A note on transversals *

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Abstract

Let B be an $m \times n$ array in which each symbol appears at most k times. We show that if $k \leq \frac{n(n-1)}{8(m+n-2)} + 1$ then B has a transversal.

1 Introduction

Let m and n be integers, $2 \leq m \leq n$. An $m \times n$ array is a table of mn cells, arranged in m rows and n columns and each cell contains exactly one symbol. A section of an array consists of m cells, one from each row and no two from the same column. A transversal is a section in which no two cells contain the same symbol.

Stein [3] introduced the following interesting notion. Let $L(m,n)$ be the largest integer z such that every $m \times n$ array in which no symbol appears more than z times has a transversal. He determined $L(2,n)$ and $L(3,n)$. Upper and lower bounds on $L(m,n)$ are also obtained in [3], namely $n-m+1 \leq L(m,n) \leq \lfloor \frac{mn-1}{m-1} \rfloor$. It is shown [1] that this upper bound on $L(m,n)$ is tight when n is large enough compared to m. Erdös and Spencer [2] showed that an $n \times n$ array in which each symbol appears at most $k \leq (n-1)/16$ times has a transversal. This gives also a lower bound on $L(n,n)$. They used a new version of the Lovász Local Lemma. Here we use their method to obtain the following lower bound on $L(m,n)$:

Theorem 1. *If* $2 \leq m \leq n$, *then* $\lfloor \frac{n(n-1)}{8(m+n-2)} \rfloor + 1 \leq L(m,n)$.

We note that for n slightly larger than m, Theorem 1 improves on existing inequalities on $L(m,n)$. Similar reasoning establishes the following result.

Theorem 2. *If* $2 \leq m \leq n$ *and each symbol appears at most* k *times with* $k^2 \leq n(n-1)(n-2)/12(m+n)$, *then* B *has a section in which each symbol appears at most twice.*

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2 Proofs

Let us first recall the extended version of the Lovász Local Lemma proved in [2].

Lemma 1. *Let $A_1, \ldots, A_r$ be events in a probability space and G be a graph with the vertex set $V(G) = \{1, \ldots, r\}$ such that for any i, $1 \le i \le r$, $\Pr(A_i) \le p$ and $\deg(i) \le d$. Suppose that for any $S \subset V(G)$ and for any $i \in V(G)\backslash S$ which is adjacent to no vertex of S, $\Pr(A_i | \bigcap_{j \in S} \bar{A}_j) \le \Pr(A_i)$. If $4dp \le 1$, then $\bigcap_{i=1}^{r} \bar{A}_i \ne \emptyset$.*

To prove Theorem 1 fix an $m \times n$ array B with no entry $b(i,j)$ appearing more than $k \le \frac{n(n-1)}{8(m+n-2)} + 1$ times. We follow the lines of [2]. Suppose that U is the set of all sections of B. Let σ be uniformly chosen from U. Let T denote the set of (i,j,i',j') with $b(i,j) = b(i',j')$, $1 \le i < i' \le m$ and $j \ne j'$, $1 \le j, j' \le n$. For each $(i,j,i',j') \in T$ let $A_{iji'j'}$ denote the *event* $\sigma(i) = j$ and $\sigma(i') = j'$. The existence of a transversal is equivalent to the statement $\bigcap_T \bar{A}_{iji'j'} \ne \emptyset$.

We have $\Pr(A_{iji'j'}) = \frac{(n-2)(n-3)\cdots(n-m+1)}{n(n-1)\cdots(n-m+1)} = \frac{1}{n(n-1)}$. Let G be the graph with the vertex set T in which (i,j,i',j') is adjacent to (x,y,x',y') if and only if the four cells (i,j), (i',j'), (x,y), (x',y') lie on fewer than four rows or on fewer than four columns. For a given vertex (i,j,i',j') we have at most $2m + 2n - 4$ choices of (x,y) with a common coordinate and then $k-1$ choices for (x',y') with $b(x,y) = b(x',y')$ giving either (x,y,x',y') or (x',y',x,y) adjacent to (i,j,i',j'). This implies that G has maximal degree at most $(2m+2n-4)(k-1) \le n(n-1)/4$. Thus our result follows as a direct application of Lemma 1 if we can show that G satisfies the condition on $S \subset V(G)$. By symmetry it suffices to show

$$\Pr(A_{1122} | \bigcap_S \bar{A}_{iji'j'}) \le 1/n(n-1)$$

where $i,j,i',j' \notin \{1,2\}$. Following [2] call σ GOOD if it belongs to $\bigcap_S \bar{A}_{iji'j'}$. Let s_{ij} denote the number of GOOD σ with $\sigma(1) = i$, $\sigma(2) = j$. We can suppose that $m \ge 4$ since we know Theorem 1 is true for $m = 2, 3$. Note that $s_{ij} = s_{ji}$.

We claim that $s_{12} \le s_{ij}$ for all possible $i \ne j$. We show this, for example, only for $2 < i < j$. Let σ be GOOD with $\sigma(1) = 1$, $\sigma(2) = 2$ and $\sigma(r) = n_r$ for $3 \le r \le m$ where $3 \le n_3, n_4, \ldots, n_m \le n$. We have four distinct cases, but will treat only cases 1 and 4.

1. We have $n_r \ne i, j$ for $3 \le r \le m$.
2. There exists x with $\sigma(x) = j$ but $n_r \ne i$ for $3 \le r \le m$.
3. There exists y with $\sigma(y) = i$ but $n_r \ne j$ for $3 \le r \le m$.
4. There exist x, y with $\sigma(x) = i$, $\sigma(y) = j$.

In case 1, define σ^* by $\sigma^*(1) = i$, $\sigma^*(2) = j$, $\sigma^*(t) = \sigma(t)$ for $t \ne 1, 2$. In case 4, define σ^* by $\sigma^*(1) = i$, $\sigma^*(2) = j$, $\sigma^*(x) = 1$, $\sigma^*(y) = 2$, $\sigma^*(t) = \sigma(t)$ for $t \ne 1, 2, x, y$. Then in any case, σ^* is a section of B that is GOOD as the new values $(1,i)$, $(2,j)$, $(x,1)$, $(y,2)$ cannot be part of any member of S and σ was GOOD. Moreover, it is easy to check that the map σ to σ^* is injective from

GOOD σ with $\sigma(1) = 1$, $\sigma(2) = 2$ to GOOD σ^* with $\sigma^*(1) = i$, $\sigma^*(2) = j$. This proves the claim for $i, j > 2$.

Hence we have

$$\begin{aligned} \Pr(A_{1122}\ \text{GOOD}\ ) &= s_{12}/\Sigma_{i\neq j} s_{ij} \\ &\leq s_{12}/\Sigma_{i\neq j} s_{12} = 1/n(n-1) \end{aligned}$$

which completes the proof of Theorem 1.

The proof of Theorem 2 is similar, with the vertices having six entries (i, j, i', j', i'', j'') instead of four entries.

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