Determinants involving generalized Stirling numbers

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Abstract: In a previous paper the first author introduced two classes of generalized Stirling numbers, $s_m(n,k,p)$, $S_m(n,k,p)$ with $m=1$ or 2, called p-Stirling numbers. In this paper, we discuss their determinant properties.

Keywords: Determinant, Generalized Stirling Numbers.

1. Introduction

It is well know that the first kind of unsigned Stirling numbers, $|s(n,k)| = (-1)^{n-k}s(n,k)$, count the number of permutations in the symmetric group $\mathbf{S}_n$ with k cycles [4, P18], and the second kind, $S(n,k)$, count the number of partitions of $[n]=\{1,2,\ldots,n\}$ into k disjoint nonempty blocks [4, P33]. In the literature, there exist many beautiful determinants involving the classical Stirling numbers of the first kind $s(n,k)$ and the second kind $S(n,k)$ [2, P228], [3], [6]. For examples, for any integer $r \geq 0$, there hold

$$\begin{aligned}
\det_{1\leq i,j\leq k}\Big(|s(r+i,j)|\Big) &= (r!)^k, \\
\det_{0\leq i,j\leq k}\Big(\frac{(ri)!}{(ri+j)!}s(ri+j,ri)\Big) &= \Big(-\frac{r}{2}\Big)^{\binom{k+1}{2}}, \\
\det_{0\leq i,j\leq k}\Big(S(r+i+j,r+j)\Big) &= \prod_{i=0}^{k}(r+i)^i,
\end{aligned}$$

$$\det_{0\le i,j\le k}\left(\frac{(ri)!}{(ri+j)!}S(ri+j,ri)\right) = \left(\frac{r}{2}\right)^{\binom{k+1}{2}},$$
$$\det_{0\le i,j\le k}\left(\frac{S(r+i+j,r+i)}{(r+i+j)!}\right) = \prod_{v=0}^{k}\frac{1}{(r+v)!}.$$

In a previous paper [5], the first author introduced the concept of k-matrix partition (permutation) on a $p\times n$ matrix $M(n,p)=(M_{ij})$ with $M_{ij}=j$. The number of k-matrix partitions (permutations) of $M(n,p)$ is counted by the generalized Stirling numbers $S_1(n,k,p)(|s_1(n,k,p)| = (-1)^{n-k}s_1(n,k,p))$, and the number of strong $(k+p-1)$-matrix partitions (permutations) of $M(n+p-1,p)$ corresponds to $S_2(n,k,p)(|s_2(n,k,p)| = (-1)^{n-k}s_2(n,k,p))$. They satisfy respectively the recursive formulas:

$$S_1(n+1,k,p) = k^pS_1(n,k,p)+S_1(n,k-1,p), \tag{1.1}$$
$$|s_1(n+1,k,p)| = n^p|s_1(n,k,p)|+|s_1(n,k-1,p)|, \tag{1.2}$$
$$S_2(n+1,k,p) = \binom{k+p-1}{p}S_2(n,k,p)+S_2(n,k-1,p), \tag{1.3}$$
$$|s_2(n+1,k,p)| = \binom{n+p-1}{p}|s_2(n,k,p)|+|s_2(n,k-1,p)|, \tag{1.4}$$

with the initial conditions for $m=1$ or 2,

$$S_m(n,k,p)=\begin{cases}0 & \text{if } n<k,\\ & \text{or } k<0,\\ 1 & \text{if } n=k\ge 0.\end{cases}\qquad |s_m(n,k,p)|=\begin{cases}0 & \text{if } n<k,\\ & \text{or } k<0,\\ 1 & \text{if } n=k\ge 0.\end{cases}$$

Note that the case $p=1$ reduces to the classical Stirling numbers, and the case $p=0$ reduces to the binomial coefficients.

The goal of this article is to evaluate determinants involving the generalized Stirling numbers $S_m(n,k,p)$ and $s_m(n,k,p)$ with $m=1$ or 2, which extends the results in [1].

2. Determinantal properties of generalized Stirling numbers

Theorem 2.1 *For any integer $r \geq 0$, we have*

$$\det_{1\leq i,j\leq k}\Big(|s_1(r+i,j,p)|\Big) = (r!)^{pk}, \tag{2.1}$$

$$\det_{1\leq i,j\leq k}\Big(S_1(r+i,j,p)\Big) = (k!)^{pr}, \tag{2.2}$$

$$\det_{0\leq i,j\leq k}\Big(S_1(r+i+j,r+j)\Big) = \prod_{v=1}^{k}(r+v)^{pv}, \tag{2.3}$$

$$\det_{1\leq i,j\leq k}\Big(|s_2(r+i,j,p)|\Big) = \prod_{v=1}^{r}\binom{v+p-1}{p}^{k}, \tag{2.4}$$

$$\det_{1\leq i,j\leq k}\Big(S_2(r+i,j,p)\Big) = \prod_{v=1}^{k}\binom{v+p-1}{p}^{r}, \tag{2.5}$$

$$\det_{0\leq i,j\leq k}\Big(S_2(r+i+j,r+j)\Big) = \prod_{v=1}^{k}\binom{r+v+p-1}{p}^{v}. \tag{2.6}$$

Proof. We just prove (2.1)-(2.3), and (2.4)-(2.6) follow similarly. Let R_α, R_β be the α-th and β-th rows, and $R_\alpha \leftarrow \theta R_\alpha + \vartheta R_\beta$ mean the standard row operation on determinants, namely, to replace the row R_α by $\theta R_\alpha + \vartheta R_\beta$, where θ, ϑ are some constants. Let C_α, C_β and $C_\alpha \leftarrow \theta C_\alpha + \vartheta C_\beta$ denote the same meaning for columns.

For $\beta = 2, 3, \ldots, k$ and $\alpha = k, k-1, \ldots, \beta$, by (1.2), the operation $R_\alpha \leftarrow R_\alpha - (n+\alpha-\beta+1)^p R_{\alpha-1}$ can transform the matrix in (2.1) to a simpler form, an upper-triangular matrix with the diagonal entries $|s_1(r+1,1,p)| = (r!)^p$, then (2.1) holds.

For $\alpha = k, k-1, \ldots, 2$, by (1.1) and $S_1(n,1,p) = 1$ for $n \geq 1$, the operation $C_\alpha \leftarrow \alpha^p C_\alpha + C_{\alpha-1}$ can induce the recursive relation:

$$\det_{1\leq i,j\leq k}\Big(S_1(r+i,j,p)\Big) = \frac{1}{(k!)^p}\det_{1\leq i,j\leq k}\Big(S_1(r+1+i,j,p)\Big),$$

then (2.2) follows by iteration on r.

For $\beta = 2, 3, \ldots, k+1$ and $\alpha = k+1, k, \ldots, \beta$, by (1.1) and $S_1(n,n,p) = 1$ for $n \geq 1$, the operation $C_\alpha \leftarrow C_\alpha - \frac{(r+\alpha-1)^{p(\beta-2)}}{(r+\alpha-2)^{p(\beta-2)}} C_{\alpha-1}$ can transform the matrix in (2.3) to a simpler form, a lower-triangular matrix with the (i,i)-entries $(r+i)^{pi} S_1(r+i,r+i,p) = (r+i)^{pi}$, thus (2.3) follows. □

Theorem 2.2 *For any integers $r, k \geq 1$, let $A(m), B(m)$ and $C(m)$ denotes six $k \times k$ matrices whose (i,j)-entries are respectively $A(m)_{ij}$ =*

$|s_m(r+i-1, r-2k+i+j)|$, $B(m)_{ij} = |s_m(r+i-1, r-k+j)|$ and $C(m)_{ij} = |s_m(r+i-1, r-k-1+i+j)|$ for $1 \le i,j \le k$ with $m=1$ or 2, then we have

$$\det A(m) \quad = \quad \det B(m) \cdot \det C(m).$$

Proof. We just prove the case $m=1$, and the case $m=2$ holds similarly. For $\beta = 2,3,\ldots,k$ and $\alpha = k, k-1, \ldots, \beta$, by (1.2), the operation $R_\alpha \leftarrow R_\alpha - \frac{(r+\alpha-2)^p}{(r+\alpha-\beta)^p} R_{\alpha-1}$ can transform the matrix $A(1)$ to a simpler form,

$$A(1)_{ij} = |s_1(r, r-2k+i+j)| \prod_{v=0}^{i-2} (r+v)^p,$$

then we get

$$\det A(1) = \det_{1\le i,j\le k} \Big(|s_1(r, r-2k+i+j)|\Big) \cdot \prod_{v=0}^{k-2} (r+v)^{p(k-v-1)}.$$

For $\beta = 2,3,\ldots,k$ and $\alpha = k, k-1, \ldots, \beta$, by (1.2), the operation $R_\alpha \leftarrow R_\alpha - (r+\alpha-\beta)^p R_{\alpha-1}$ can transform the matrix $B(1)$ to a simpler form,

$$B(1)_{ij} = |s_1(r, r-k-i+j+1)|.$$

Note that

$$B(1)^t \cdot M = \Big(|s_1(r, r-k-i+j+1)|\Big)^t \cdot M = \Big(|s_1(r, r-2k+i+j)|\Big),$$

where $B(1)^t$ denotes the transposed matrix of $B(1)$ and M is the $k \times k$ anti-unit matrix. Then we obtain

$$\det B(1) = (-1)^{\binom{k}{2}} \det_{1\le i,j\le k} \Big(|s_1(r, r-2k+i+j)|\Big).$$

For $\beta = 2,3,\ldots,k$ and $\alpha = k, k-1, \ldots, \beta$, by (1.2), the operation $R_\alpha \leftarrow R_\alpha - \frac{(r+\alpha-2)^p}{(r+\alpha-\beta)^p} R_{\alpha-1}$ can transform the matrix $C(1)$ to a simpler form,

$$C(1) = \Big(C(1)_{ij}\Big) = \Big(|s_1(r, r-k+i+j-1)| \prod_{v=0}^{i-2} (r+v)^p\Big),$$

which is an anti-upper-triangular matrix with the anti-diagonal $(i, k+1-i)$-entries $s_1(r,r,p) \prod_{v=0}^{i-2} (r+v)^p = \prod_{v=0}^{i-2} (r+v)^p$. Then we have

$$\det C(1) = (-1)^{\binom{k}{2}} \prod_{v=0}^{k-2} (r+v)^{p(k-v-1)}.$$

Summarizing these facts, we obtain the desired result. □

Theorem 2.3 *For any integers $r \geq 1, k \geq 0$, let $E(m), F(m)$ and $G(m)$ denotes six $(k+1) \times (k+1)$ matrices whose (i,j)-entries are respectively $E(m)_{ij} = S_m(r+i+j, r+j)$, $F(m)_{ij} = S_m(r+k+i, r+j)$ and $G(m)_{ij} = S_m(r+k+i+j, r+j)$ for $0 \leq i,j \leq k$ with $m = 1$ or 2, then we have*

$$\det G(m) \quad = \quad \det E(m) \cdot \det F(m).$$

Proof. We just prove the case $m = 1$, and the case $m = 2$ holds similarly. For $\beta = 2, 3, \ldots, k+1$ and $\alpha = k+1, k, \ldots, \beta$, by (1.1), the operation $C_\alpha \leftarrow C_\alpha - \frac{(r+\alpha-1)^{p(\beta-2)}}{(r+\alpha-2)^{p(\beta-2)}} C_{\alpha-1}$ can transform the matrix $G(1)$ to a simpler form,

$$G(1)_{ij} = S_1(r+k+i, r+j)(r+j)^{pj} = F(1)_{ij}(r+j)^{pj},$$

then we get

$$\det G(1) = \det F(1) \cdot \prod_{v=1}^{k} (r+v)^{pv}.$$

But (2.3) tells us that $\det E(1) = \prod_{v=1}^{k} (r+v)^{pv}$, then the result holds. □

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