A Note on Strongly Graceful Trees [1]

Bing YAO[a,2] Hui CHENG[a,3] Ming YAO[b] Meimei ZHAO [a]

a. College of Mathematics and Information Science, Northwest Normal University, Lanzhou, 730070, P.R.China

b. Department of Information Process and Control Engineering, Lanzhou Petrochemical College of Vocational Technology, 730060, P.R.China

Abstract

A tree T with n vertices and a perfect matching M is *strongly graceful* if T admits a graceful labeling f such that $f(u)+f(v) = n-1$ for every edge $uv \in M$. Broersma and Hoede [5] conjectured that every tree containing a perfect matching is strongly graceful in 1999. We prove that a tree T with diameter $D(T) \leq 5$ supports the strongly graceful conjecture on trees. We show several classes of basic seeds and some constructive methods for constructing large scale of strongly graceful trees.

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1 Introduction and concepts

Golomb (cf. [7], [9]) poses the question of how to notch a metal bar k units in length at a minimum number of integer points in such a way that the distances between any two notches, or between a notch and an endpoint, are distinct and generate the set $\{1, 2, \ldots, k\}$. The above question could be investigated by some graph labellings. A graceful labeling of a simple graph G with q edges is assignment of distinct labels from $\{0, 1, \ldots, q\}$ to vertices of G, where edges are labeled by absolute values of difference of labels of adjacent vertices and every label from $1, 2, \ldots, q$ is used exactly once as an edge label (cf. [9]).

All graphs mentioned in this article are simple, undirected and finite. The undefined terminologies will follow [1]. For the sake of simplicity, the shorthand symbol $[m, n]$ stands for a set $\{m, m+1, \ldots, n\}$, where m and n are non-negative integers with $m \leq n$. We formulate formally the definition of a graceful graph in the following:

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[2] The corresponding author. E-mail: yybb918@163.com

[3] The writer.

Definition 1. [1] [10] Let f be a labelling of a connected graph G with p vertices and q edges. Each vertex u of G is assigned a number $f(u) \in [0, q]$ such that $f(u) \neq f(v)$ for $u \neq v$ in G, and the label of an edge uv of G is $f(uv) = |f(u) - f(v)|$ and the set of all edge labels is equal to $[1, q]$. Then f is called a *graceful labelling*, so say, G is *graceful*.

Let f be a graceful labelling of a connected graph G having q edges. The labelling $h(x) = q - f(x)$ for all $x \in V(G)$ is called the *dual graceful labelling* of f. By Definition 1, a *graceful tree* T on p vertices admits a graceful labelling f such that vertex label set $\{f(u) : u \in V(T)\} = [0, p-1]$ (denoted by $f(V(T))$) and edge label set $\{f(uv) : uv \in E(T)\} = [1, p-1]$ (denoted by $f(E(T))$). The following long-standing Graceful Tree Conjecture (GTC) was found by many researchers (cf. [1, 10]):

Conjecture 1. (Alexander Rosa, 1966) [11] *Each tree is graceful.*

Rosa discovered that if each tree admits a graceful labelling, then this will settle a longstanding, well-known Ringel-Kotzig Decomposition Conjecture in popularization: K_{2n+1} can be decomposed into $2n+1$ subgraphs which are isomorphic with a given tree with n edges (cf. [10], [11]).

Despite the tremendous work of many literatures, GTC is still open up to now. A *caterpillar* is a tree T such that the graph obtained by deleting all leaves from T is just a path, where a vertex of degree one is called a *leaf*. A *lobster* H is a tree such that the graph obtained by deleting all leaves from H is a caterpillar. One may consider: Every lobster is graceful (cf. [3]).

In this article we will focus on strongly graceful lobsters. The definition of a strongly graceful tree is formulated in the following:

Definition 2. Let T be a tree with n vertices and a perfect matching M. T is *strongly graceful* if T admits a graceful labeling f such that $f(u) + f(v) = n - 1$ for every edge $uv \in M$.

Conjecture 2. (H. J. Broersma and C. Hoede, 1999) [5] *Every tree containing a perfect matching is strongly graceful.*

Definition 3. [4] A *bipartite labeling* of a tree T on n vertices is a bijection f: $V \to [0, n-1]$ for which there exists a positive number k such that whenever $f(u) \leq k \leq f(v)$, then u and v have different colors. The *λ-size* $\lambda(T)$ of the tree T is the maximum number of elements in the sets $\{|f(u) - f(v)| : uv \in E\}$, taken over all bipartite labelings f of T.

The quantity $\lambda(n)$ is defined as the minimum of $\lambda(T)$ over all trees with n vertices. In an earlier article [12], Rosa et al. proved that $5n/7 \leq \lambda(n) \leq (5n+4)/6$ for all $n \geq 4$; the upper bound is believed to be the asymptotically correct value of $\lambda(n)$. Let $\lambda_3(n)$ be the smallest λ-size among all trees with n vertices, each of degree at most three. They proved that $\lambda_3(n) \geq 5n/6$ for all $n \geq 12$, thus supporting the belief above. This result can be seen as an approximation toward GTC. Using a computer search, they also established that $\lambda_3(n) \geq n-2$ for all $n \leq 17$.

Clearly, $\lambda(T) = n-1$ if a tree T on n vertices is graceful. Furthermore. we define a *bipartite graceful* tree T if it admits a graceful labelling f such that $f(u) < f(v)$ for all $u \in V_1$ and $v \in V_2$, where $V(T) = V_1 \cup V_2$ and both V_1, V_2 are independent. For the sake of convenience, we write this case as $f(V_1) < f(V_2)$ throughout this paper.

In Section 2 we prove that a tree T with diameter $D(T) \leq 5$ supports Conjecture 2, and some lemmas for constructing strongly graceful trees. In Section 3 we show several classes of basic seeds and a so-called recurrent labelling that is useful for constructing strongly graceful trees. In Section 4 we introduce a new labelling on trees, called the k-strongly graceful labelling, and we propose some problems. By our experience, we present a conjecture that any tree T on n vertices contains a certain largest matching M and a graceful labelling f such that $f(u) + f(v) = n-1$ for each edge $uv \in M$.

2 Lemmas and theorems

Let w be a vertex of a tree T and let $\deg_T(w)$ stand for the degree of w in T. A *defect matching* M^* of T is a matching that saturates each vertex of $V(T) \setminus \{w\}$. We say that T is *defect strongly graceful* if T admits a graceful labelling h such that $h(x) + h(y) = |T| - 1$ for each edge xy of the defect matching M^*, also, h is called a *defect strongly graceful labelling.* An *end-node* u of T is a such vertex u that has its own neighbour set $N(u) = \{v, u_1, u_2, \ldots, u_{\deg_T(u)-1}\}$, where every u_i is a leaf of T for $1 \leq i \leq \deg_T(u) - 1$, and degree $\deg_T(v) \geq 2$.

The following lemma shows some properties of a tree with a perfect matching:

Lemma 1. *Let T be a tree with a perfect matching M and n vertices. Suppose that f is a strongly graceful labelling of T. Then*

(i) *The maximum degree $\Delta(T) \leq n/2$; each end-node of T has degree 2; and $|S| = |U|$ where S, U are independent sets of T such that $V(T) = S \cup U$.*

(ii) *For $uv \in M$ there are $f(u) = n-1-k$ and $f(v) = k$ for $0 \leq k \leq n/2$.*

(iii) *There is a path $P = uvxy$ in T such that $uv, xy \in M$ and $f(u) = 0$, $f(v) = n-1$, $f(x) = 1$ and $f(y) = n-2$, or $f(u) = n-1$, $f(v) = 0$, $f(x) = n-2$ and $f(y) = 1$.*

(vi) *There are no two different perfect matchings in T.*

Proof. The assertions (i), (ii) and (iii) are obvious, so we only verify the assertions (vi). By the induction. Let M be a perfect matching of T. Since each end-node of T has degree 2 we have an edge set $S_e = \{x_i y_i : 1 \leq i \leq k\}$ where each x_i is an end-node and each y_i is a leaf of T. Clearly, $S_e \subseteq M$ and $H = T - S_e$ is a tree with a perfect matching M'. Note that S_e is unique in T, and M' is unique in H by the induction hypothesis, we are done. □

Theorem 2. *Let T be a tree with a perfect matching M and n vertices. Then T is strongly graceful if the diameter $D(T) \leq 5$.*

Proof. Let T be a tree with a perfect matching M and n vertices. Since T is a path P_2 on 2 vertices if $D(T) = 1$ and a path P_4 on 4 vertices if $D(T) = 3$, so the result is obvious. For $D(T) = 4$, it is easy to see that the center of T is only a vertex w, and w also is the unique vertex of maximum degree. Therefore, the order n of T must be even, without loss of generalization, let $n = 2k + 2$. We can describe T as this: $V(T) = \{w, z, x_i, y_i : 1 \leq i \leq k\}$ and $E(T) = \{wz, wx_i, x_i y_i : 1 \leq i \leq k\}$, and a perfect matching $M = \{wz, x_i y_i : 1 \leq i \leq k\}$. Note that the set of all leaves of T is $\{z, y_i : 1 \leq i \leq k\}$.

It is straightforward to give T a strongly graceful labelling f in the following: $f(w) = 2k+1$, $f(z) = 0$; and $f(x_i) = 2i-1$, $f(y_i) = 2k+1-f(x_i)$ for $1 \leq i \leq k$.

We, now, consider case $D(T) = 5$. There are two classes of trees with diameter 5 and a perfect matching, denoted by $\mathcal{F}_1$, $\mathcal{F}_2$ respectively. Note that the center of a tree T with diameter 5 has just two vertices. We write one member of $\mathcal{F}_1$ by $T^{(I)}_{m,n}$, where $1 \leq m \leq n$. Furthermore, $V(T^{(I)}_{m,n}) = \{w_1, w_2, x_i, y_i, u_j, v_j : 1 \leq i \leq m, 1 \leq j \leq n\}$, where w_1, w_2 are the center vertices and $|V(T^{(I)}_{m,n})| = 2(m+n+1)$; $E(T^{(I)}_{m,n}) = X \cup \{w_1 w_2\} \cup U$, where $X = \{w_1 x_i, x_i y_i : 1 \leq i \leq m\}$ and $U = \{w_2 u_j, u_j v_j : 1 \leq j \leq n\}$; and a perfect matching $M = \{x_i y_i, w_1 w_2, u_j v_j : 1 \leq i \leq m, 1 \leq j \leq n\}$. It is straightforward to show a labelling f of $T^{(I)}_{m,n}$ by setting $f(y_i) = 2(i-1)$ and $f(x_i) = 2(m+n+1) - 1 - 2(i-1)$ for $1 \leq i \leq m$; $f(w_1) = 1$ and $f(w_2) = 2(m+n+1)-2$; $f(v_j) = 1+2j$ and $f(u_j) = 2(m+n+1)-2(j+1)$ for $1 \leq j \leq n$. Note that $f(x_i) + f(y_i) = 2(m+n+1) - 1$ $(1 \leq i \leq m)$ and $f(u_j) + f(v_j) = 2(m+n+1) - 1$ $(1 \leq j \leq n)$, and furthermore there are

$0 = f(y_1) < f(w_1) < f(y_2) < f(v_1) < f(y_3) < f(v_2) < f(y_4) < \cdots f(v_{m-2}) < f(y_m) = 2m - 2$ and $f(v_{m+t-1}) < f(u_{n-t})$ for $0 \leq t \leq (m+n-1)/2 - 1$, $m \geq 2$; and

$2(m+n+1) - 1 = f(x_1) > f(w_2) > f(x_2) > f(u_1) > f(x_3) > f(u_2) > f(x_4) > \cdots > f(u_{m-2}) > f(x_m) = 2n + 3$ and $f(u_{m+t-1} > f(v_{n-t})$ for $0 \leq t \leq (m+n-1)/2$.

Thereby, we claim that f is strongly graceful. See for an example shown in Figure 1.(a).

Similarly, $T^{(II)}_{m,n}$ $(1 \le m \le n)$ stands for one member of $\mathcal{F}_2$. We write the basic characters of $T^{(II)}_{m,n}$ in the following: $V(T^{(II)}_{m,n}) = \{w_1, w_2, w'_1, w'_2, x_i, y_i, u_j, v_j : 1 \le i \le m, 1 \le j \le n\}$, where w_1, w_2 are the center vertices; $E(T^{(II)}_{m,n}) = X \cup \{w_1w'_1, w_1w_2, w_2w'_2\} \cup U$, where $X = \{w_1x_i, x_iy_i : 1 \le i \le m\}$ and $U = \{w_2u_j, u_jv_j : 1 \le j \le n\}$, and a perfect matching $M = \{x_iy_i, w_1w'_1, w_2w'_2, u_jv_j : 1 \le i \le m, 1 \le j \le n\}$. Clearly, $|T^{(II)}_{m,n}| = |T^{(I)}_{m,n}| + 2 = 2(m+n+2)$. We define a labelling h of $T^{(II)}_{m,n}$ in the following: $h(w'_1) = 0$, $h(w_1) = 2(m+n+2)-1$, $h(w_2) = 2m+1$ and $h(w'_2) = 2(n+1)$; $h(y_i) = 2(m+n+2) - 2i$ and $h(x_i) = 1 + 2(i-1)$ for $1 \le i \le m$; $h(v_j) = 2(n+1) - 2j$ and $h(u_j) = 2m + 1 + 2j$ for $1 \le j \le n$. It follows that proof on $T^{(I)}_{m,n}$ above, so that h is strongly graceful. A such example $T^{(I)}_{3,5}$ is shown in Figure 1. (b). □

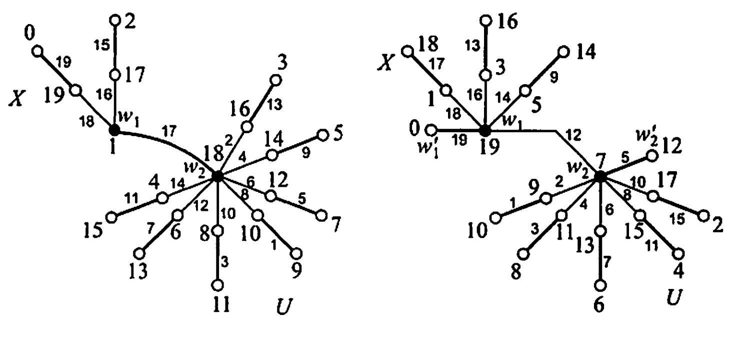


Figure 1: (a) A tree $T^{(I)}_{2,7}$; (b) a tree $T^{(II)}_{3,5}$.

A tree T with 8 vertices, diameter 4 and a perfect matching is strongly graceful (see the proof of Theorem 2), but bipartite graceful. Thereby, we can claim that there are some trees that admit no labelling which is strongly graceful, and bipartite graceful simultaneously.

Lemma 3. (String linking lemma) *Let T be a bipartite graceful tree and let H be a graceful tree. There exist vertices $u \in V(T)$ and $v \in V(H)$ such that linking u with v together by an edge yields a graceful tree.*

Proof. For the sake of simplicity, we define a (A)-tree in the following:

> A (A)-tree is a bipartite graceful tree T with n vertices and $V(T) = U \cup V$ where $U \cap V = \emptyset$, and any edge xy of T

holds $x \in U$ and $y \in V$. Let $U = \{u_1, u_2, \dots, u_s\}$ and $V = \{v_1, v_2, \dots, v_t\}$. Furthermore, T admits a graceful labelling f such that $f(u_i) = i - 1$ for $u_i \in U$ and $1 \le i \le s$ and $f(v_j) = s + j - 1$ for $v_j \in V$ and $1 \le j \le t$.

Let H be a tree on m vertices that admits a graceful labelling h. Assume that $h(w) = 0$ for $w \in V(H)$, and w may be not a leaf of H.

We construct a tree G^* by using an edge wv_1 to adjoin v_1 with w together, and make a labelling α as $\alpha(u_i) = f(u_i)$ for $1 \le i \le s$ and $\alpha(v_j) = f(v_j) + m$ for $1 \le j \le t$ (as a result, $\{|\alpha(x) - \alpha(y)| : xy \in E(T) \subset E(G^*)\} = [m+1, m+n-1]$); and $\alpha(x) = h(x) + s$ for all $x \in V(H)$ (it contributes $\{|\alpha(x) - \alpha(y)| : xy \in E(H) \subset E(G^*)\} = [1, m-1]$). Note that $\alpha(v_1) - \alpha(w) = (s+m) - s = m$. Hence, the labelling α is exactly a graceful labelling of G^*. □

Be careful to check the proof of the string linking lemma (Lemma 3), we are easy to verify the following results on strongly graceful trees:

Theorem 4. *Let T be a (A)-tree defined in the proof of Lemma 3.*

(i) *If f is strongly graceful for the perfect matching M of T and H is a (respectively, a bipartite) strongly graceful tree with a leaf w labelled by zero, then linking $v_1 \in V \subset V(T)$ with w by an edge yields a (respectively, a bipartite) strongly graceful tree.*

(ii) *If $f(u) + f(v) = n - 1$ for each edge uv of a defect matching M' of $V(T) \setminus \{u_s\}$ and H is a (respectively, a bipartite) strongly graceful tree with a leaf w labelled by zero, then identifying u_s with w into one provides a (respectively, a bipartite) strongly graceful tree.*

Lemma 5. *Let T be a (A)-tree defined in the proof of Lemma 3 and let H be a graceful tree with a leaf w labelled by zero.*

(i) (Vertex-identified linking lemma) *There exists a vertex $x \in V(T)$ such that identifying x with w into one results a graceful tree.*

(ii) (Edge-identified linking lemma) *If u_s is adjacent to v_1 that is a leaf in T, then there is an edge $xy \in V(H)$ such that identifying the edge $u_s v_1$ with the edge xy into one yields a graceful tree.*

Proof. Let T be a (A)-tree described in the proof of Lemma 3. Let h be a graceful labelling of the tree H on m vertices. Assume that there is a leaf w of H such that $h(w) = 0$. We are ready to run the proof of this lemma.

(i) There is a tree G obtained from identifying the leaf w of H with the vertex u_s of T into a vertex z. Next we define a labelling α for G by setting $\alpha(z) = s - 1$ $(= h(w) + s - 1 = f(u_s))$; and $\alpha(u_i) = f(u_i)$ for $1 \le i \le s-1$ and $\alpha(v_j) = f(v_j) + m - 1$ for $1 \le j \le t$ (it implies that $\{|\alpha(x) - \alpha(y)| : xy \in E(T) \subset E(G)\} = [m, m+n-2]$); and $\alpha(x) = h(x) + s - 1$ for all $x \in V(H)$ (it shows that $\{|\alpha(x) - \alpha(y)| : xy \in E(H) \subset E(G)\} = [1, m-1]$). Therefore, G is graceful.

(ii) Without loss of generalization, let a vertex w' be adjacent to w in H, thus, $h(w') = m-1$. We identify the edge $u_s v_1$ of T with the edge ww' of H into one, so we obtain a new tree G. To consider the gracefulness of G, we define directly a graceful labelling γ for G in the following. We set $\gamma(u_i) = f(u_i)$ for $1 \le i \le s$ and $\gamma(v_j) = f(v_j) + m - 2$ for $1 \le j \le t$; and $\gamma(x) = h(x) + f(u_s)$ for all $x \in V(H)$. Furthermore, it is not difficult to evaluate $\{|\gamma(x) - \gamma(y)| : xy \in (E(T) \setminus \{u_s v_1\}) \subset E(G)\} = [m, m+n-2]$ and $\{|\gamma(x) - \gamma(y)| : xy \in (E(H) \setminus \{ww'\}) \subset E(G)\} = [1, m-2]$ and $\gamma(v_1) - \gamma(u_s) = \gamma(w') - \gamma(w) = m - 1$.

The proof of the lemma is completed. □

Lemma 6. (Edge-symmetric linking lemma) *Let T be a graceful tree and let T' be a copy of T. Linking any vertex $x \in V(T)$ with its corresponding vertex x' in T' by an edge yields a bipartite graceful tree.*

Proof. Let T be a graceful tree on m vertices. Let $S = \{x_1, x_2, \dots, x_s\}$ and $U = \{y_1, y_2, \dots, y_t\}$ such that $V(T) = S \cup U$ and $S \cap U = \emptyset$. We take a graceful labelling f with $f(x_1) = 0$ and $f(y_t) = m-1$. Correspondingly, $V(T') = U' \cup S'$ and $U' \cap S' = \emptyset$, where $U' = \{y'_1, y'_2, \dots, y'_t\}$ and $S' = \{x'_1, x'_2, \dots, x'_s\}$. And, f' is the corresponding copy of f with $f'(x'_1) = 0$ and $f'(y'_t) = m-1$.

First, we have a new tree H obtained from linking $y_t \in U$ with $y'_t \in U'$ together by using an edge, so that $E(H) = E(T) \cup E(T') \cup \{y_t y'_t\}$ and $V(H) = (S \cup U') \cup (U \cup S')$ where $(S \cup U') \cap (U \cup S') = \emptyset$. Furthermore, it is straightforward to define a labelling α of H in the following. Let $\alpha(x_i) = f(x_i)$ for $1 \le i \le s$ and $\alpha(y_j) = f(y_j) + m$ for $1 \le j \le t$; and let $\alpha(y'_j) = f'(y_j)$ for $1 \le j \le t$ and $\alpha(x'_i) = f'(x_i) + m$ for $1 \le i \le s$. Clearly, $\alpha(u) \ne \alpha(v)$ if $u \ne v$ for $u, v \in V(H)$. We are easy to see that $\{|\alpha(u) - \alpha(v)| : uv \in E(T) = E(H - y_t y'_t) \setminus (E(T'))\} = [m+1, 2m-1]$, $\{|\alpha(u) - \alpha(v)| : uv \in E(T') = E(H - y_t y'_t) \setminus (E(T))\} = [1, m-1]$, and $\alpha(y_t) - \alpha(y'_t) = 2m - 1 - (m-1) = m$. Clearly, the labelling α is just a bipartite graceful labelling since $\alpha(u) < \alpha(v)$ for $u \in S \cup U'$ and $v \in U \cup S'$.

Second, observe that $\alpha(x'_i) - \alpha(x_i) = m$ for $1 \le i \le s$ and $\alpha(y_j) - \alpha(y'_j) = m$ for $1 \le j \le t$. We delete the edge $y_t y'_t$ from H and then adjoin two vertices x_i, x'_i (or y_j, y'_j) together, the resulting tree $H - y_t y'_t + x_i x'_i$ (or $H - y_t y'_t + y_j y'_j$) is bipartite graceful too. □

It should be pointed out that the tree H' constructed in the proof of Lemma 6 may be not bipartite graceful. And the structures of two trees H, H' constructed in the proof of Lemma 6 may be distinct in each other. Immediately, as a direct sequence of Lemma 6 we have the following theorem:

Theorem 7. *Let H be a tree constructed from a tree T and its copy T' by means of the edge-symmetric linking lemma (Lemma 6).*

(i) *If T is strongly graceful, so is H.*

(ii) *If T is bipartite graceful, so is H.*

(iii) *If T admits a labelling that is simultaneously strongly graceful and bipartite graceful, so does H.*

We define a class of (B)-trees as follows.

A (B)-tree is a tree T which satisfies the following conditions:

(i) T contains a perfect matching M.

(ii) T contains a vertex subset $S = \{w', w_0, w'', x_i, y_i : 1 \leq i \leq m\}$ such that w_0 is adjacent to w', w'' and each x_i for $1 \leq i \leq m$, where degrees $\deg_T(w') \neq 2$, $\deg_T(w'') \neq 2$, $\deg_T(x_i) = 2$, $\deg_T(y_{m-i+1}) = 1$ and edges $x_i y_{m-i+1} \in M$ for $1 \leq i \leq m$. If $\deg_T(w') = 2$ (or degree $\deg_T(w'') = 2$), then no path $P = xw'w_0$ such that degree $\deg_T(x) = 1$ (or no path $P = yw''w_0$ such that degree $\deg_T(y) = 1$).

(iii) T admits a bipartite strongly graceful labelling f such that there are integers $s, t \geq 0$, we have a case C_I: $f(y_i) = 2(s+i)$ and $f(x_i) = 2(t+i) - 1$, or another case C_{II}: $f(y_i) = 2(t+i) - 1$ and $f(x_i) = 2(s+i)$ for $1 \leq i \leq m$.

Lemma 8. (Growing lemma) *Let T be a (B)-tree and let $P_j = u_j v_{b-j+1}$ be paths for $1 \leq j \leq b$. A tree H obtained by adjoining w_0 with each vertex u_j is strongly graceful.*

Proof. Clearly, $\deg_H(z) = \deg_T(z)$ if $z \neq w_0, u_j, v_{b-j+1}$, $\deg_H(u_j) = 2$ and $\deg_H(u_{b-j+1}) = 1$ for $1 \leq j \leq b$. H contains a perfect matching $M' = M \cup \{u_j v_{b-j+1} : 1 \leq j \leq b\}$. Since the proofs about two cases C_I, C_{II} are very similar, we apply the case C_I to define a labelling h of H in the following.

(1) $h(y_i) = f(y_i) = 2(s+i)$ and $h(x_i) = 2(t+b+i) - 1$ for $1 \leq i \leq m$.

(2) $h(u_j) = 2(t+j) - 1$ and $h(v_j) = 2(s+m+j)$ for $1 \leq j \leq b$.

(3) For $z \in V(H) \setminus \{x_i, y_i, u_j, v_j : 1 \leq i \leq m,\ 1 \leq j \leq b\}$, let $h(z) = f(z)$ if $f(z) < f(y_1)$; $h(z) = f(z) + 2b$ if $f(z) > f(y_m)$ and $f(z)$ is even; and $h(z) = f(z) + 2b$ if $f(z) > f(x_m)$ and $f(z)$ is odd.

Let $n = |T|$. Note that $f(y_{m-i+1}) + f(x_i) = 2(s+m-i+1) + 2(t+i) - 1 = 2(s+m+t+1) - 1 = n - 1$. We have

$$h(y_{m-i+1}) + h(x_i) = 2(s+m-i+1) + 2(t+b+i) - 1 = (n+2b) - 1 = |H| - 1,$$

$$h(v_{b-j+1}) + h(u_j) = 2(s+m+b-j+1) + 2(t+j) - 1 = (n+2b) - 1 = |H| - 1,$$

as a result, $h(x) + h(y) = |H| - 1$ for $xy \in M'$.

Suppose that $|h(x)-h(y)| = |g(u)-h(v)|$ for some edges $xy, uv \in E(H)$. If $h(x) = f(x) + 2b$, $h(y) = f(y)$, $h(u) = f(u) + 2b$, $h(v) = f(v)$, by the definition of the labelling h we have $f(x) > f(y)$ and $f(u) > f(v)$, in turn that $|f(x) - f(y)| = |f(u) - f(v)|$, a contradiction. If $h(x) = f(x) + 2b$, $h(y) = f(y)$, $h(u) = f(u)$, $h(v) = f(v)$, it is impossible, we must have $h(u) = f(u)+2b$ or $h(v) = f(v)+2b$ since T is strongly graceful. Therefore, h is a strongly graceful labelling of H. □

3 Constructing strongly graceful trees

3.1 Basic seeds

Let T be a tree with n vertices and a perfect matching M. Since $\Delta(T) \leq n/2$ and each end-node of T has degree 2, we can make some seeds for growing strongly graceful trees.

1. Path seeds contains two classes of paths.

(1) P_{2m} is a path on $2m$ vertices depicted by $P_{2m} = u_1u_2\cdots u_{2m}$. Thus, $V(P_{2m}) = S \cup U$ where $S = \{u_{2i-1} : 1 \leq i \leq m\}$ and $U = \{u_{2i} : 1 \leq i \leq m\}$. The path P_{2m} contains a *perfect matching* $M = \{u_{2i-1}u_{2i} : 1 \leq i \leq m\}$, and has a strongly graceful labelling f defined in the way: $f(u_{2i-1}) = i - 1$ and $f(u_{2i}) = 2m - i$ for $1 \leq i \leq m$.

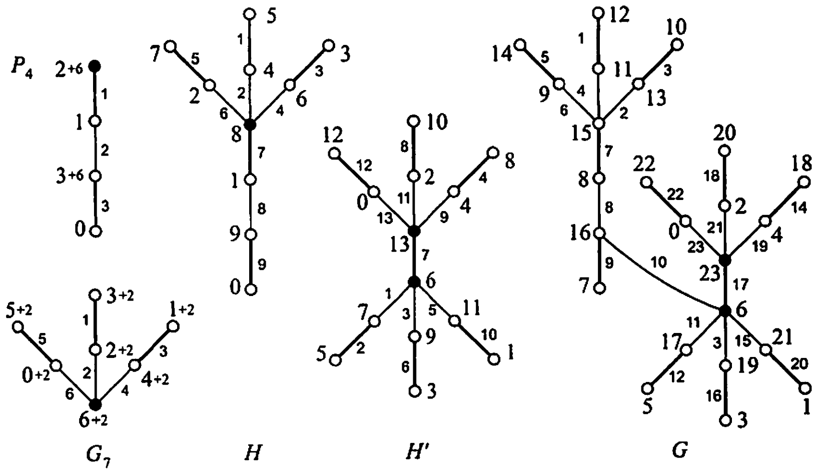
(2) P_{2m+1} stands for a path $P_{2m+1} = u_1u_2\cdots u_{2m}u_{2m+1}$ on $2m + 1$ vertices. It contains a *defect matching* $M = \{u_{2i-1}u_{2i} : 1 \leq i \leq m\}$, and admits a defect strongly graceful labelling h defined by setting $h(u_{2i-1}) = i - 1$ for $1 \leq i \leq m$ and $f(u_{2j}) = 2m + 1 - j$ for $1 \leq j \leq m + 1$.

2. Spider seeds are denoted as G_{2k+1} and G_{2k+2} ($k \geq 1$) that are defined in the following.

(1) G_{2k+2} is a tree of $2k+2$ vertices with $V(G_{2k+2}) = \{w, z, x_i, y_i : 1 \leq i \leq k\}$, $E(G_{2k+2}) = \{wz, wx_i, x_iy_i : 1 \leq i \leq k\}$, and a *perfect matching* $M = \{wz, x_iy_i : 1 \leq i \leq k\}$. Each y_i is a leaf and each x_i of degree 2 is an end-node. $D(G_{2k+2}) = 4$ and $\Delta(G_{2k+2}) = k + 1$. It admits a strongly graceful α defined in the way that $\alpha(w) = 2k + 1$ and $\alpha(z) = 0$; $\alpha(x_i) = 2i - 1$ and $\alpha(y_i) = 2k - 2(i - 1)$ for $1 \leq i \leq k$.

(2) G_{2k+1} is a tree of $2k + 1$ vertices with $V(G_{2k+1}) = \{w, x_i, y_i : 1 \leq i \leq k\}$, $E(G_{2k+1}) = \{wx_i, x_iy_i : 1 \leq i \leq k\}$, and a *defect matching* $M = \{x_iy_i : 1 \leq i \leq k\}$, where only w is not saturated by M. The vertex w is called the *center* of G_{2k+1}. Each y_i is a leaf of G_{2k+1}, and each x_i of degree 2 is an end-node of G_{2k+1}. $D(G_{2k+1}) = 4$ and $\Delta(G_{2k+1}) = k$. G_{2k+1} admits a graceful labelling α defined by $\alpha(w) = 0$, $\alpha(x_i) = 2k - 2(i - 1)$ and $\alpha(y_i) = 2i - 1$ for $1 \leq i \leq k$.

3. An *osg*-seed H_i is a tree which contains a perfect matching M and



(a) (b) (c) (d)

Figure 2: (a) A path seed P_4 and a spider seed G_7; (b) a strongly graceful lobster H grown from G_7 and P_4 by using the vertex-identified linking lemma (Lemma 5); (c) a graceful lobster H' constructed by applying the edge-symmetric linking lemma (Lemma 6) to G_7; (d) a graceful lobster G grown from H, H' by using the string linking lemma (Lemma 3).

$V(H_i) = U_i \cup V_i$ where the *small vertex set* $U_i = \{u_{i,1}, u_{i,2}, \dots, u_{i,s_i}\}$ and the *large vertex set* $V_i = \{v_{i,1}, v_{i,2}, \dots, v_{i,t_i}\}$ such that any edge xy satisfies $x \in U_i$ and $y \in V_i$. And, H_i admits a bipartite and strongly graceful labelling f such that $f(u_{i,l}) = l - 1$ for $1 \le l \le s_i$ and $f(v_{i,j}) = s_i + j - 1$ for $1 \le j \le t_i$. We say that $u_{i,1}$ and $v_{i,1}$ are the *S-head* and *L-head* of each *osg*-seed H_i, and u_{i,s_i} and v_{i,t_i} are the *S-tail* and *L-tail* of each *osg*-seed H_i, respectively.

3.2 The recurrent labelling

In this subsection we use a so-called *recurrent labelling* to yield a bipartite and strongly graceful labelling h for a so-called *super caterpillar* H obtained in the following way.

Let H_i be *osg*-seeds for $1 \le i \le m$, thus, we have a tree H obtained by adjoining every S-head $u_{i,1} \in V(H_i)$ with the L-head $v_{i-1,1} \in V(H_{i-1})$ by an edge for $2 \le i \le m$, here, we call H a *super caterpillar*.

Theorem 9. *For osg-seeds H_i for $1 \le i \le m$, a super caterpillar H resulted by H_i is bipartite and strongly graceful.*

Proof. First, we define the recurrent labelling h for H in the following:

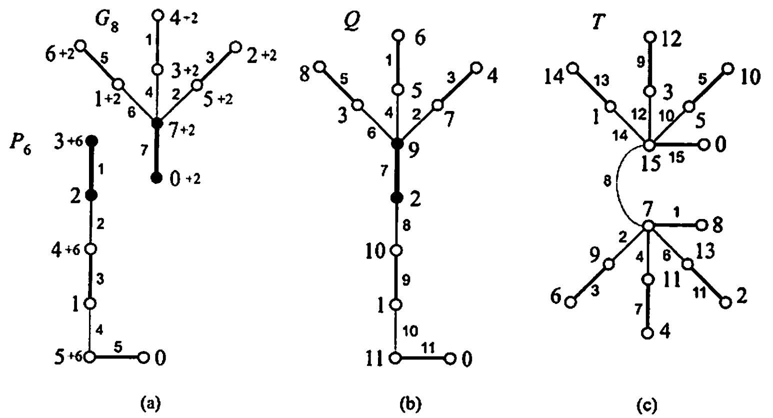


Figure 3: (a) A spider seed G_8 and a path seed P_6; (b) a strongly graceful lobster Q grown from seeds G_8 and P_6 by using the edge-identified linking lemma (Lemma 5); (c) a bipartite and strongly graceful lobster T constructed by applying the edge-symmetric linking lemma (Lemma 6) to a strongly graceful tree G_8 that is not bipartite.

$h(u_{i,r}) = (r-1) + \sum_{l=1}^{i-1} s_l$, $1 \le r \le s_i$, $1 \le i \le m$, where let $\sum_{l=1}^{0} s_l = 0$; and

$h(v_{k,j}) = (j-1) + \sum_{l=1}^{m} s_l + \sum_{l=k+1}^{m} t_l$, $1 \le j \le t_k$, $1 \le k \le m$.

Let $M_i = \sum_{l=1}^{i-1} s_l$ and $N_i = \sum_{l=1}^{m} s_l + \sum_{l=i+1}^{m} t_l$ for $2 \le i \le m$, we have

$h(V(H_i)) = \{M_i, M_i+1, \dots, M_i+(s_i-1), N_i, N_i+1, \dots, N_i+(t_i-1)\}$;

$h(E(H_i)) = \left[\sum_{l=i+1}^{m} s_l + \sum_{l=i+1}^{m} t_l + 1, \sum_{l=i}^{m} s_l + \sum_{l=i}^{m} t_l - 1\right]$; and

$h(v_{i,1}) - h(u_{i+1,1}) = \sum_{l=1}^{m} s_l + \sum_{l=i+1}^{m} t_l - \sum_{l=1}^{i} s_l = \sum_{l=i+1}^{m} s_l + \sum_{l=i+1}^{m} t_l$.

For the first *osg*-seed H_1, we have

$h(V(H_1)) = \{0, 1, \dots, s_1-1, N_1, N_1+1, \dots, N_1+(t_1-1)\}$, where $N_1 = \sum_{l=1}^{m} s_l + \sum_{l=2}^{m} t_l$,

$h(E(H_1)) = [\sum_{l=2}^{m} s_l + \sum_{l=2}^{m} t_l + 1, \sum_{l=1}^{m} s_l + \sum_{l=1}^{m} t_l - 1]$, and

$h(v_{1,1}) - h(u_{2,1}) = N_1 - \sum_{l=1}^{1} s_l = \sum_{l=2}^{m} s_l + \sum_{l=2}^{m} t_l$.

It is not difficult to see that $h(x) \neq h(y)$ as if $x \neq y$ for vertices $x, y \in V(H)$. Clearly, $h(V(H)) = [0, \sum_{l=1}^{m} s_l + \sum_{l=1}^{m} t_l - 1]$ and $h(E(H)) = [1, \sum_{l=1}^{m} s_l + \sum_{l=1}^{m} t_l - 1]$.

In fact, we label vertices in small vertex sets U_i from H_1 to H_m by the order from S-head to S-tail in one *osg*-seed tree H_i; and after labelling vertices in all small vertex sets, we then label vertices in large vertex sets V_i from H_m to H_1 by the order from L-head to L-tail in one *osg*-seed H_i. This is the reason that the above labelling h is named as the recurrent labelling.

The trueness of the bipartite and strongly graceful tree H introduced above is based on Lemma 3. It is not hard to see that we can adjoin $u_{i,j} \in V(H_i)$ with $v_{i-1,j} \in V(H_{i-1})$ (since $h(v_{i-1,j}) - h(u_{i,j}) = h(v_{i-1,1}) - h(u_{i,1})$) and delete the edge $v_{i-1,1}u_{i,1}$ in H, as a result, the resulting super caterpillar may be not equal to H.

Based on the above recurrent labelling, the proof of this theorem is completed. □

We consider a particular case in the proof of Theorem 9, that is, let $s_i = t_i = s \geq 1$ for each *osg*-seed H_i, $1 \leq i \leq m$. Immediately, we have

$h(u_{i,j}) = (i-1)s + (j-1),\ 1 \leq j \leq s,\ 1 \leq i \leq m$; and

$h(v_{i,j}) = (2m-i)s + (j-1),\ 1 \leq j \leq s,\ 1 \leq i \leq m.$

For this particular case, we have two approaches to form graceful trees in the following. The first way is to adjoin a S-head $u_{i,1} \in V(H_i)$ with a L-head $v_{i-1,1} \in V(H_{i-1})$ by an edge, $2 \leq i \leq m$, so that we get a super caterpillar H that is bipartite and strongly graceful. The second way is to construct a so-called *super spider* H' that is defect strongly graceful by adding a new vertex w with labels $2ms$ such that w is adjacent to S-heads $u_{2t-1,1} \in V(H_{2t-1})$ for $1 \leq t \leq \lfloor \frac{m+1}{2} \rfloor$, and to L-heads $v_{2r,1} \in V(H_{2r})$ for $1 \leq r \leq \lfloor \frac{m}{2} \rfloor$ (see an example described in Figure 4). This is the proof of the following Theorem 10:

Theorem 10. (Vertex-symmetric linking lemma) *Let H_i $(1 \leq i \leq m)$ be osg-seeds and let $s \geq 1$ be a fixed integer. If $s_i = t_i = s$ for each H_i, then there is a super spider H obtained by adding a vertex w and adjoin w with a certain vertex of H_i $(1 \leq i \leq m)$ such that H is defect strongly graceful.*

In Figure 4, each bipartite and strongly graceful tree T_i can be constructed by spider seeds and path seeds, where $V(T_i) = U_i \cup V_i$, the vertices in U_i are in black, and $|U_i| = |V_i| = 8$ $(1 \leq i \leq 3)$. There are two methods for constructing graceful trees from T_1, T_2, T_3. One is to adjoin vertex 40 with vertex 8, and vertex 32 with vertex 16 (in fact, this way may product many bipartite and strongly graceful trees); and another one is to adjoin a new vertex w (labelled by 48) with vertices 0, 32 and 16, and then it yields a defect strongly graceful tree.

By checking the proof of Lemma 8, we have a generalization of Lemma 8. Before stating this result, we need a particupar class of trees defined as follows.

A *super* (B)-*tree* is a tree T that satisfies the following conditions:

(i) T contains a perfect matching M.

(ii) T contains a vertex w_0 which is adjacent to w', w'', where degrees $\deg_T(w') \neq 2$, $\deg_T(w'') \neq 2$. And T has *osg*-seeds

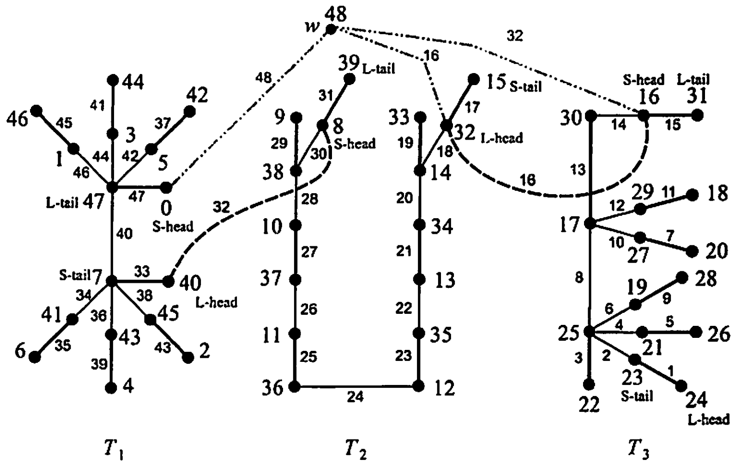


Figure 4: Two strongly graceful trees made by the recurrent labelling.

H_i with $s_i = t_i = s$ for $1 \leq i \leq m$. w_0 is adjacent to each S-heads $u_{2t-1,1} \in V(H_{2t-1})$ for $1 \leq t \leq \lfloor \frac{m+1}{2} \rfloor$, and to each L-heads $v_{2r,1} \in V(H_{2r})$ for $1 \leq r \leq \lfloor \frac{m}{2} \rfloor$.

(iii) T admits a bipartite strongly graceful labelling f such that there are integers $\alpha, \beta \geq 0$, we have $f(u_{i,j}) = 2(\alpha + (i-1)s + (j-1))$, $1 \leq j \leq s$, $1 \leq i \leq m$ and $f(v_{i,j}) = 2(\beta + (2m-i)s + (j-1)) - 1$, $1 \leq j \leq s$, $1 \leq i \leq m$.

Theorem 11. (Super growing lemma) *Let T be a super (B)-tree and let T_j be osg-seeds with $s_j = t_j = s$ for $1 \leq j \leq b$. We have a bipartite strongly graceful tree H obtained by adjoining w_0 of T to each S-heads $u_{2t-1,1} \in V(T_{2t-1})$ for $1 \leq t \leq \lfloor \frac{b+1}{2} \rfloor$, and to each L-heads $v_{2r,1} \in V(T_{2r})$ for $1 \leq r \leq \lfloor \frac{b}{2} \rfloor$.*

4 Further works

Clearly, the results, here, are suitable for constructing graceful trees or bipartite graceful trees. Most of proofs of the results in this note are constructive, so that they can be transferred into algorithms for computation. We define a k-strongly graceful labelling of a tree T with a perfect matching M and n vertices in the following.

Definition 4. A tree T with n vertices and a perfect matching M is k-strongly graceful if it admits a graceful labelling f such that $f(u) + f(v) \geq$

$n-k$ for each edge $uv \in M$ and $1 \leq k \leq n-1$, also, f is called a k-strongly graceful labelling of T.

Therefore, Conjecture 2 can be restated as this: Any tree with a perfect matching is 1-strongly graceful. Obviously, the k-strongly gracefulness is weaker than the strongly gracefulness as if $k \geq 2$. However, determining the k-strongly gracefulness of a tree seems to be very hard since GTC is still open up to now.

Problem 1. A tree T with n vertices and a perfect matching M admits a strongly graceful labelling f. For each edge $uv \in E(T) \setminus M$, does both integers $f(u) + f(v)$ and n have the same odevity?

According to GTC our experience shows a conjecture in the following:

Conjecture 3. *Any tree T with n vertices contains a certain largest matching M and a graceful labelling f such that $f(u) + f(v) = n - 1$ for each edge $uv \in M$.*

Note that a tree may have largest matchings more than one, so the Conjecture 3 is not true for any largest matching of T.

Problem 2. Let T be a strongly graceful tree on $n \geq 4$ vertices and let H be a subtree of T that has a perfect matching. Is H strongly graceful?

Lemma 12. (Self-transmuted lemma) *Let T be a (strongly) graceful tree on $n \geq 4$ vertices, and let f be a graceful labelling of T. Then there are an edge uv and a vertex w such that $H = T - uv + uw$ or $H' = T - uv + vw$ is a graceful tree on $n \geq 4$ vertices.*

Proof. In fact, it is sufficient to prove that there is vertex w such that $|h(u) - h(v)| = |h(u) - h(w)|$ or $|h(u) - h(v)| = |h(w) - h(v)|$. We, without loss of generalization, may assume that $|h(u) - h(v)| = 1$. If $h(u) < n - 1$, we then take a vertex w with $h(w) = h(u) + 1$. If $h(v) > 1$, we have a vertex w that holds $h(w) = h(v) - 1$. If $h(u) = n - 1$ and $h(v) = 1$, thus, $1 = |h(u) - h(v)| = (n - 1) - 1$, it turns out $n = 3$, a contradiction.

We consider the tree T on n vertices is strongly graceful, so it contains a perfect matching M that means n is even. For the sake of convenience, let $n = 2m$. Similarly, we take an edge uv such that $|h(u) - h(v)| = 1$, where $h(u) > h(v)$. If $uv \notin M$, we are done by the above proof. As for $uv \in M$, we have $h(u) + h(v) = 2m - 1$, and furthermore $2h(v) = 2m - 2$ from $|h(u) - h(v)| = 1$, and $h(u) = m$. We may assume that u is adjacent to u' in T. If $h(u') < h(u) = m$, we take a vertex w which holds $h(w) = m + h(u) - h(u')$. Hence, $H = T - uu' + uw$ or $H' = T - uu' + u'w$ is a strongly graceful tree by which component of $T - uu'$ the vertex w belongs to. If $h(u') > h(u) = m$, we take a vertex w which holds $h(w) = m - (h(u) - h(u'))$, and furthermore we have a strongly graceful tree $H = T - uu' + uw$ or $H' = T - uu' + u'w$. □

Suppose that $H = T - uv + uw$ is obtained from a graceful tree T, where $w \in V(T)$ and $uv \in E(T)$. H is a *graceful* 1*-neighbour* of T if H is graceful. We say that H is a *graceful k-neighbour* of T if H is obtained by applying Lemma 12 k times.

Problem 3. Let T, H be graceful trees with the same number of vertices. Is H a graceful k-neighbour of T for some integer $k \geq 1$?

Problem 4. Let $\mathcal{F}_N = \{T : T \text{ is a graceful tree of order } n \leq N\}$. Can any tree $T \in \mathcal{F}_N$ be obtained by several fixed classes of trees of $\mathcal{F}_N$ and some fixed operations of constructing graphs?

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