Counting embeddings of a chain into a binary tree

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Abstract

Let T be a partially ordered set whose Hasse diagram is a binary tree and let T posses a unique maximal element 1_T. For a natural n we compare the number A_T^n of those chains of length n in T that contain 1_T and the number B_T^n of those chains that do not contain 1_T. We show that if the depth of T is greater or equal to $2n+[n \ln n]$ then $B_T^n > A_T^n$.

1 Introduction and notation

A partially ordered set T will be called a *tree* if T is finite, T has only one maximal element and the Hasse diagram of T is a tree in the graph-theoretic sense. The maximal element of T is called the *root* of T.

Let D and T be trees. An *embedding* of D into T is any subset of T whose induced order is that of D. If the root of this subset is the same as the root of T, the embedding is called *good*; otherwise, it is called *bad*.

Motivated by looking for optimal best choice algorithms on complete binary trees and investigating some properties of such algorithms the authors of [M] and [KLM1] considered the ratio of the number of good embeddings of D into T_n (the complete binary tree of depth n) to the number of all embeddings of D into T_n. In [KLM2], the problem of embedding chains into complete binary trees was considered. Compare also a very interesting paper [G] that develops the subject of the aforementioned papers and proposes new methods. In [KMN] the subject was extended to embeddings of chains into any trees.

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Let the number of all linearly ordered subsets of cardinality k of a tree T be denoted C_T^k. Of course, it is the number of all embeddings of a chain of length k into T_k. Let the number of all good (bad) embeddings of such a chain be called A_T^k (B_T^k respectively).

It is proved in [KMN] for a given $k \in N$ that if T is not too "bushy" at its top part and if T is sufficiently deep then the ratio $\frac{A_T^k}{B_T^k}$ is smaller than 1. The precise formulation of the above statement was possible using the average depth of T, $AD(T)$, which is the arithmetic mean of the depths of all leaves of T. Namely, the conclusion $\frac{A_T^k}{C_T^k} \leq \frac{1}{2}$ holds whenever $AD(T) \geq 2k$ ([Th.2.2, KMN]; actually a better upper bound is given involving the structure of T, [Th.2.4, KMN]).

For some families of trees the condition "not too bushy at its top" is naturally satisfied. This happens if there is some restriction imposed on their branching. Of course, here the most natural family is that of binary trees. Thus in the case of binary trees it should be possible to replace the assumption on the average depth of T by some assumption on simply the depth of T, $dp(T)$.

For a tree T, the *depth* of T is the number

$$\mathrm{dp}(T) = \max\{\mathrm{dp}(l) : l \text{ is a leaf of } T\}.$$

In this paper we show that, indeed, the conclusion $\frac{A_T^k}{C_T^k} < \frac{1}{2}$ is achieved if $dp(T) \geq 2k + [k \ln k]$. This assumption on T was suggested by numerical experiments.

2 Main result

Our main result is the following:

Theorem 2.1 *If T is a binary tree and* $\mathrm{dp}(T) \geq [n \ln n] + 2n$, *then* $B_T^n > A_T^n$.

The proof of Theorem 2.1 is based on Lemma 2.4 and, in turn, the proof of Lemma 2.4 is based on an analytic inequality of Lemma 2.5. To make the order of our reasoning more clear, we first present the proof of the theorem, then that of Lemma 2.4 and next we present Lemma 2.5 and then series of technical lemmas leading to the proof of Lemma 2.5 that concludes this paper.

The worst situation from the point of view of considering the difference $A_T^n - B_T^n$, i.e. the situation when this difference attains its maximum for a given depth j, occurs for the family of trees defined below.

Let j and n be two positive integers and let $j > 2n - 1$. We define a family $\mathcal{T}_{(j,n)}$ of binary trees by $T \in \mathcal{T}_{(j,n)}$ if and only if the following conditions are met

i) T has only one leaf of depth j;
ii) all leaves of T but one have depth $2n - 2$ or $2n - 1$;
iii) all vertices of T that have depth smaller than $2n - 2$ have two sons.

For such two integers n, j, let $T_{(j,n)}$ be a binary tree of exactly $2^{2n-2} - 1$ leaves of depth $2n - 1$ and one leaf of depth j. One can easily see that this is a complete binary tree of depth $2n - 1$ with one leaf extended by a chain of length $j - 2n + 1$ (see Fig.1).

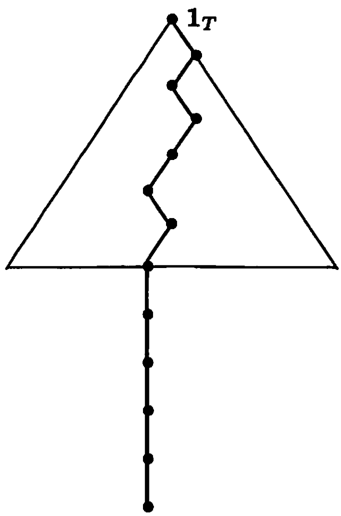

Fig.1. A complete binary tree of depth $2n - 1$ and a chain of depth j

As $T_{(j,n)}$ can be obtained from any other tree $T \in \mathcal{T}_{(j,n)}$ by extending some leaves of depth $2n - 2$ by new leaves of depth $2n - 1$, and as such an extension adds the same number of bad and good chains, we see that the following lemma is true.

Lemma 2.2. *Let n and j be two positive integers such that $j > 2n - 1$. Then*

$$A_T^n - B_T^n = A_{T_{(j,n)}}^n - B_{T_{(j,n)}}^n,$$

for each $T \in \mathcal{T}_{(j,n)}$. $\square$

Lemma 2.3. *Let n and j be two positive integers such that $j > 2n-1$. Let T be a binary tree such that* $\mathrm{dp}(T) = j$ *and*

$$A_T^n - B_T^n = \max\{A_S^n - B_S^n : S \text{ is a a binary tree and } \mathrm{dp}(S) = j\}.$$

Then $T \in \mathcal{T}_{(j,n)}$.

Proof. Aiming at contradiction, assume that $T \notin \mathcal{T}_{(j,n)}$. Then there are two cases we have to consider: the first when there is a vertex (possibly a leaf) of depth smaller than $2n-2$ that does not have two sons, and the second when there are two leaves of depth greater than $2n-1$.

Assume first that there exists a vertex v of T such that $m = \mathrm{dp}(v) < 2n-2$. If $m \geq n-1$ let us extend T of v's new son l. Let $T' = T \cup \{l\}$. We have

$$A_{T'}^n - B_{T'}^n = (A_T^n - B_T^n) + \left(\binom{m-1}{n-2} - \binom{m-1}{n-1}\right) > A_T^n - B_T^n.$$

If $m < n-1$ let us extend T to a T' by adding under l a chain of new vertices of length $n-m$. We have now

$$A_{T'}^n - B_{T'}^n = (A_T^n - B_T^n) + 1 > A_T^n - B_T^n.$$

In both cases we get a contradiction.

Assume now that there are at least two leaves of T of depth greater than $2n-1$. As $\mathrm{dp}(T) = j$ one of them must be of depth j. Let us pick another one and call it l. Let $r = \mathrm{dp}(l)$. Let now $T' = T \setminus \{l\}$. We have

$$A_{T'}^n - B_{T'}^n = (A_T^n - B_T^n) - \left(\binom{r-2}{n-2} - \binom{r-2}{n-1}\right) > A_T^n - B_T^n.$$

A contradiction again. □

Lemma 2.4. *For $i = 1, 2, \ldots n-2, n \geq 3$,*

$$\sum_{j=[(i+1)\ln(i+1)]+1}^{[(i+2)\ln(i+2)]} (j+1)\binom{2n-2+j}{n-2} > (n-1-i)\binom{n-2+i}{n-2}2^{n-1+i}. \quad (1)$$

Proof of Theorem 2.1 Let $f(n) = [n \ln n] + 2n$. Let $j = \mathrm{dp}(T)$. By Lemmas 2.3 and 2.2,

$$A_T^n - B_T^n \leq A_{T_{(j,n)}}^n - B_{T_{(j,n)}}^n.$$

Using the argument consisting in deleting the deepest leaf and comparing the difference between the numbers of good and bad embeddings of a chain of length n, we get

$$B_{T_{(j,n)}}^n - A_{T_{(j,n)}}^n \geq B_{T_{(f(n),n)}}^n - A_{T_{(f(n),n)}}^n.$$

Thus it is enough to prove the theorem for $j = f(n)$.

For $T_{(f(n),n)}$ we have

$$A_T^n = \sum_{i=n-1}^{2n-2} \binom{i-1}{n-2} 2^i + \sum_{i=0}^{f(n)-2n} \binom{2n-2+i}{n-2}$$

and

$$B_T^n = \sum_{i=n}^{2n-2} \binom{i-1}{n-1} 2^i + \sum_{i=0}^{f(n)-2n} \binom{2n-2+i}{n-1}.$$

We want to prove that for every $n \geq 2$

$$\sum_{i=0}^{f(n)-2n} \left[\binom{2n-2+i}{n-1} - \binom{2n-2+i}{n-2} \right] > \sum_{i=0}^{n-2} \binom{n-2+i}{n-2} \frac{n-1-i}{n-1} 2^{n-1+i}$$

which is equivalent to

$$\sum_{i=0}^{[n \ln(n)]} (i+1) \binom{2n-2+i}{n-2} > \sum_{i=0}^{n-2} (n-1-i) \binom{n-2+i}{n-2} 2^{n-1+i}. \qquad (2)$$

Note that for every $n \geq 2$

$$\binom{2n-2}{n-2} + 2\binom{2n-1}{n-2} > (n-1)2^{n-1}. \qquad (3)$$

Indeed, inequality (3) is equivalent to the inequality

$$(2n-3)!!(5n-1) > (n+1)!,$$

because $(2n-2)! = 2^{n-1}(n-1)!(2n-3)!!$, where $(2n-3)!! = 1 \cdot 3 \cdot \ldots \cdot (2n-3)$. Inequality (3) is obvious if we write it in the following form:

$$(5n-1) \prod_{i=1}^{n-2} (2i+1) < 2(n+1) \prod_{i=1}^{n-2} (i+2).$$

Let us notice that the left-hand side of inequality (3) is the sum of two first terms of the left-hand side of inequality (2). Because the right-hand side of inequality (3) is the first term of the right-hand side of inequality (2) we get inequality (2) as a sum of inequality (3) and $n-2$ inequalities of the form (1) of Lemma 2.4. □

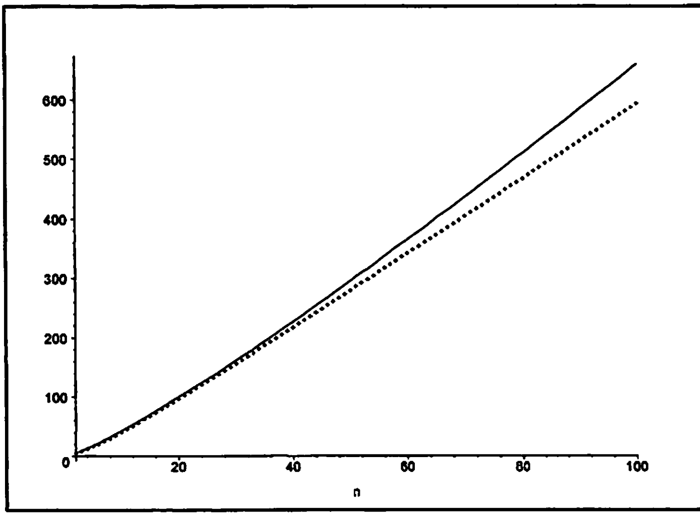

Fig.2. Calculation and plot by Maple

In Fig.2 we can compare the graph of the function

$$g(n)=\min\left\{s:\sum_{i=n}^{2n-2}\binom{i-1}{n-1}2^i+\sum_{i=0}^{s-2n}\binom{2n-2+i}{n-1}-\right.$$

$$\left.\left(\sum_{i=n-1}^{2n-2}\binom{i-1}{n-2}2^i+\sum_{i=0}^{s-2n}\binom{2n-2+i}{n-2}\right)\right\}$$

(the starred line) with the graph of the function $f(n)$ (the continuous line).

Proof of Lemma 2.4. Let $k=[(i+2)\ln(i+2)]$. It is obvious that the left-hand side of inequality(1) is greater than the sum of the two last terms of it:

$$\sum_{j=[(i+1)\ln(i+1)]+1}^{[(i+2)\ln(i+2)]}(j+1)\binom{2n-2+j}{n-2}>k\binom{2n-3+k}{n-2}+(k+1)\binom{2n-2+k}{n-2}.$$

Obviously,

$$k\binom{2n-3+k}{n-2}+(k+1)\binom{2n-2+k}{n-2}>k\left(\binom{2n+k-3}{n-2}+\binom{2n-2+k}{n-2}\right)=$$

$$k\frac{(2n+k-3)!(3n+2k-2)}{(n-2)!(n+k)!}.$$

Thus in order to prove inequality (1) it is enough to prove that for every $n \geq 3$, $i = 1 \dots n-2$ and $k = [(i+2)\ln(i+2)]$

$$k(3n+2k-2)(2n+k-3)!i! > (n-i-1)(n+i-2)!(n+k)!2^{n+i-1} \quad (4)$$

For $n = 3 \dots 200$ we check it directly using a computer program. We have used Maple. For $n > 200$ we prove it using some approximations.

Inequality (4) is equivalent to

$$k(3n+2k-2)(n+k+1)...(n+k+n-3) > (n-i-1)(i+1)...(i+n-2)2^{n+i-1}.$$

Taking logarithms of both sides we get

$$\ln k + \ln(3n+2k-2) + \left[\ln(n+k+1) + ... + \ln(n+k+n-3)\right] >$$

$$\ln(n-i-1) + (n+i-1)\ln(2) + \left[\ln(i+1) + ... + \ln(n+i-2)\right].$$

Let us notice that

$$\int_{n+k}^{2n+k-3} \ln x dx < \ln(n+k+1) + ... + \ln(2n+k-3)$$

and

$$\int_{i+1}^{n+i-1} \ln x dx > \ln(i+1) + ... + \ln(n+i-2).$$

Thus to prove (4) it is enough to prove that

$$\ln k + \ln(3n+2k-2) + (2n+k-3)\ln(2n+k-3) -$$

$$(2n+k-3) - (n+k)\ln(n+k) + (n+k) > \ln(n-i-1) +$$

$$(n+i-1)\ln(2) + (n+i-1)\ln(n+i-1) - (n+i-1) - (i+1)\ln(i+1) + (i+1).$$

Let $x = i+2$, $i = 0, \dots, n-2$. In this notation the inequality above takes the form:

$$\ln([x\ln x]) + \ln(3n+2[x\ln x]-2) + (2n+[x\ln x]-3)\ln(2n+[x\ln x]-3) +$$

$$(x-1)\ln(x-1) + 1 - \ln(n-x+1) - (n+x-3)\ln(2(n+x-3)) -$$

$$(n+[x\ln x])\ln(n+[x\ln x]) > 0.$$

Now, the conclusion clearly follows from following Lemma 2.5. □

Lemma 2.5. *Let*

$$f_n(x) = \ln(x\ln x - 1) + \ln(3n+2x\ln x-4) + (2n+x\ln x-4)\ln(2n+x\ln x-4) +$$

$(x-1)\ln(x-1)+1-\ln(n-x+1)-(n+x-3)\ln(2(n+x-3))-(n+x\ln x)\ln(n+x\ln x$

For every $x \in [2,n]$, $n \geq 200$, $f_n(x) > 0$.

Proof. Let $n \geq 200$ and $x \in [2,n]$. First let us notice that

$$\ln\frac{3n+2x\ln x-4}{n-x+1} > 1.$$

Now let us scale the interval $[2,n]$ to the interval $[0,1]$ by taking $t = \frac{x-2}{n-2}$. The function $f_n(x) - \ln\frac{3n+2x\ln x-4}{n-x+1}$ takes the form:

$$F(t,n) = \ln(g(t,n)-1) + (2n+g(t,n)-4)\ln(2n+g(t,n)-4)+$$

$$+(1+t(n-2))\ln(1+t(n-2))+1$$

$$-(n-1+t(n-2))\ln(2(n-1+t(n-2))) - (n+g(t,n))\ln(n+g(t,n)),$$

where $g(t,n) = (2+t(n-2))\ln(2+t(n-2))$. Note that we treat t as an independent variable. We shall prove the following two facts:

(A) $F(t,n)$ is positive for every $t \in [0,1]$ and $n = 200$;

(B) $F(t,n)$ is an increasing function with respect to n, for every fixed $t \in [0,1]$.

Fact (A) is obvious by Lemma 3.1 because $F(t,200) = f(x)$ if we put $x = 2+198t$ in $f(x)$. To prove (B) let us take the partial derivative of $F(t,n)$ with respect to n:

$$\frac{\partial F}{\partial n}(t,n) =$$

$$\frac{\frac{\partial g}{\partial n}(t,n)}{g(t,n)-1} + t\ln(1+t(n-2)) + \left(2+\frac{\partial g}{\partial n}(t,n)\right)\ln(2n+g(t,n)-4)-$$

$$(1+t)\ln(2(n-1+t(n-2))) - \left(1+\frac{\partial g}{\partial n}(t,n)\right)\ln(n+g(t,n)),$$

where $\frac{\partial g}{\partial n}(t,n) = t(\ln(2+t(n-2))+1)$.

We can write it in the following form:

$$\frac{\frac{\partial g}{\partial n}(t,n)}{g(t,n)-1} + s_1(t,n) + s_2(t,n),$$

where

$$s_1(t,n) = t\left[\ln\frac{1+t(n-2)}{2(n-1+t(n-2))} + \left(\ln(2+t(n-2))+1\right)\ln\frac{2n+g(t,n)-4}{n+g(t,n)}\right]$$

and

$$s_2(t,n) = \ln\frac{(2n+g(t,n)-4)^2}{2(n-1+t(n-2))(n+g(t,n))}.$$

By Lemma 3.5, it is easy to see that $s_1(t,n) > -1/2$, because $t\ln\frac{1+t(n-2)}{2(n-1+t(n-2))} = h_n(x)$ and $t\big(\ln(2+t(n-2))+1\big)\ln\frac{2n+g(t,n)-4}{n+g(t,n)} = k_n(x)$ if we put $x = 2+t(n-2)$ in $h_n(x)$ and $k_n(x)$.

By Lemma 3.2, we get $s_2(t,n) > 1/2$ if we put $x = 2+t(n-2)$ in (5). Hence $s_1(t,n)+s_2(t,n) > 0$, for every $t \in [0,1]$ and $n \geq 200$.

It is obvious that

$$\frac{\frac{\partial g}{\partial n}(t,n)}{g(t,n)-1} = t\frac{\ln(2+t(n-2))+1}{(2+t(n-2))\ln(2+t(n-2))-1} \geq 0.$$

Thus we have (B). By (A) and (B) $F(t,n)$ is positive, which completes the proof that $f_n(x) > 0$ for $n \geq 200$ and $x \in [2,n]$. □

3 Technical lemmas

In this section we present several technical lemmas needed to prove Lemma 2.5.

Lemma 3.1. *Let*

$$f(x) = \ln(x\ln x - 1) + (x-1)\ln(x-1) + 1 + (396 + x\ln x)\ln(396 + x\ln x) -$$

$$(197+x)\ln(394+2x) - (200+x\ln x)\ln(200+x\ln x).$$

If $x \in [2,200]$, *then* $f(x) > 0$.

Proof. Let

$$\tilde{g}(x) = \ln(x\ln x - 1) + (x-1)\ln(x-1) + 1$$

and

$$\tilde{f}(x) = (396+x\ln x)\ln(396+x\ln x) - (197+x)\ln(394+2x) -$$

$$(200+x\ln x)\ln(200+x\ln x).$$

Thus $f(x) = \tilde{g}(x) + \tilde{f}(x)$. It is easy to see that $\tilde{g}(x)$ is an increasing positive function. Now we shall prove that $\tilde{f}(x)$ is decreasing. Let us take the derivative of $\tilde{f}(x)$:

$$\tilde{f}'(x) = (\ln x + 1)\ln\left(1 + \frac{196}{200+x\ln x}\right) - \ln(394+2x) - 1.$$

The function $\ln(1+\frac{196}{200+x\ln x})$ is decreasing and positive, whence

$$(\ln x + 1)\ln\left(1 + \frac{196}{200 + x\ln x}\right) < (\ln(200) + 1)\ln\left(1 + \frac{98}{100 + \ln 2}\right).$$

Moreover,

$$(\ln 200 + 1)\ln\left(1 + \frac{98}{100 + \ln 2}\right) < \ln 398 + 1.$$

Hence

$$(\ln x + 1)\ln\left(1 + \frac{196}{200 + x\ln x}\right) < \ln(394 + 2x) + 1,$$

because $\ln(394 + 2x)$ is increasing and positive.

Now let us split the interval $[2, 200]$ into 6 subintervals in the following way: $[2, 25], [25, 40], [40, 60], [60, 80], [80, 120], [120, 200]$ and let us find values of functions $\tilde{f}(x)$ and $\tilde{g}(x)$ at the ends of these intervals (Maple).

We can see that $\tilde{f}(25) > 0$, thus $\tilde{g}(x) + \tilde{f}(x) > 0$ for $x \in [2, 25]$ by the fact that $\tilde{g} > 0$ and by monotonicity of $\tilde{f}(x)$. Now we can see that $\tilde{g}(25) > 81$ and $\tilde{f}(40) > -80.5$, whence, by monotonicity of $\tilde{g}(x)$ and $\tilde{f}(x)$, we have $\tilde{g}(x) + \tilde{f}(x) > 0$ for $x \in [25, 40]$. Analogously, we have $\tilde{g}(40) + \tilde{f}(60) > 0$, $\tilde{g}(60) + \tilde{f}(80) > 0$, $\tilde{g}(80) + \tilde{f}(120) > 0$ and $\tilde{g}(120) + \tilde{f}(200) > 0$. Thus, by monotonicity of $\tilde{g}(x)$ and $\tilde{f}(x)$, we have $f(x) > 0$ for every $x \in [2, 200]$.□

Lemma 3.2. *Let $n \geq 200$. For every $x \in [2, n]$,*

$$(2n + x\ln x - 4)^2 \geq 3.4(n - 3 + x)(n + x\ln x). \tag{5}$$

Proof. Let us treat inequality (5) as a quadratic inequality with respect to n:

$$0.6n^2 + n(0.6x\ln x - 5.8 - 3.4x) + x^2(\ln x)^2 - 3.4x^2\ln x + 2.2x\ln x + 16 \geq 0.$$

Let

$$b(x) = 0.6x\ln x - 5.8 - 3.4x,\ c(x) = x^2(\ln x)^2 - 3.4x^2\ln x + 2.2x\ln x + 16$$

and $\Delta(x) = b^2(x) - 2.4c(x)$. Let us fix x, where $x \geq 2$. If $\Delta(x) < 0$, then the quadratic inequality above trivially holds for every n. If $\Delta(x) \geq 0$, then this inequality is valid for every $n > \frac{-b(x)+\sqrt{\Delta(x)}}{1.2}$. To complete the proof it is enough to show that

$$\frac{-b(x) + \sqrt{\Delta(x)}}{1.2} < 200$$

which is equivalent to

$$\Delta(x) < (240 + b(x))^2$$

(it is easy to check that $b(x) + 240 > 0$).

Thus we are going to prove the following inequality for $x \geq 2$

$$2.4x^2 \ln x(\ln x - 3.4) + x(293.28 \ln x - 1632) + 54854.4 > 0. \quad (6)$$

Let

$$v(x) = x(293.28 \ln x - 1632) + 54854.4$$

and

$$w(x) = 2.4x^2 \ln x(\ln x - 3.4).$$

Let x_1 satisfy $\ln x_1 = \frac{1632}{293.28} - 1$. We can see that $v(x)$ is decreasing for $x \in [2, x_1]$ and increasing for $x \geq x_1$ and, hence, the minimum of $v(x)$ is equal to $v(x_1) > 0$ (Maple). Thus $v(x) > 0$.

Let us notice that $w(x) \geq 0$, for $x \in [x_2, \infty)$, where $\ln x_2 = 3.4$. Hence $v(x) + w(x) > 0$ for $x \in [x_2, \infty)$. For $x \in [2, x_2]$, $w(x) \leq 0$ and $v(x)$ is decreasing, because $x_2 < x_1$. Obviously $2.4x^2 \ln x$ is increasing and positive, whereas $\ln x - 3.4$ is increasing and negative for $x \in [2, x_2]$, whence

$$w(x) + v(x) \geq w(x) + v(x_2) \geq$$

$$2.4x_2^2(\ln x_2)(\ln x - 3.4) + v(x_2) > 2.4x_2^2 \ln x_2(\ln 2 - 3.4) + v(x_2) > 0$$

(Maple), which completes the proof of (6). □

Lemma 3.3. *Let* $m \in \{1, \frac{10}{9}, \frac{4}{3}, 2, 5\}$ *and* $x \geq 200$. *Then*

$$\tilde{k}_m(x) = \frac{\frac{x}{m} - 2}{x - 2}\left(\ln \frac{x}{m} + 1\right) \ln\left(1 + \frac{x - 4}{x + \frac{x}{m}\ln\frac{x}{m}}\right)$$

is positive and increasing, and

$$\tilde{h}_m(x) = \frac{\frac{x}{m} - 2}{x - 2} \ln \frac{\frac{x}{m} - 1}{2(\frac{x}{m} + x - 3)}$$

is negative and decreasing.

Proof. We can write $\tilde{k}_m(x)$ in the following form:

$$\tilde{k}_m(x) = \frac{1}{m}\left(1 - \frac{2m - 2}{x - 2}\right) \ln\left[\left(1 + \frac{1}{g_m(x)}\right)^{g_m(x)}\right]^{f_m(x)},$$

where $g_m(x) = \frac{x + \frac{x}{m}\ln\frac{x}{m}}{x-4} > 0$ and $f_m(x) = \frac{(x-4)(\ln\frac{x}{m}+1)}{x + \frac{x}{m}\ln\frac{x}{m}} > 0$. The function $g_m(x)$ is increasing because its derivative is positive: $g_m'(x) =$

$\frac{-4(\ln\frac{x}{m}+m+1)+x}{m(x-4)^2} > 0$. The function $f_m(x)$ is also increasing. It is obvious if we write it in the following form:

$$f_m(x) = m + \frac{1-m}{1+\frac{1}{m}\ln\frac{x}{m}} - \frac{4}{\frac{x}{\ln\frac{x}{m}}+\frac{x}{m}} - \frac{4}{x+\frac{x}{m}\ln\frac{x}{m}}.$$

Thus $\tilde{k}_m(x)$ is increasing and positive.

Now we prove that $\tilde{h}_m(x)$ is decreasing. First, let us notice that

$$0 < \frac{\frac{x}{m}-1}{2(\frac{x}{m}+x-3)} < 1.$$

Thus $\ln\frac{\frac{x}{m}-1}{2(\frac{x}{m}+x-3)} < 0$. Next,

$$\left(\frac{\frac{x}{m}-1}{2(\frac{x}{m}+x-3)}\right)' = \frac{-\frac{4}{m}+2}{(2(\frac{x}{m}+x-3))^2} \le 0.$$

Hence, for $m \le 2$, $\frac{\frac{x}{m}-2}{x-2}\ln\frac{\frac{x}{m}-1}{2(\frac{x}{m}+x-3)}$ is negative and decreasing, because $\frac{\frac{x}{m}-2}{x-2}$ is increasing and positive.

Hence, for $m \le 2$ the conclusion holds. For $m = 5$ we have to present a separate proof.

For $m = 5$ we have the following function

$$\tilde{h}_5(x) = \frac{\frac{x}{5}-2}{x-2}\ln\frac{\frac{x}{5}-1}{2(\frac{x}{5}+x-3)} =$$

$$\frac{1}{5}\left(-\frac{x-10}{x-2}\ln 2 + \frac{x-10}{x-2}\ln\frac{x-5}{6x-15}\right).$$

To prove that this function is decreasing it is enough to prove that

$$\varphi(x) = \frac{x-10}{x-2}\ln\frac{x-5}{6x-15}$$

is decreasing. The derivative of this function is equal to

$$\varphi'(x) = \frac{1}{(x-2)^2}\left(8\ln\frac{x-5}{6x-15} + 15\frac{(x-10)(x-2)}{(x-5)(6x-15)}\right).$$

$\varphi'(x)$ is negative because the function $8\ln\frac{x-5}{6x-15}+15\frac{(x-10)(x-2)}{(x-5)(6x-15)}$ is increasing ($\left(8\ln\frac{x-5}{6x-15}+15\frac{(x-10)(x-2)}{(x-5)(6x-15)}\right)' = 125\frac{x^2-6x+8}{(2x-5)^2(x-5)^2}$) and

$$\lim_{x\to\infty}\left(8\ln\frac{x-5}{6x-15} + 15\frac{(x-10)(x-2)}{(x-5)(6x-15)}\right) < 0. \ \square$$

Let

$$k_n(x) = \frac{x-2}{n-2}(1+\ln x)\ln\frac{2n + x\ln x - 4}{n + x\ln x}$$

and

$$h_n(x) = \frac{x-2}{n-2}\ln\frac{x-1}{2(n-3+x)}.$$

Lemma 3.4. *Let $n \geq 200$ and $x \geq 2$. Then $k_n(x)$ is increasing and $h_n(x)$ is decreasing.*

Proof. Let us write $k_n(x)$ in the following form:

$$k_n(x) = \frac{(x-2)(n-4)}{(n-2)(n+\ln x)}(1+\ln x)\ln\left(1+\frac{n-4}{n+\ln x}\right)^{\frac{n+\ln x}{n-4}}.$$

It is easy to see that

$$\frac{(x-2)(n-4)}{(n-2)(n+\ln x)}(1+\ln x)$$

is increasing when we take its derivative. Now it is already obvious that $k_n(x)$ is increasing, because $\frac{n+\ln x}{n-4}$ is positive and increasing.

In order to prove that $h_n(x)$ is decreasing, we calculate its derivative

$$h_n'(x) = \frac{1}{n-2}\left(\frac{(x-2)(n-2)}{(x-1)(n-3+x)} - \ln\left(2+\frac{2(n-2)}{x-1}\right)\right).$$

It is easy to see that $h_n'(x)$ is negative. Namely, we first notice that $0 \leq \frac{x-2}{x-1} < 1$ and $0 < \frac{n-2}{n-3+x} < 1$. Now, if $2(n-2)/(x-1) \geq 1$ then $\ln(2+\frac{2(n-2)}{x-1}) > 1$ and $h_n'(x) < 0$. In the opposite case we have $(n-2)/(x-1) < \frac{1}{2}$ and all the more $(n-2)/(n+x-3) < \frac{1}{2}$ and, as $\ln 2 > \frac{1}{2}$, $h_n'(x)$ is negative again. □

Lemma 3.5. *For every $x \geq 2$ and $n \geq 200$,*

$$k_n(x) + h_n(x) + 1/2 > 0. \tag{7}$$

Proof. Let $x \geq 2$ and $n \geq 200$. By Lemma 3.4, we know that $k_n(x)$ is increasing and $h_n(x)$ is decreasing. Thus $k_n(x)$ is positive because $k_n(2) = 0$, and $h_n(x)$ is negative because $h_n(2) = 0$.

Let us notice that $k_n(\frac{n}{m}) = \tilde{k}_m(n)$ and $h_n(\frac{n}{m}) = \tilde{h}_m(n)$.

By Lemma 3.3, for $n \geq 200$ and $m \in \{1, \frac{10}{9}, \frac{4}{3}, 2, 5\}$ fixed, $\tilde{k}_m(n)$ is a positive and increasing sequence and $\tilde{h}_m(n)$ is a negative and decreasing sequence.

Let

$$\alpha_m = \lim_{n\to\infty} \tilde{h}_m(n) = \frac{-\ln(2+2m)}{m}.$$

Thus for each m as above $\tilde{h}_m(n) > \alpha_m$.

Let us split the interval $[2, n]$ into 5 subintervals in the following way:

$$\left[2, \frac{n}{5}\right], \left[\frac{n}{5}, \frac{n}{2}\right], \left[\frac{n}{2}, \frac{3}{4}n\right], \left[\frac{3}{4}n, \frac{9}{10}n\right], \left[\frac{9}{10}n, n\right].$$

Let us take the interval $[2, \frac{n}{5}]$. We have there $k_n(x) + \frac{1}{2} \geq \frac{1}{2}$ and $h_n(x) > h_n(\frac{n}{5}) = \tilde{h}_5(n)$ by monotonicity of these functions. In order to prove (7) in $[2, \frac{n}{5}]$ it is enough to prove that $\tilde{h}_5(n) > -1/2$ for every $n \geq 200$.

$$\tilde{h}_5(n) > \alpha_5 = \frac{1}{5}\ln\frac{1}{12} > -\frac{1}{2}.$$

Hence $h_n(x) > -1/2$ for every $x \in \left[2, \frac{n}{5}\right]$.

Now let us take the next interval $\left[\frac{n}{5}, \frac{n}{2}\right]$. We have there

$$k_n(x) + h_n(x) + 1/2 > k_n\left(\frac{n}{5}\right) + h_n\left(\frac{n}{2}\right) + 1/2 = \tilde{k}_5(n) + \tilde{h}_2(n) + 1/2$$

(by monotonicity of these sequences)

$$> \tilde{k}_5(200) + \tilde{h}_2(n) + 1/2 > \tilde{k}_5(200) + \alpha_2 + 1/2 > 0.$$

Analogously, for $x \in \left[\frac{n}{2}, \frac{3}{4}n\right]$:

$$k_n(x) + h_n(x) + 1/2 > \tilde{k}_2(n) + \tilde{h}_{\frac{4}{3}}(n) + 1/2 > \tilde{k}_2(200) + \alpha_{\frac{4}{3}} + 1/2 > 0,$$

for $x \in \left[\frac{3}{4}n, \frac{9}{10}n\right]$:

$$k_n(x) + h_n(x) + 1/2 > \tilde{k}_{\frac{4}{3}}(n) + \tilde{h}_{\frac{10}{19}}(n) + 1/2 > \tilde{k}_{\frac{4}{3}}(200) + \alpha_{\frac{10}{9}} + 1/2 > 0$$

and for $x \in [\frac{9}{10}n, n]$:

$$k_n(x) + h_n(x) + 1/2 > \tilde{k}_{\frac{10}{9}}(n) + \tilde{h}_1(n) + 1/2 > \tilde{k}_{\frac{10}{9}}(200) + \alpha_1 + 1/2 > 0.$$

Thus the inequality (7) holds for every $n \geq 200$ and $x \geq 2$.□

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