:





**Acknowledgment**

Author would likes to thank the referees for their helpful comments and suggestions.


# References


[1] A.Rosa, On certain valuations of the vertices of a graph Theory of graphs, Proc. of Intemational Symposium, Rome 1966(Gordon Breach, New York 1967), 349-355.

[2] S.W.Golomb, How to number a graph? in "Graph theory and computing", edited by R.C.Read, 23-27,Academic Press, New York 1972.

[3] R.E.L.Aldred, B.D.McKay, Graceful and harmonious labellings of trees, Bulletin ICA, 23(1998), 69-72.

[4] J.A.Gallian, A dynamic survey of graph labeling, Electronic Journal of Combinatorics, 5(2002),DS#6 .

[5] J.C.Bermond, Graceful graphs, radio antennae and Fench windmills, Graph Theory and Combinatorics, R.J.Wilson, Ed., Pitman, London (1979), 13-37.

[6] A.K.Dewdney, The search for an invisible ruler that will help radio astronomers to measure the earth, Scientific American, Dec.(1986), 16-19.

[7] S.M.Lee, K.Y.Lai, Y.S.Wang and M.K.Kiang, On the graceful permutation graphs conjecture, Congressus Numerantium, 103(1994), 193-201.

[8] R.E.L.Aldred, J.Siran and M.Siran, A note on the number of graceful labellings of paths, Discrete Mathematices, 261(2003), 27-30.

[9] M.Maheo, Strongly graceful graphs, Discrete Math,29(1980), 39-46.

[10] C.Delorme, Two sets of graceful graphs, J.Graph Theory, 4(1980), 247-250.

[11] R.W.Frucht and J.A.Gallian, Labelling prisms, Ars Combin., 26(1988), 69-82.

[12] J.A.Gallian, Labelling prisms and prism related graphs, Congressus Numerantium, 59(1987), 89-100.

[13] Z. Liang, H. Zhang, N. Xu, S. Ye, Y. Fan and H. Ge, Gracefulness of five permutation graphs of paths. Utilitas Mathematica, to appear.