On the Folded Hypercube and Bi-folded Hypercube*

Ying Xu, Jixiang Meng †

College of Mathematics and Systems Science, Xinjiang University,

Urumqi, Xinjiang 830046, P. R. China

Abstract

We determine the automorphism group and the spectrum of folded hypercube. In addition, we define the Bi-folded hypercube and determine its spectrum.

Keywords: Folded hypercube; Bi-folded hypercube; Automorphism group; Spectrum.

1 Introduction

The *spectrum* of a graph X is the set of numbers which are eigenvalues of $A(X)$, together with their multiplicities. If the distinct eigenvalues of $A(X)$ are $\lambda_1, \lambda_2, \cdots, \lambda_s$, and their multiplicities are $m(\lambda_1), m(\lambda_2), \cdots, m(\lambda_s)$, then we shall write

$$spec(X) = \begin{pmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_s \\ m(\lambda_1) & m(\lambda_2) & \cdots & m(\lambda_s) \end{pmatrix}.$$

Let Γ be a finite group with identity 1 and S be the subset of $\Gamma\backslash 1$. Denote by $X(\Gamma, S)$ the *Cayley graph* of Γ with respect to S, where Γ is its vertex set and edge set $E(X(\Gamma, S)) = \{uv : vu^{-1} \in S\}$. If S is an arbitrary subset

*The research is supported by NSFC (No.10671165) and XJEDU (No.2004G05)

†Corresponding author.E-mail: mjx@xju.edu.cn (J.Meng),xuying1209@163.com(Y.Xu).

of Γ, then we can define a directed graph $X(\Gamma, S)$ with vertex set Γ and arc set $\{uv : vu^{-1} \in S\}$. If S is *inverse closed*, this is, $S^{-1} = \{s^{-1} | \, s \in S\} = S$, then this graph is undirected and has no loops.

Let $Q_k = (V, E)$ be a *k-dimensional hypercube*, where $V = \{a = (a_1, a_2, \cdots, a_k) | \; a_i = 0 \text{ or } 1\}$ and $a \sim b$ is an edge if and only if they differ in precisely one coordinate position. Let Z_n be the cyclic group of integers modulo n and Z_2^k be the directed product of k copies of Z_2 and $C = \{e_i | i = 1, 2, \cdots, k\}$. It is easy to see that Q_k is isomorphic to the Cayley graph $X(\Gamma, S)$ where $\Gamma = Z_2^k$ and $S = C$.

It is well-known that Q_k has $k+1$ distinct eigenvalues $\lambda_r = k - 2r$ $(r = 0, 1, \cdots, k)$ with multiplicities $m(\lambda_r) = \binom{k}{r}$.

The *k-dimensional folded hypercube*, denoted by FQ_k is an undirected graph obtained from Q_k by adding all complementary edges. For two vertices $x = (x_1, x_2, \cdots, x_k)$ and $y = (y_1, y_2, \cdots, y_k)$ of FQ_k, $xy \in E(FQ_k)$ is a complementary edge if and only if their bits are the complement of each other, i.e., $y_i = \overline{x_i}$ for each $i = 1, 2, \cdots, k$. Let $H = \{e_i | i = 1, 2, \cdots, k\} \cup \{e_{k+1} = (1, 1, \cdots, 1)\} \subset Z_2^k$, it is easy to see that FQ_k is isomorphic to the Cayley graph $X(\Gamma, S)$ where $\Gamma = Z_2^k$ and $S = H$.

In [5], L.Lovász determine the spectrum of graph with transitive automorphism group. In [2], L.Babai derive an expression for the spectrum of the Cayley graph $X(\Gamma, S)$ in terms of irreducible characters of the group Γ. In this paper, we give a formula of the spectrum of Cayley graph on Abel group by an explicit expression of primitive roots of unity. In terms of the formula, we derive a explicit expression of the spectrum of the folded hypercube FQ_k. In addition, we define the Bi-folded hypercube and determine its spectrum.

2 Main results

Let Γ be a group, for any element $p \in \Gamma$, let $R_i(p)$ be the irreducible representation of p, the *irreducible character* $\chi_i(p)$ of p is defined as the trace of $R_i(p)$, that is

$$\chi_i(p) = tr(R_i(p)).$$

Let G be an Abel group. It is well-known that G can be decomposed uniquely into a directed product of cyclic groups, say

$$\begin{aligned} G &= Z_{n_1} \times Z_{n_2} \times \cdots \times Z_{n_k} \\ &= (a_1) \times (a_2) \times \cdots \times (a_k), \end{aligned}$$

where $a_i (i = 1, \cdots, k)$ is the generator of the cyclic group Z_{n_i}. In this section, we focus on to determine the spectrum of Cayley graph $X(G, S)$ on Abel group .

Lemma 2.1 (Babai [2]). *Denoting by $\{\lambda_1 \cdots, \lambda_n\}$ the spectrum of the Cayley graph $X = X(G, S)$ of the Abel group G we have*

$$\lambda_i = \sum_{g \in S} \chi_i(g) (i = 1, \cdots, n),$$

where $\chi_i(g)$ denotes the irreducible characters of g.

For any $g \in G = Z_{n_1} \times Z_{n_2} \times \cdots \times Z_{n_k}$, we have $g = a_1^{r_1} a_2^{r_2} \cdots a_k^{r_k}$, $r_i = 1, 2, \cdots, n_i$, $i = 1, \cdots, k$. Then

$$\begin{aligned} \chi(g) &= \chi(a_1^{r_1} a_2^{r_2} \cdots a_k^{r_k}) \\ &= \chi(a_1^{r_1}) \cdots \chi(a_k^{r_k}) \\ &= \chi(a_1)^{r_1} \cdots \chi(a_k)^{r_k}. \end{aligned}$$

Note that $(\chi(a_i))^{n_i} = \chi(a_i^{n_i}) = \chi(1) = 1$, then $\chi(a_i) = \omega_{n_i}^{j_i}$, $j_i = 1, \cdots, n_i$, where $\omega_{n_i} = e^{\frac{2\pi l}{n_i}}$ denotes the primitive unity roots of order n_i, $l = \sqrt{-1}$. Hence

$$\chi_{j_1, j_2, \cdots, j_k}(g) = \omega_{n_1}^{j_1 r_1} \omega_{n_2}^{j_2 r_2} \cdots \omega_{n_k}^{j_k r_k}.$$

Let $\lambda_{j_1 j_2 \cdots j_k}$ denote the eigenvalues of Cayley graph on Abel group, where $j_i = 1, \cdots, n_i, i = 1, \cdots, k$. By Lemma 2.1, we have

$$\begin{aligned} \lambda_{j_1 j_2 \cdots j_k} &= \sum_{g \in S} \chi_{j_1 j_2 \cdots j_k}(g) \\ &= \sum_{g \in S} \omega_{n_1}^{j_1 r_1} \omega_{n_2}^{j_2 r_2} \cdots \omega_{n_k}^{j_k r_k}, \end{aligned}$$

where $g = a_1^{r_1} a_2^{r_2} \cdots a_k^{r_k}$.

We summarize the above facts in the following Corollary.

Corollary 2.2. *Let $X = X(G,S)$ be a Cayley graph on Abel group G, then the eigenvalues of X are*

$$\lambda_{j_1 j_2 \cdots j_k} = \sum_{g \in S} \omega_{n_1}^{j_1 r_1} \omega_{n_2}^{j_2 r_2} \cdots \omega_{n_k}^{j_k r_k},$$

where $g = a_1^{r_1} a_2^{r_2} \cdots a_k^{r_k}$, ω_{n_i} denotes the primitive unity roots of order n_i and $j_i = 1, \cdots, n_i, i = 1, 2, \cdots, k$.

By Corollary 2.2, we have

Theorem 2.3. *If k is even, then the spectrum of folded hypercube is*

$$spec(FQ_k) = \begin{pmatrix} k+1 & k-2\cdot 1-1 & k-2\cdot 3-1 & \cdots & k-2\cdot(k-1)-1 \\ 1 & \binom{k}{1}+\binom{k}{2} & \binom{k}{3}+\binom{k}{4} & \cdots & \binom{k}{k-1}+\binom{k}{k} \end{pmatrix};$$

If k is odd, then the spectrum of folded hypercube is

$$spec(FQ_k) = \begin{pmatrix} k+1 & -k-1 & k-2\cdot 1-1 & k-2\cdot 3-1 & \cdots & k-2\cdot(k-2)-1 \\ 1 & 1 & \binom{k}{1}+\binom{k}{2} & \binom{k}{3}+\binom{k}{4} & \cdots & \binom{k}{k-2}+\binom{k}{k-1} \end{pmatrix}.$$

Proof. In fact, since FQ_k is a Cayley graph $X(Z_2^k, H)$ on Abel group Z_2^k, where $H = \{e_i | i = 1, 2, \cdots, k\} \cup \{e_{k+1} = (1, 1, \cdots, 1)\}$, then $spec(FQ_k) = spec(X)$. Note that $\omega_2 = -1$, by Corollary 2.2, the eigenvalues of FQ_k are

$$\begin{aligned} \lambda_{j_1 j_2 \cdots j_k} &= \sum_{g \in H} \omega_2^{j_1 r_1} \omega_2^{j_2 r_2} \cdots \omega_2^{j_k r_k} \\ &= (-1)^{j_1} + (-1)^{j_2} + (-1)^{j_3} + \cdots + (-1)^{j_k} + (-1)^{j_1+j_2+\cdots+j_k}, \end{aligned}$$

where $j_i = 1, 2, i = 1, 2, \cdots, k$.

Let $\lambda'_{j_1 j_2 \cdots j_k}$ and $\lambda''_{j_1 j_2 \cdots j_k}$ be eigenvalues of folded hypercube FQ_k, then $\lambda'_{j_1 j_2 \cdots j_k} = \lambda''_{j_1 j_2 \cdots j_k}$ if and only if they have the same number of $j_i = 1$ $(1 \le i \le k)$. Let r' denote the number of 1 in $\{j_1, j_2, \cdots, j_k\}$ and then $r' = 0, 1, \cdots, k$. Thus, FQ_k has eigenvalues $\lambda_{r'} = r'(-1)^1 + (k-r')(-1)^2 + (-1)^{k+(k-r')} = k - 2r' + (-1)^{2k-r'}$ $(r' = 0, 1, \cdots, k)$ with multiplicities $m(\lambda_{r'}) = \binom{k}{r'}$, obviously, $\lambda_0 = k+1$ is a simple eigenvalue.

If k is even, then $\lambda_{2t+1} = k - 2(2t+1) - 1$ and $\lambda_{2t+2} = k - 2(2t+2) + 1 = k - 2(2t+1) - 1 = \lambda_{2t+1}$ where $t = 0, 1, \cdots, \frac{k-2}{2}$. Therefore, we can get $\frac{k}{2}$ distinct $\lambda_r = k - 2r - 1$ $(r = 1, 3, \cdots, k-1)$ with multiplicities $m(\lambda_r) = \binom{k}{r} + \binom{k}{r+1}$.

If k is odd, similarly, there are $\frac{k+1}{2}$ distinct $\lambda_r = k - 2r - 1$ $(r = 1, 3, \cdots, k-2)$ with multiplicities $m(\lambda_r) = \binom{k}{r} + \binom{k}{r+1}$ and $\lambda_k = k - 2k - 1 = -(k+1)$ is a simple eigenvalue. □

Example. Let $G = Z_2^4$ and the Folded hypercube FQ_4. By Theorem 2.3, the spectrum of FQ_4 is

$$spec(FQ_4) = \begin{pmatrix} 5 & 1 & -3 \\ 1 & 10 & 5 \end{pmatrix}.$$

A graph X is called the *integral graph* if it has an integral spectrum.

Theorem 2.4. *All the Cayley graph on Abel group Z_2^k are integral.*

For a group Γ, and a subset S (possibly, contains the identity element) of Γ, the *Bi-Cayley graph* $BC(\Gamma, S)$ of Γ with respect to S is defined as the bipartite graph with vertex set $\Gamma \times \{0,1\}$ and edge set $\{\{(g,0),(sg,1)\} | g \in \Gamma, s \in S\}$. The *Bi-folded hypercube* is a Bi-Cayley graph $BC(\Gamma, S)$, where $\Gamma = Z_2^k$ and $S = H$, we denote Bi-folded hypercube by BFQ_k.

Theorem 2.5. *Let $\lambda_1, \lambda_2, \cdots, \lambda_n$ be the eigenvalues of folded hypercube FQ_k, then the eigenvalues of Bi-folded hypercube BFQ_k are $\pm|\lambda_1|, \pm|\lambda_2|, \cdots, \pm|\lambda_n|$.*

Proof. Let A and B be adjacency matrices of FQ_k and BFQ_k, respectively. It is easy to see that

$$B = \begin{pmatrix} 0 & A \\ A & 0 \end{pmatrix}.$$

Therefore, we have

$$|\lambda I - B| = \left| \lambda I - \begin{pmatrix} 0 & A \\ A & 0 \end{pmatrix} \right| = \begin{vmatrix} \lambda I & -A \\ -A & \lambda I \end{vmatrix} = |\lambda^2 I - A^2|.$$

Since the eigenvalues of A are $\lambda_1, \lambda_2, \cdots, \lambda_n$, the eigenvalues of A^2 are $\lambda_1^2, \lambda_2^2, \cdots, \lambda_n^2$, then the eigenvalues of BFQ_k are $\pm|\lambda_1|, \pm|\lambda_2|, \cdots, \pm|\lambda_n|$. □

It is easy to see that FQ_k is also a Cayley graph on Abel group Z_2^k. Let $FQ_k = X(Z_2^k, H)$, where $H = \{e_i | i = 1, 2, \cdots, k\} \cup \{e_{k+1} = (1, 1, \cdots, 1)\} \subset Z_2^k$. The automorphism group of folded hypercube FQ_k is denoted as $Aut(FQ_k)$.

Lemma 2.6 (Xu [6]). *Suppose that $X(\Gamma, S)$ is strongly connected and $X(\Gamma, S) \cong X(\Gamma, T)$. If for every isomorphism σ from $X(\Gamma, S)$ to $X(\Gamma, T)$ with $\sigma(1) = 1$, we have $\sigma(ab) = \sigma(a)\sigma(b)$ for all a and b in S, then $\sigma \in Aut(\Gamma)$ for all such σ.*

Let FQ_k be the folded hypercube and $G = Z_2^k$. Let $R(G) = \{r_a : x \longrightarrow x + a(\forall x \in G)\}$, then $R(G)$ is a subgroup of $Aut(FQ_k)$ which is isomorphic to G and acts transitively on vertices of FQ_k. Let G_0 denote the subgroup of $Aut(FQ_k)$ which fixes the zero element of G, i.e., $G_0 = \{\tau \in Aut(FQ_k)|\tau(0) = 0\}$. It is well known that $Aut(FQ_k) = R(G) \cdot G_0$.

Let σ be linear transformation on linear space Z_2^k over binary field F_2 and $\sigma H = H$. Let L be the set of all these linear transformation. If u is adjacent to v, then $u = v + e_i$ and $\sigma(u) = \sigma(v) + \sigma(e_i)$, thus $\sigma(u)$ is adjacent to $\sigma(v)$. Therefore, σ is automorphism of folded hypercube.

Conversely, if σ is a automorphism of folded hypercube and $\sigma(0) = 0$, then $\sigma H = H$. Obviously, e_i and e_j have two common neighbors 0 and $e_i + e_j$ in folded hypercube, then $\sigma(e_i + e_j)$ should be the common neighbor of $\sigma(e_i)$ and $\sigma(e_j)$. Since both $\sigma(e_i + e_j) \neq 0$ and $\sigma(e_i) + \sigma(e_j) \neq 0$, hence $\sigma(e_i + e_j) = \sigma(e_i) + \sigma(e_j)$.

For any $u, v \in FQ_k$, we have $u = \sum_i e_i$, $v = \sum_j e_j$, and then $\sigma(u + v) = \sigma(\sum_i e_i + \sum_j e_j) = \sigma(\sum_i e_i) + \sigma(\sum_j e_j) = \sigma(u) + \sigma(v)$. Therefore, σ is linear transformation on linear space Z_2^k over binary field F_2.

Then L is the subgroup of $Aut(FQ_k)$ stabilizing H setwise and $L = G_0$. Thus, we have $Aut(FQ_k) = R(G) \cdot L$. We at once check that $|L| = (k+1)!$, therefore, $|Aut(FQ_k)| = 2^k(k + 1)!$.

From above discussion, we have the following result.

Theorem 2.7. *Let FQ_k be the folded hypercube and $G = Z_2^k$. Then $Aut(FQ_k) = R(G) \cdot L$ and $|Aut(FQ_k)| = 2^k(k + 1)!$.*

By Theorem 2.7, it is easy to see that

Corollary 2.8. *Folded hypercube is edge-transitive.*

References

[1] J.L.Alperin, R.B.Bell, Groups and Representations, Springer-Verlag New York, 1997.

[2] Babai,L., Spectra of Cayley graph, J.Combin.Theory Ser.B 27(1979), 180-189.

[3] D.Cvetković, M.Doob, H.Sachs. Spectra of graphs, Academic Press, New York 1980.

[4] C.Godsil, G.Royle, Algebraic Graph Theory, Springer-Verlag New York, Inc,2001.

[5] L.Lovasz, Spectra of graphs with transitive groups, Period. Math. Hungar. 6(1975), 191-196.

[6] Mingyao Xu, Automorphism of Groups and Isomorphisms of Cayley Digraphs, Discrete Mathematics 182(1998), 309-319.

[7] Wasin So, Integral circulant graphs, Discrete Mathematics 306(2005), 153-158.

[8] Junming Xu, Topological Structure and Analysis of Interconnection Networks, Kluwer Academic Publishers, (2001).

[9] Jin Xu, Ruibin Qu, The spectra of Hypercubes, Journal of Engneering Mathematics, 4(1999), 1-5.

[10] H.Zou, J.X.Meng, Some algebraic properties of Bi-Cayley Graphs, Acta Mathematica Sinica, Chinese Series 50 (5)(2007), 1075-1080.