Properties of Lucas Trees

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ABSTRACT

In this paper we consider a class of recursively defined, full binary trees called Lucas trees and investigates their basic properties. In particular, the distribution of leaves in the trees will be carefully studied. We then go on to show that these trees are 2-splittable, i.e. they can be partitioned into two isomorphic subgraphs. Finally we investigate the total path length and external path length in these trees, the Fibonacci trees, and other full m-ary trees.

KEYWORDS

Fibonacci Trees, Lucas Trees, splittable, isomorphic factorization, path length, full binary trees, m-ary trees.

0. INTRODUCTION

The first Fibonacci "tree" was pictured in Steinhaus' *Mathematical Snapshots* in 1938 [7, p. 28]. Since then, at least two distinct trees (graphs) have come to be called "Fibonacci Trees." Both the Fibonacci trees of Donald Knuth [6, pp. 414-15] and those of Atkins and Geist [1, pp. 334-35] have been used in computer science for efficient searching algorithms. Those of Knuth are the same as those in [4], [5], and [8] where they have been studied as purely combinatorial/graph theoretical objects. It is this second type of study that generated our interest in Fibonacci trees and in trying to develop a similar structure based on the sequence of Lucas numbers.

Initially, the paper of Knisely, Wallis & Domke [5] in which they show that Fibonacci trees can be split (their edge sets can be partitioned into two sets so that the induced subtrees are isomorphic) led us to develop the definition of a Lucas tree in Section 1 below and show that such trees are also splittable [Section 4]. Subsequently, the paper by Ralph Grimaldi [4] in which he investigates many combinatorial properties of Fibonacci trees caused us to look at similar properties for our Lucas trees and that study constitutes the bulk of this paper. Most of the results are very similar with the Lucas numbers substituting for the Fibonacci numbers in the results. When we begin investigating the distribution of internal nodes [Section 3], however, the results begin to look quite different.

The similarity of the results on total path length and external path length in Fibonacci and Lucas trees [Section 5] led us to a similar general result for all full (complete) binary trees [Section 6] and even more generally for full m-ary trees [Section 7].

1. DEFINITIONS AND BASIC PROPERTIES

The Lucas trees (LT_n) are defined in a manner similar to the Fibonacci trees (FT_n). LT_0 is the binary tree with root, one left child, and one right child; it is the same as FT_3. LT_1 is the tree consisting of a single vertex, the root. Then for $n \geq 2$, LT_n is the rooted, full binary tree with LT_{n-1} as its left subtree and LT_{n-2} as its right subtree. The first six Lucas trees are shown in *Figure 1.*

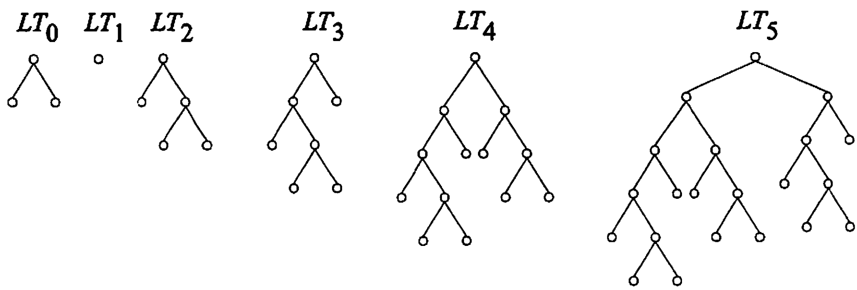


Figure 1

We shall use the following notation for the basic features of the Lucas trees:

In Lucas tree LT_n, where n is any non-negative integer,

$v_n =$ the number of vertices (nodes),

$\ell_n =$ the number of leaves,

$i_n =$ the number of internal (non-leaf) nodes,

$e_n =$ the number of edges,

$h_n =$ the height of the tree.

Many of the properties of Lucas trees relate to the Lucas numbers and we will often encounter variations of the recurrence relation used to define this sequence of numbers. We take a moment now to note some of the ideas related to the Lucas numbers.

The Lucas numbers are similar to the better known Fibonacci numbers. They are defined using the same recurrence relation but different initial conditions. $L_0 = 2$, $L_1 = 1$, and for all $n \geq 2$, $L_n = L_{n-1} + L_{n-2}$. Thus the sequence looks like $2, 1, 3, 4, 7, 11, 18, 29, 47$, ... and shows up in discrete mathematics almost as often as the Fibonacci numbers (see [3]). It is well known that the general solution to the difference equation defining these sequences can be written as $A\alpha^n + B\beta^n$ where $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$. Imposing the initial conditions for the Lucas sequence we find that $A = B = 1$ so that we can write $L_n = \alpha^n + \beta^n$. (Note: for the Fibonacci sequence, $A = \frac{1}{\sqrt{5}}$ and $B = \frac{-1}{\sqrt{5}}$ so that $F_n = \frac{\alpha^n}{\sqrt{5}} - \frac{\beta^n}{\sqrt{5}}$.)

The basic properties of the Lucas trees are contained in our first theorem.

Theorem 1. For the Lucas tree LT_n, n any non-negative integer, the quantities defined above can be expressed as follows:

(*i*) $\ell_n = L_n$,

(*ii*) $h_0 = 1$, $h_1 = 0$, and for $n \geq 2$, $h_n = n$

(*iii*) $v_n = 2L_n - 1$,

(*iv*) $i_n = L_n - 1$,

(*v*) $e_n = 2L_n - 2$.

Proof. From the definition of these trees it is obvious that $\ell_n = \ell_{n-1} + \ell_{n-2}$ and $\ell_0 = 2 = L_0$, and $\ell_1 = 1 = L_1$. Thus we have (*i*). The height of one of these trees, from LT_3 on, will be one more than the height of the taller of its two subtrees, which will always be the previous Lucas tree. Thus $h_n = h_{n-1} + 1$, $h_2 = 2$ and we have (*ii*).

The number of vertices will be the sum of the number of vertices in the two preceding trees plus 1 for the new root, so to determine v_n we need to solve the recurrence $v_n = v_{n-1} + v_{n-2} + 1$. Following the standard techniques for solving such difference equations (see [3]) we need to find a solution ($v_n^{(h)}$) to the related homogeneous equation and find a particular solution ($v_n^{(p)}$) to the non-homogeneous equation. It is easy to see from inspection that a particular solution to the non-homogeneous equation is $v_n^{(p)} = -1$. The homogeneous equation is exactly that defining the Fibonacci and Lucas numbers so we know the solution must be given by $v_n^{(h)} = A\alpha^n + B\beta^n$ and $v_n = A\alpha^n + B\beta^n - 1$. The initial conditions give

$$v_0 = 3 = A + B - 1 \quad \text{and}$$
$$v_1 = 1 = A\alpha + B\beta - 1\,.$$

Solving these equations gives $A = B = 2$ and hence
$v_n = 2\alpha^n + 2\beta^n - 1 = 2L_n - 1$, result (*iii*) of the theorem.

Now $i_n = v_n - \ell_n = 2L_n - 1 - L_n = L_n - 1$, and part (*iv*) is done. Since the number of edges in any tree is one less than the number of vertices we also have (*v*) and the theorem is proved. □

These properties are completely analogous to those for Fibonacci trees [4, pp. 22-23]

2. DISTRIBUTION OF LEAVES IN LUCAS TREES

We next investigate how the leaves and internal (non-leaf) nodes are distributed at the various levels in the Lucas trees. To this end we define (after [4]) the following quantities.

Let n be any non-negative integer. Then in the Lucas tree LT_n,

$L(n,i)$ = the number of leaves at level i,
$LL(n,i)$ = the number of leaves at level i that are left children, and
$LR(n,i)$ = the number of leaves at level i that are right children.

Since the height of LT_n is n, there can be no leaves at level greater than n so we have, for $i > n$, $L(n,i) = LL(n,i) = LR(n,i) = 0$ and most of these 0's are not entered in the tables below. As the trees grow we will not find leaves at the lower levels and most of these 0's are also omitted from the tables. The exact level at which leaves begin to appear will be calculated shortly.

Because of the recursive definition of the Lucas trees, a leaf at level i in the n^{th} Lucas tree had to have been a leaf at level $i-1$ in one of the two previous Lucas trees and vice versa. Consequently we have the following recurrence relations for the above quantities.

In the Lucas tree LT_n, for $n \geq 2$ and $i \geq 2$,

$$L(n,i) = L(n-1,i-1) + L(n-2,i-1)\,, \tag{2.1}$$
$$LL(n,i) = LL(n-1,i-1) + LL(n-2,i-1)\,, \tag{2.2}$$
$$LR(n,i) = LR(n-1,i-1) + LR(n-2,i-1)\,. \tag{2.3}$$

Note: recurrence (2.1) also works for $i = 1$.

Some of the values for these quantities are shown in Tables 1 (LL), 2 (LR), and 3 (L). The first two rows and columns were entered by inspection and the rest of the tables generated from the above recurrence relations using an *Excel*™ spreadsheet.

We begin investigating LL and LR since if we can find nice expressions for these quantities, an expression for L will just be their sum. The corresponding tables for Fibonacci trees [4, p. 25] contain exactly the rows of

Pascal's Triangle, i.e. their values are exactly binomial coefficients. That is obviously not the case for Lucas trees.

$LL(n, i)$, the number of leaves at level i that are left children in LT_n

$i\backslash n$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	0	1	0													
1	1	0	1	0	0											
2	0	0	1	1	1	0										
3			0	1	2	2	1	0								
4				0	1	3	4	3	1	0						
5					0	1	4	7	7	4	1	0				
6						0	1	5	11	14	11	5	1	0		
7							0	1	6	16	25	25	16	6	1	0
8								0	1	7	22	41	50	41	22	7
9									0	1	8	29	63	91	91	63
10										0	1	9	37	92	154	182
11											0	1	10	46	129	246
12												0	1	11	56	175
13													0	1	12	67
14														0	1	13
15															0	1

Table 1

The rows in Table 1, for $LL(n, i)$, are at least symmetric and the following observations motivate our result.

$$LL(7,6) = 5 = 4 + 1 = \binom{4}{1} + \binom{5}{0},$$
$$LL(8,6) = 11 = 6 + 5 = \binom{4}{2} + \binom{5}{1},$$
$$LL(9,6) = 14 = 4 + 10 = \binom{4}{3} + \binom{5}{2},$$
$$LL(10,6) = 11 = 1 + 10 = \binom{4}{4} + \binom{5}{3},$$
$$LL(11,6) = 5 = 0 + 5 = = \binom{4}{5} + \binom{5}{4}.$$

This pattern is followed in other rows as well and it appears that $LL(n, i) = \binom{i-2}{n-i} + \binom{i-1}{n-i-1}$. This is easily verified by induction for $n \geq 1, i \geq 2$. If $n = 1$ and $i \geq 2$, then $1 - i$ and $1 - i - 1$ are both negative while $i - 2$ and $i - 1$ are both at least 0 making both binomial coefficients 0 and $LL(1, i) = 0$ for all $i \geq 2$. When $n = 2$, we have $LL(2, 2) = 1 = 1 + 0$ $= \binom{0}{0} + \binom{1}{-1} = \binom{2-2}{2-2} + \binom{2-1}{2-2-1}$. If $i > 2$, $LL(2, i) = \binom{i-2}{2-i} + \binom{i-1}{2-i-1} = 0$ since $2 - i$ and $2 - i - 1$ are negative making both binomial coefficients 0. Next consider the case for $i = 2$ and $n \geq 3$.

$LL(3,2) = 1 = 0 + 1 = \binom{0}{1} + \binom{1}{0} = \binom{2-2}{3-2} + \binom{2-1}{3-2-1}$ and
$LL(4,2) = 1 = 0 + 1 = \binom{0}{2} + \binom{1}{1} = \binom{2-2}{4-2} + \binom{2-1}{4-2-1}$. If $n > 4$ then $n - i = n - 2 > 2 = i - 2$ and $n - i - 1 = n - 2 - 1 > 1 = i - 1$ so both binomial coefficients are 0 making $LL(n,2) = 0$.

Now we assume the relation holds true for $2 \le m < n$ and $2 \le j < i$. From the recurrence relation (2.2) for LL we get

$$\begin{aligned} LL(n,i) &= LL(n-1,i-1) + LL(n-2,i-1) \\ &= \binom{i-3}{n-i} + \binom{i-2}{n-i-1} + \binom{i-3}{n-i-1} + \binom{i-2}{n-i-2} \\ &= \binom{i-3}{n-i} + \binom{i-3}{n-i-1} + \binom{i-2}{n-i-1} + \binom{i-2}{n-i-2} \\ &= \binom{i-2}{n-i} + \binom{i-1}{n-i-1}. \end{aligned}$$

We would also like to know at what levels we actually find leaves that are left children. $LL(n,i) \neq 0$ as long as either of the binomial coefficients in the above expression is nonzero, i.e. as long as $0 \le n - i \le i - 2$ or $0 \le n - i - 1 \le i - 1$. These reduce to $i \le n \le 2i$ and so $\lceil \frac{n}{2} \rceil \le i \le n$. There will be left-child leaves at the $n - \lceil \frac{n}{2} \rceil + 1$ consecutive levels $\lceil \frac{n}{2} \rceil$, $\lceil \frac{n}{2} \rceil$+1, ..., n

$LR(n,i)$, the number of leaves at level i that are right children in LT_n

$i \backslash n$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	0	0	0													
1	1	0	0	1	0											
2	0	0	1	0	1	1	0									
3			0	1	1	1	2	1	0							
4				0	1	2	2	3	3	1	0					
5					0	1	3	4	5	6	4	1	0			
6						0	1	4	7	9	11	10	5	1	0	
7							0	1	5	11	16	20	21	15	6	1
8								0	1	6	16	27	36	41	36	21
9									0	1	7	22	43	63	77	77
10										0	1	8	29	65	106	140
11											0	1	9	37	94	171
12												0	1	10	46	131
13													0	1	11	56
14														0	1	12
15															0	1

Table 2

The pattern in Table 2 is less obvious. If we consider the fifth row as an example, it reads $1, 3, 4, 5, 6, 4, 1$. From the left it starts out like the binomial coefficients for the third power and from the right it looks like those for the fourth power. If we write these two rows of binomial coefficients, shifted two places, above one another and add we get the row in this table.

1	3	3	1			
		1	4	6	4	1
1	3	4	5	6	4	1

A bit more checking shows that this pattern seems to hold for all rows in the table. Thus it appears that $LR(n,i) = \binom{i-2}{n-i} + \binom{i-1}{n-i-2}$ for $n \geq 1, i \geq 2$. Again the induction follows easily from the recurrence relation (2.3) for LR and the basic identity for binomial coefficients (with the messiest part being the verification of the base cases) and we omit the details.

Now $LR(n,i) \neq 0$ as long as either of the binomial coefficients is nonzero. This means $0 \leq n-i \leq i-2$ or $0 \leq n-i-2 \leq i-1$ which simplify to $i \leq n \leq 2i+1$. For a particular n this gives $\frac{n-1}{2} \leq i \leq n$ and we will have right-child leaves at the $n - \lceil \frac{n-1}{2} \rceil + 1$ consecutive levels from $\lceil \frac{n-1}{2} \rceil$ up to n.

The total numbers of leaves at each level are shown in Table 3. Since these values are just the sums of the two preceding values, we do not need to analyze them on their own.

$L(n,i)$, the number of leaves at level i in LT_n

$i \backslash n$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	0	1	0													
1	2	0	1	1	0											
2	0	0	2	1	2	1	0									
3			0	2	3	3	3	1	0							
4				0	2	5	6	6	4	1	0					
5					0	2	7	11	12	10	5	1	0			
6						0	2	9	18	23	22	15	6	1	0	
7							0	2	11	27	41	45	37	21	7	1
8								0	2	13	38	68	86	82	58	28
9									0	2	15	51	106	154	168	140
10										0	2	17	66	157	260	322
11											0	2	19	83	223	417
12												0	2	21	102	306
13													0	2	23	123
14														0	2	25
15															0	2

Table 3

We have

$$L(n,i) = LL(n,i) + LR(n,i) = 2\binom{i-2}{n-i} + \binom{i}{n-i-1} \text{ for } n \geq 1, i \geq 2.$$

The range of levels where we find right-child leaves includes that for left-child leaves, so the levels where any leaves occur is the same as that for LR. We summarize these results in the next theorem.

Theorem 2. For the Lucas tree LT_n, if $n \geq 1$ and $i \geq 2$, then there are

(*i*) $LL(n,i) = \binom{i-2}{n-i} + \binom{i-1}{n-i-1}$ leaves at level i, for $\lceil \frac{n}{2} \rceil \leq i \leq n$, that are left children,

(*ii*) $LR(n,i) = \binom{i-2}{n-i} + \binom{i-1}{n-i-2}$ leaves at level i, for $\lceil \frac{n-1}{2} \rceil \leq i \leq n$, that are right children,

(*iii*) $L(n,i) = 2\binom{i-2}{n-i} + \binom{i}{n-i-1}$ leaves at level i, for $\lceil \frac{n-1}{2} \rceil \leq i \leq n$.

If we consider the total number of leaves in LT_n that are left children or that are right children, we discover another interesting pattern. Summing the columns of Tables 1 and 2, beginning with $n = 2$, we get the sequences $2, 2, 4, 6, 10, 16, \ldots$ and $1, 2, 3, 5, 8, 13$. Letting $LL(n)$ and $LR(n)$ be the total number of leaves in LT_n that are left or right children, respectively, we have that $LL(n) = 2F_{n-1}$ and $LR(n) = F_n$ for $n \geq 2$. These results follow immediately from the observation that $LL(n)$ and $LR(n)$ obey the same recurrence as the Fibonacci and Lucas numbers. Combining these with the result from Theorem 1 that the total number of leaves in LT_n is the n^{th} Lucas number gives the familiar identity $L_n = F_n + 2F_{n-1}$.

Next we consider the total number of leaves appearing at level i over all Lucas trees for left-child leaves, right-child leaves, and all leaves. Thus, $TL(i) = \sum_{n=0}^{\infty} LL(n,i)$, $TR(i) = \sum_{n=0}^{\infty} LR(n,i)$, and $T(i) = \sum_{n=0}^{\infty} L(n,i)$. The values we observe in the tables suggest that $TL(i) = TR(i) = 3 \cdot 2^{i-2}$ and so $T(i) = 3 \cdot 2^{i-1}$. The fact that $TL(i) = TR(i)$ is obvious since the first terms in their formulas from Theorem 2, parts (*i*) and (*ii*), are exactly the same and the second terms contribute the same nonzero values to the total since the upper number of the binomial coefficient is the same and the bottom numbers range from some negative value to values beyond the top number. Thus, if we establish that $TL(i) = 3 \cdot 2^{i-2}$, we will have all the results. For $i \geq 2$,

$$TL(i+1) = \sum_{n=0}^{\infty} LL(n,i+1) = \sum_{n=3}^{\infty} LL(n,i+1)$$

$$= \sum_{n=3}^{\infty} \left[\binom{i-2}{n-i-1} + \binom{i-2}{n-i-2} + \binom{i-1}{n-i-2} + \binom{i-1}{n-i-3} \right],$$

by recurrence 2.2 and Theorem 2 (*i*),

$$= \sum_{n=3}^{\infty} \left[\binom{i-2}{n-i-1} + \binom{i-1}{n-i-2} \right] + \sum_{n=3}^{\infty} \left[\binom{i-2}{n-i-2} + \binom{i-1}{n-i-3} \right]$$

$$= \sum_{n=2}^{\infty} \left[\binom{i-2}{n-i} + \binom{i-1}{n-i-1} \right] + \sum_{n=1}^{\infty} \left[\binom{i-2}{n-i} + \binom{i-1}{n-i-1} \right]$$

$$= TL(i) + TL(i) = 2\,TL(i) \quad \text{for } i \geq 2.$$

Therefore $TL(i) = c\,2^i$ for some constant c. $TL(2) = 3 = c\,2^2$ so $c = \frac{3}{4}$ and $TL(i) = 3 \cdot 2^{i-2}$ as claimed. We summarize these results on the total numbers of leaves in the following theorem.

Theorem 3. (a) For the Lucas tree LT_n, $n \geq 2$, the total number of

(*i*) left-child leaves is $LL(n) = 2\,F_{n-1}$,

(*ii*) right-child leaves is $LR(n) = F_n$,

(*iii*) leaves is $\ell_n = F_n + 2\,F_{n-1} = L_n$.

(b) Summing over all Lucas trees, for $i \geq 2$, the total number of

(*i*) left- or right-child leaves at level i is $TL(i) = TR(i) = 3 \cdot 2^{i-2}$

(*ii*) leaves at level i is $T(i) = 3 \cdot 2^{i-1}$.

The recurrence relations (2.1-2.3) for $LL(n,i)$, $LR(n,i)$, and $L(n,i)$ presented at the beginning of this section indicate that each element in any of these tables is the sum of two consecutive elements from the row above. This also means that each element in a row contributes the same amount to the value of two neighboring elements in the row below. Thus, if we add up every other element in a row, the sum we get will be exactly the sum of all the elements in the row above. Since the recurrences only hold for $n \geq 2$, the result of this paragraph only works for rows 3 and beyond.

Theorem 4. For the Lucas Trees LT_n, if $i \geq 3$, then

(*i*) $$\sum_{n\,\text{even}} LL(n,i) = \sum_{n\,\text{odd}} LL(n,i) = \sum_{\text{all}\,n} LL(n,i-1)$$

(ii) $\sum_{n\text{ even}} LR(n,i) = \sum_{n\text{ odd}} LR(n,i) = \sum_{\text{all } n} LR(n,i-1)$

(iii) $\sum_{n\text{ even}} L(n,i) = \sum_{n\text{ odd}} L(n,i) = \sum_{\text{all } n} L(n,i-1)$

These recurrences also allow us to express the entries in the tables as a linear combination of entries in each of the rows above with coefficients being the binomial coefficients in several ways. For example,

$$
\begin{aligned}
L(12,9) &= 106 \\
&= 38 + 68 \\
&= (11+27) + (27+41) = 11 + 2(27) + 41 \\
&= (2+9) + 2(9+18) + (18+23) = 2 + 3(9) + 3(18) + 23 \\
&\vdots \\
&= \tbinom{7}{7}0 + \tbinom{7}{6}0 + \tbinom{7}{5}0 + \tbinom{7}{4}0 + \tbinom{7}{3}2 + \tbinom{7}{2}1 + \tbinom{7}{1}2 + \tbinom{7}{0}1.
\end{aligned}
$$

Since the recurrences do not apply to the element in the $(1,2)$ position, we cannot extend back above the second row of the tables. The proof of these results is an easy (finite) induction and they are stated in general in the next theorem (cf. [4, p. 27, (3)]).

Theorem 5. For the Lucas Tree LT_n,

(i) $LL(n,i) = \sum_{k=0}^{t} \binom{t}{k} LL(n-t-k, i-t)$ for $0 \le t \le i-2$;

(ii) $LR(n,i) = \sum_{k=0}^{t} \binom{t}{k} LR(n-t-k, i-t)$ for $0 \le t \le i-2$;

(iii) $L(n,i) = \sum_{k=0}^{t} \binom{t}{k} L(n-t-k, i-t)$ for $0 \le t \le i-2$.

The numbers along the diagonals from upper left to lower right, beginning with row $i = 2$, exhibit some nice patterns as well, though they are not immediately obvious. One thing that is obvious is that for $n \ge 2$ $LL(n,n) = LR(n,n) = 1$ and this is easily verified. The next diagonal up from this, for the left and right children, contains the non-negative integers in order but shifted one or two places. In particular $LL(n+1,n) = n-1$ and $LR(n+1,n) = n-2$. The entries along the next diagonal are the same for both LL and LR but shifted from one table to the next. The entries themselves, $1, 2, 4, 7, 11, 16, \ldots$, are seen to be 1 more than the triangular numbers, $0, 1, 3, 6, 10, 15, \ldots$, which are exactly the binomial coefficients $\binom{n}{2}$. This gives

$LL(n+2,n) = 1 + \binom{n-1}{2}$, $LR(n+2,n) = 1 + \binom{n-2}{2}$, and so $L(n+2,n) = 2 + \binom{n-1}{2} + \binom{n-2}{2}$. These can, again, be verified by an easy induction.

From here on, the numbers appearing in the tables follow unfamiliar patterns, but we can write and solve recurrences that will enable us to write down expressions for the elements. Our basic recurrence (2.2) tells us $LL(n+3,n) = LL(n+2,n-1) + LL(n+1,n-1)$. This looks nicer if we shift the index up one and write it as $LL(n+4,n+1) - LL(n+3,n) = LL(n+2,n)$. This is now $\Delta\Big(LL(n+3,n)\Big) = \frac{n^2-3n+2}{2}$ by Theorem 2. While we could solve this non-homogeneous difference equation directly, it is easier to solve if we notice that the right hand side can be expressed as a sum of factor polynomials $n^{(k)}$. $\frac{n^2-3n+2}{2} = \frac{1}{2}n^{(2)} - n^{(1)} + 2$, where $n^{(k)} = n(n-1)(n-2)\cdots(n-k+1)$. (See [2, p. 52].) This difference equation can then be solved by summation (the discrete analog of integration) and gives $LL(n+3,n) = \frac{1}{6}n^{(3)} - \frac{1}{2}n^{(2)} + 2n^{(1)} + c$ for some arbitrary constant c. The initial value forces $c = -3$. A similar process applied to the right child leaves gives $LR(n+3,n) = \frac{1}{6}n^{(3)} - n^{(2)} + 4n^{(1)} - 5$ and adding these two gives $L(n+3,n) = \frac{1}{3}n^{(3)} - \frac{3}{2}n^{(2)} + 6n^{(1)} - 8$. The diagonals beyond this can be dealt with in a similar fashion but the patterns in the coefficients become more cumbersome. The expressions we are getting for the factor polynomials are, in fact, exactly binomial coefficients. For example, $\frac{1}{6}n^{(3)} = \binom{n}{3}$ and $\frac{1}{2}n^{(2)} = \binom{n}{2}$. If we go back and write the expressions for the previous diagonal elements in these same terms as well as the results for the next diagonal, we can see a very nice general pattern appear.

(*i*) $LL(n,n) = LR(n,n) = 1 = \binom{n}{0}$;

(*ii*) $LL(n+1,n) = n-1 = \binom{n}{1} - \binom{n}{0}$,

$LR(n+1,n) = n-2 = \binom{n}{1} - 2\binom{n}{0}$;

(*iii*) $LL(n+2,n) = 1 + \binom{n-1}{2} = \binom{n}{2} - \binom{n}{1} + 2\binom{n}{0}$,

$LR(n+2,n) = 1 + \binom{n-2}{2} = \binom{n}{2} - 2\binom{n}{1} + 4\binom{n}{0}$;

(*iv*) $LL(n+3,n) = \binom{n}{3} - \binom{n}{2} + 2\binom{n}{1} - 3\binom{n}{0}$,

$LR(n+3,n) = \binom{n}{3} - 2\binom{n}{2} + 4\binom{n}{1} - 5\binom{n}{0}$;

(*v*) $LL(n+4,n) = \binom{n}{4} - \binom{n}{3} + 2\binom{n}{2} - 3\binom{n}{1} + 4\binom{n}{0}$,

$LR(n+4,n) = \binom{n}{4} - 2\binom{n}{3} + 4\binom{n}{2} - 5\binom{n}{1} + 6\binom{n}{0}$.

This pattern does continue throughout the tables which is our next result.

Theorem 6. For the Lucas Trees LT_n with $n \geq 2$ and $k \geq 0$

$$(i)\ LL(n+k,n) = \binom{n}{k} - \binom{n}{k-1} + \sum_{j=2}^{k}(-1)^j j \binom{n}{k-j},$$

$$(ii)\ LR(n+k,n) = \binom{n}{k} - 2\binom{n}{k-1} + \sum_{j=2}^{k}(-1)^j (j+2) \binom{n}{k-j},$$

$$(iii)\ L(n+k,n) = 2\binom{n}{k} - 3\binom{n}{k-1} + \sum_{j=2}^{k}(-1)^j (2j+2) \binom{n}{k-j}$$

$$= \binom{n}{k-1} + 2\sum_{j=0}^{k}(-1)^j (j+1) \binom{n}{k-j}.$$

Proof. We prove *(i)* by induction on n. The proof of *(ii)* is similar and *(iii)* follows by addition. When $n = 2$, we see from Table 1 that $LL(2+k,2) = \begin{cases} 1 & \text{if } k = 0,1,2 \\ 0 & \text{if } k \geq 3 \end{cases}$. The right hand side of *(i)* reduces to $\binom{2}{0} = 1$ for $k = 0$, and $\binom{2}{1} - \binom{2}{0} = 1$ for $k = 1$. When $k = 2$ we get $\binom{2}{2} - \binom{2}{1} + 2\binom{2}{0} = 1$ and when $k = 3$, $\binom{2}{3} - \binom{2}{2} + 2\binom{2}{1} - 3\binom{2}{0} = 0$. For all values of $k \geq 4$, the first two binomial coefficients are 0 as are all the terms in the sum except for the last three, when the lower numbers are 2, 1, and 0. Thus the right hand side reduces to

$$(-1)^k k\binom{2}{0} + (-1)^{k-1}(k-1)\binom{2}{1} + (-1)^{k-2}(k-2)\binom{2}{2}$$

$$= (-1)^{k-2}[k - (k-1)\,2 + (k-2)] = 0.$$

Now assuming the equation holds true for n and all k we get

$$LL(n+1+k,n+1) = LL(n+k,n) + LL(n+k-1,n)$$

$$= \binom{n}{k} - \binom{n}{k-1} + \sum_{j=2}^{k}(-1)^j j \binom{n}{k-j} + \binom{n}{k-1} - \binom{n}{k-2} + \sum_{j=2}^{k-1}(-1)^j j \binom{n}{k-1-j}$$

$$= \binom{n+1}{k} - \binom{n+1}{k-1} + \sum_{j=2}^{k-1}(-1)^j j \binom{n+1}{k-j} + (-1)^k k \binom{n}{0}.$$

But $\binom{n}{0} = \binom{n+1}{0}$, so this last term can be incorporated into the sum giving the desired result. The case in the induction step for $k = 0$ gives $LL(n+1, n+1)$, which is 1 and the right hand side reduces to $\binom{n+1}{0} = 1$ and the proof is complete. □

For our final result on the distribution of leaves we note that these same diagonal elements in Table 3 can also be expressed nicely as sums of squares (though not as nicely as those for Fibonacci trees in [4, pp. 27-28]). Looking first at the diagonal elements $L(n+4,n)$ we find

$$L(7,3) = 1 = 1^2,$$
$$L(8,4) = 4 = 0(1^2) + 2^2,$$
$$L(9,5) = 10 = 1(1^2) + 0(2^2) + 3^2,$$
$$L(10,6) = 22 = 2(1^2) + 1(2^2) + 0(3^2) + 4^2,$$
$$L(11,7) = 45 = 3(1^2) + 2(2^2) + 1(3^2) + 0(4^2) + 5^2.$$

Thus,

$$\begin{aligned} L(n+4,n) &= \sum_{i=1}^{n-3} (n-3-i)\, i^2 + (n-2)^2 \\ &= (n-3)\sum_{i=1}^{n-3} i^2 - \sum_{i=1}^{n-3} i^3 + (n-2)^2 \\ &= \tfrac{(n-2)(n^3-10n^2+45n-60)}{12} \\ &= 2\tbinom{n}{4} - 3\tbinom{n}{3} + 6\tbinom{n}{2} - 8\tbinom{n}{1} + 10\tbinom{n}{0}, \end{aligned}$$

the expression obtained in Theorem 6 for $L(n+4,n)$.

Investigating the next diagonal we find

$$L(9,4) = 1 = 1^2,$$
$$L(10,5) = 5 = 1^2 + 2^2,$$
$$L(11,6) = 15 = 2(1^2) + 2^2 + 3^2,$$
$$L(12,7) = 37 = 4(1^2) + 2(2^2) + 3^2 + 4^2,$$
$$L(13,8) = 82 = 7(1^2) + 4(2^2) + 2(3^2) + 4^2 + 5^2.$$

Except for the last square in each sum, the coefficients are exactly consecutive triangular numbers plus one. So $L(n+5,n) = \sum_{i=1}^{n-4} (t_{n-4-i}+1)\, i^2 + (n-3)^2$, where t_j is the j^{th} triangular number, and this is easily verified by evaluating this and the expression from Theorem 6 as polynomials in n. If we express the elements along the next diagonal as sums of squares we discover that the coefficients involve sums of triangular numbers. Letting $s_j = t_1 + \cdots + t_j$, the j^{th} tetrahedral number, we can write

$$L(n+6,n) = \sum_{i=1}^{n-5} (s_{n-5-i} + (n-4-i) + 1)\, i^2 + (n-4)^2.$$

To discover the overriding pattern, however, we notice that using the recursion for LL, each element along one of these diagonals is the sum of the elements in the previous diagonal that lie above the current line. This means that the coefficient of i^2 in the new diagonal will be the sum of the coefficients from the previous diagonal. Thus, in going from $LL(n+5,n)$ to $LL(n+6,n)$, adding up the t_{n-4-i} gives the s_{n-4-i}, adding up 1 gives $n-4-i$, and each of the final squares provides the $+1$. The coefficients in the next diagonal are a sum of s_j's (a fourth dimensional figurate number), a t_j

(from the $n-4-i$), $n-5-1$ (from the 1), and 1 (from $(n-4)^2$). In order to generalize, we need to realize that we do not need to invent new notation for triangular, tetrahedral, or fourth dimensional figurate numbers. They are all just binomial coefficients: $1=\binom{j-1}{0}$, $j=\binom{j}{1}$, $t_j=\binom{j+1}{2}$, $s_j=\binom{j+2}{3}$, $\sum_{i=1}^{j} s_i=\binom{j+3}{4}$, etc. We summarize all of this (without proof) in our final result of this section.

Theorem 7. For the Lucas Trees LT_n with $k\geq 4$ and $n\geq k-2$

$$L(n+k,n)=\sum_{i=1}^{n-k+1} C_i\,i^2+(n-k+2)^2 \quad\text{where}$$

$$C_i=\sum_{j=0}^{k-5}\binom{n-k+j-i+1}{j}+\binom{n-3-i}{k-3}.$$

3. DISTRIBUTION OF INTERNAL NODES IN LUCAS TREES

We now turn our attention briefly to the distribution of the internal nodes in the Lucas Trees following along the same lines as [4, pp. 30-32] and motivated by the patterns appearing in Table 4. In this table, $I(n,i)$ represents the number of internal nodes at level i in Lucas Tree LT_n and satisfies the recurrence relation $I(n,i)=I(n-1,i-1)+I(n-2,i-1)$. We state the results without proof and leave the reader to discover more relationships.

Theorem 8. For the Lucas Trees LT_n with $I(n,i)$ internal nodes at level i we have the following.

(*i*) Let E_n and O_n be the total number of internal nodes in LT_n appearing at even levels and odd levels respectively. Then:

$$E_n=O_n \text{ if } n\equiv 1,2 \bmod 3 \text{ and}$$

$$E_n=O_n+1 \text{ if } n\equiv 0 \bmod 3.$$

(*ii*) Considering the diagonals that run from upper left to lower right we have for $n\geq 1$:

$$I(n+k+1,n)$$

$$=\frac{n^{(k)}}{k!}+2\left[\frac{n^{(k-2)}}{(k-2)!}+\frac{n^{(k-4)}}{(k-4)!}+\frac{n^{(k-6)}}{(k-6)!}+\cdots+\begin{Bmatrix}2 \text{ if } k \text{ is even}\\ 2n \text{ if } k \text{ is odd}\end{Bmatrix}\right]$$

$$=\binom{n}{k}+2\left[\binom{n}{k-2}+\binom{n}{k-4}+\cdots+\begin{Bmatrix}1 \text{ if } k \text{ is even}\\ n \text{ if } k \text{ is odd}\end{Bmatrix}\right]$$

$I(n, i)$, the number of internal nodes at level i in LT_n

$i \backslash n$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
0	1	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
1	0	0	1	1	2	2	2	2	2	2	2	2	2	2	2	2	2	2
2		0	0	1	2	3	4	4	4	4	4	4	4	4	4	4	4	4
3			0	0	1	3	5	7	8	8	8	8	8	8	8	8	8	8
4				0	0	1	4	8	12	15	16	16	16	16	16	16	16	16
5					0	0	1	5	12	20	27	31	32	32	32	32	32	32
6						0	0	1	6	17	32	47	58	63	64	64	64	64
7							0	0	1	7	23	49	79	105	121	127	128	128
8								0	0	1	8	30	72	128	184	226	248	255
9									0	0	1	9	38	102	200	312	410	474
10										0	0	1	10	47	140	302	512	722
11											0	0	1	11	57	187	442	814
12												0	0	1	12	68	244	629

Table 4

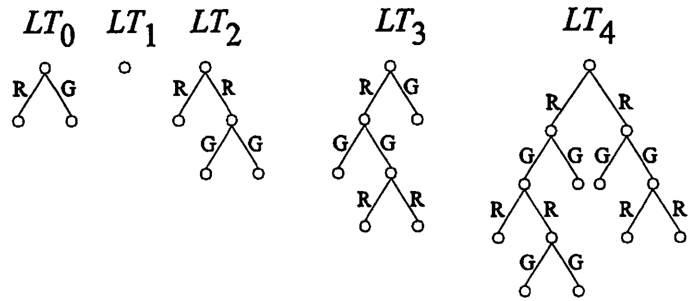
4. LUCAS TREES ARE 2-SPLITTABLE

We say that a graph is **2-splittable** (or just **splittable**) if it can be partitioned into two isomorphic subgraphs. Earlier terminology for this concept included determining **isomorphic factorizations** of a graph and called the graph **bisectable.** This is commonly described in terms of coloring the edges of the graph red and green so that the red subgraph is isomorphic to the green subgraph.

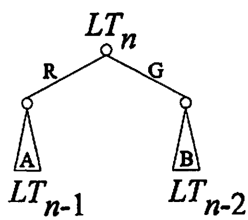
It has already been shown [5] that the Fibonacci trees can be 2-split. The Lucas trees also have this property as shown in the following theorem. This theorem is actually a special case of a more general result on recursively generated trees [8], but the proof here, restricted to just Lucas trees, is more direct.

Theorem 9. Let LT_n be the n^{th} Lucas tree. Then LT_n is 2-splittable and if $n \equiv 1$ or $2 \bmod 3$, then the coloring can be performed in such a way that the edges incident with the root are colored the same.

Proof. The base cases are easily handled and are shown below. For LT_0 the isomorphic subgraphs are each a P_2, a path on 2 vertices. For LT_1 each is just the single vertex and all edges incident with the root are colored the same. For LT_2 each subgraph is a P_3, a path on 3 vertices, and the two root edges are colored the same. In LT_3 the subgraphs consist of a P_2 and a P_3 and since $3 \not\equiv 1$ or $2 \bmod 3$ we do not need to worry about the coloring of the root edges. (The splitting of LT_4 is also shown but it is not needed for the induction step.)

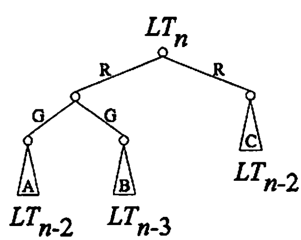


For the induction step we assume that the result is true for all LT_n with $n \leq 3$ and proceed with two cases. First consider $n \equiv 0 \bmod 3$, which is shown schematically below. Since $n-1 \equiv 2 \bmod 3$ and $n-2 \equiv 1 \bmod 3$, each of the subtrees A and B can be split with the root edges colored the same. Split A with both root edges colored green and B with both root edges colored red. Color the root edges of the main tree red and green as shown below.



We have now partitioned the entire tree into a red P_2 and a green P_2, the red subgraph of A, which is isomorphic to the green subgraph of A, and the isomorphic red and green subgraphs of B. The P_2's at the top do not interfere with the subgraphs of A and B because of the coloring of the root edges in those two subtrees.

Next consider the case of $n \equiv 1$ or $2 \bmod 3$. In this case we redraw LT_{n-1} in terms of its subtrees as shown below.



Color the entire subtree C red and all of the subtree A green. We now have two copies of LT_{n-2} with a P_3 attached to the root, one entirely red and one entirely green. Since $n \equiv 1$ or $2 \bmod 3$ so is $n-3$ and by induction LT_{n-3} may be split with both of its root edges colored the same. Split B this way with both edges incident with the root colored red. This accomplishes the splitting of LT_n with both root edges colored the same. The induction step is finished and the proof is complete. □

5. PATH LENGTH IN FIBONACCI AND LUCAS TREES

Let $T(V,E,r)$ be a tree with root r and let $p(v)$ be the length of the unique path from the root r to the vertex v. The *total path length of* $T = \sum_{v\in V} p(v)$. If L is the set of leaves of T then the *external path length of* $T = \sum_{\ell\in L} p(\ell)$. In this section and those that follow, we investigate the relationship between these two parameters.

Grimaldi [4, Theorem 2, p. 24] has calculated the total path length t_n and external path length x_n for the n^{th} Fibonacci tree and finds

$$t_n = \left(\frac{-36-16\sqrt{5}}{5\left(5+\sqrt{5}\right)}\right)\alpha^n + \left(\frac{-36+16\sqrt{5}}{5\left(5-\sqrt{5}\right)}\right)\beta^n + \left(\frac{2(\alpha+1)}{\sqrt{5}(\alpha+2)}\right)n\alpha^n - \left(\frac{2(\beta+1)}{\sqrt{5}(\beta+2)}\right)n\beta^n + 2$$

and

$$x_n = \left(\frac{-52-12\sqrt{5}}{100+20\sqrt{5}}\right)\alpha^n + \left(\frac{-52+12\sqrt{5}}{100-20\sqrt{5}}\right)\beta^n + \left(\frac{(\alpha+1)}{\sqrt{5}(\alpha+2)}\right)n\alpha^n - \left(\frac{(\beta+1)}{\sqrt{5}(\beta+2)}\right)n\beta^n .$$

Rationalizing the denominators and simplifying the coefficients involving α and β allows us to write these formulas as

$$\begin{aligned} t_n &= \left(-1-\tfrac{11}{5\sqrt{5}}\right)\alpha^n + \left(-1+\tfrac{11}{5\sqrt{5}}\right)\beta^n + \left(\tfrac{1}{\sqrt{5}}+\tfrac{1}{5}\right)n\alpha^n \\ &\quad - \left(\tfrac{1}{\sqrt{5}}-\tfrac{1}{5}\right)n\beta^n + 2 \\ &= -(\alpha^n+\beta^n) - \tfrac{11}{5}\left(\tfrac{\alpha^n-\beta^n}{\sqrt{5}}\right) + \tfrac{n}{5}(\alpha^n+\beta^n) + n\left(\tfrac{\alpha^n-\beta^n}{\sqrt{5}}\right) + 2 \\ &= \left(\tfrac{5n-11}{5}\right)F_n + \left(\tfrac{n-5}{5}\right)L_n + 2 \quad \text{and} \\ x_n &= \tfrac{1}{2}\left(-1-\tfrac{1}{5\sqrt{5}}\right)\alpha^n + \tfrac{1}{2}\left(-1+\tfrac{1}{5\sqrt{5}}\right)\beta^n + \tfrac{1}{2}\left(\tfrac{1}{\sqrt{5}}+\tfrac{1}{5}\right)n\alpha^n \\ &\quad - \tfrac{1}{2}\left(\tfrac{1}{\sqrt{5}}-\tfrac{1}{5}\right)n\beta^n \end{aligned}$$

$$= -\tfrac{1}{2}(\alpha^n + \beta^n) - \tfrac{1}{10}\left(\frac{\alpha^n - \beta^n}{\sqrt{5}}\right) + \tfrac{n}{10}(\alpha^n + \beta^n) + \tfrac{n}{2}\left(\frac{\alpha^n - \beta^n}{\sqrt{5}}\right)$$

$$= \left(\tfrac{5n-1}{10}\right)F_n + \left(\tfrac{n-5}{10}\right)L_n\,.$$

Noticing that the coefficients in t_n are nearly twice those of x_n, we compute $t_n - 2x_n = 2 - 2F_n$. This gives us the following result.

Corollary (to Grimaldi's **Theorem 2** [4, p. 24]). In the Fibonacci tree FT_n, having total path length t_n and external path length x_n, we find that

$$(i) \quad t_n = \left(\tfrac{5n-11}{5}\right)F_n + \left(\tfrac{n-5}{5}\right)L_n + 2\,,$$

$$(ii) \quad x_n = \left(\tfrac{5n-1}{10}\right)F_n + \left(\tfrac{n-5}{10}\right)L_n\,,$$

and $(iii) \quad t_n = 2x_n - 2F_n + 2.$

We now make similar calculations for Lucas trees. Since each node in LT_n (except for the root) is one unit farther from the root than it had been in its respective subtree, we can write a recurrence relation for t_n as

$$t_n = t_{n-1} + t_{n-2} + v_n - 1$$

where v_n is the number of vertices in LT_n. We showed earlier that $v_n = 2L_n - 1$ and the recurrence becomes

$$t_n = t_{n-1} + t_{n-2} + 2L_n - 2 = t_{n-1} + t_{n-2} + 2(\alpha^n + \beta^n) - 2$$

or

$$t_n - t_{n-1} - t_{n-2} = 2\alpha^n + 2\beta^n - 2\,.$$

The homogeneous difference equation is the same as that satisfied by the Fibonacci and Lucas numbers so we know the basic solutions are α^n and β^n. The non-homogeneous term consists of three pieces and we can use the method of undetermined coefficients to find a solution for each piece. Observation suffices to see that $t_n = 2$ is a solution for the -2 portion. Since α^n (or β^n) is a solution to the homogeneous case, we assume that $t_n = An\alpha^n$ (or $Bn\beta^n$) and substituting into $t_n - t_{n-1} - t_{n-2} = 2\alpha^n$ (or $2\beta^n$) gives

$$An\alpha^n - A(n-1)\alpha^{n-1} - A(n-2)\alpha^{n-2} = 2\alpha^n,$$

$$An(\alpha^n - \alpha^{n-1} - \alpha^{n-2}) + A\alpha^{n-1} + 2A\alpha^{n-2} = 2\alpha^n,$$

$$A\alpha^{n-1} + 2A\alpha^{n-2} = 2\alpha^n\,.$$

Solving, we find $A = \frac{2\alpha^n}{\alpha^{n-1}+2\alpha^{n-2}} = \frac{2\alpha^2}{\alpha+2} = \frac{2(\alpha+1)}{\alpha+2}$. Similarly, we find $B = \frac{2(\beta+1)}{\beta+2}$. The complete solution to the difference equation can now be written as

$$t_n = C_1\alpha^n + C_2\beta^n + \frac{2(\alpha+1)}{\alpha+2}n\alpha^n + \frac{2(\beta+1)}{\beta+2}n\beta^n + 2$$

where C_1 and C_2 are constants, which will be determined by the total path lengths of the first two Lucas trees, namely $t_0 = 2$ and $t_1 = 0$. Plugging in these values and solving the resulting system gives

$$C_1 = \frac{-2}{\sqrt{5}}\left(1 + \frac{2\alpha+1}{\alpha+2} + \frac{2\beta+1}{\beta+2}\right) = \frac{-4}{\sqrt{5}}$$

$$C_2 = \frac{2}{\sqrt{5}}\left(1 + \frac{2\alpha+1}{\alpha+2} + \frac{2\beta+1}{\beta+2}\right) = \frac{4}{\sqrt{5}}.$$

We can now write

$$\begin{aligned} t_n &= \frac{-4}{\sqrt{5}}\alpha^n + \frac{4}{\sqrt{5}}\beta^n + \left(1 + \frac{1}{\sqrt{5}}\right)n\alpha^n + \left(1 - \frac{1}{\sqrt{5}}\right)n\beta^n + 2 \\ &= (n-4)F_n + nL_n + 2\,. \end{aligned}$$

To calculate the external path length for a Lucas tree, we note that each leaf will be one unit farther from the root than it was in its subtree so we obtain the recurrence

$$x_n = x_{n-1} + x_{n-2} + \ell_n = x_{n-1} + x_{n-2} + L_n$$

where ℓ_n is the number of leaves in LT_n, which we know is the n^{th} Lucas number. The homogeneous difference equation is the same as before. The method of undetermined coefficients leads to the particular solution of the non-homogeneous equation:

$$\frac{(\alpha+1)}{\alpha+2}n\alpha^n + \frac{(\beta+1)}{\beta+2}n\beta^n\,.$$

Applying the initial conditions $x_0 = 2$, $x_1 = 0$ produces the complete solution

$$\begin{aligned} x_n &= \left(1 - \tfrac{2}{\sqrt{5}}\right)\alpha^n + \left(1 + \tfrac{2}{\sqrt{5}}\right)\beta^n \\ &\quad + \tfrac{1}{2}\left(1 + \tfrac{1}{\sqrt{5}}\right)n\alpha^n + \tfrac{1}{2}\left(1 - \tfrac{1}{\sqrt{5}}\right)n\beta^n \\ &= \tfrac{1}{2}(n-4)F_n + \tfrac{1}{2}(n+2)L_n\,. \end{aligned}$$

Just as with the Fibonacci trees, this appears to be close to one-half the total path length. Subtracting twice the external path length from the total path length leads to the relation $t_n = 2x_n - 2L_n + 2$ and we summarize these results for Lucas Trees in the following theorem.

Theorem 10. In the Lucas tree LT_n, having total path length t_n and external path length x_n, we find that

$$(i) \quad t_n = (n-4)F_n + nL_n + 2\,,$$

$$(ii) \quad x_n = \tfrac{1}{2}(n-4)F_n + \tfrac{1}{2}(n+2)L_n\,,$$

and

$$(iii) \quad t_n = 2x_n - 2L_n + 2.$$

6. PATH LENGTH IN FULL BINARY TREES

The two relationships between t_n and x_n [parts (*iii*) in the Corollary and Theorem 10) are enticingly similar. They become even more so when we realize that F_n is the number of leaves in a Fibonacci tree and L_n is the number of leaves in a Lucas tree. Thus both equations can be written as $t_n = 2x_n - 2\ell_n + 2$, where ℓ_n is the number of leaves in the tree.

Is this relationship the result of these two classes of trees being recursively generated by basically the same procedure or is there something even more general going on? The answer is the latter. This relationship (without the subscripts) will hold in any full binary tree. We use *full* to mean that every non-leaf has exactly two children. It should be noted that some authors use the term *complete* instead of *full* for this condition. We can easily prove the general result by induction without having to find individual formulas for the total or the external path length.

Theorem 11. In a full binary tree having ℓ leaves, the total path length t and the external path length x satisfy $t = 2x - 2\ell + 2$.

Proof. We induct on the number of nodes in the tree. If there is a single node, $t = 0$, $x = 0$, $\ell = 1$, and $0 = 2(0) - 2(1) + 2$. If there are 3 nodes ($K_{1,2}$), then $t = 2, x = 2$, $\ell = 2$, and $2 = 2(2) - 2(2) + 2$. Assuming the relation holds true for all full binary trees with n or fewer nodes, we consider a tree with $n + 2$ nodes. This tree must have two sibling nodes. Remove them and the edges back to their parent. Let this new tree have total path length t', external path length x', and ℓ' leaves. By our induction hypothesis we know $t' = 2x' - 2\ell' + 2$ and we can relate these values to those in the larger tree. Removing the two leaf nodes converted their parent into a leaf so $\ell' = \ell - 1$. To see how this action affected the path lengths, assume the parent of the removed leaves was at level h. Then each leaf contributed $h + 1$ to the total path length and to the external path length. In the new tree, their parent still contributes h to the total path length but now contributes an additional h to the external path length. Thus, $t' = t - 2(h+1) = t - 2h - 2$ and $x' = x - 2(h+1) + h = x - h - 2$. Substituting these values in the equation from the induction hypothesis produces

$$t - 2h - 2 = 2(x - h - 2) - 2(\ell - 1) + 2\,,$$

which simplifies to $t = 2x - 2\ell + 2$. □

7. PATH LENGTH IN FULL M-ARY TREES

Does this relation generalize to full m-ary trees? If we try the same relationship with ternary trees we find immediately that it does not work with $K_{1,3}$. What does work for $K_{1,3}$ is to replace each 2 with 3. (This also works in the single node case.) If we consider the next largest ternary tree with 7 nodes and $t = 9$, $x = 8$, $\ell = 5$, however, this relationship does not work.

Assuming that there is some relationship that will hold in general for full m-ary trees, we suppose that $t = ax - b\ell + c$ and try to determine values for a, b, and c. Applying this relation to the three simplest such trees we get a system of equations whose solution gives $a = \frac{m}{m-1}$ and $b = c = \frac{m}{(m-1)^2}$. This solution does in fact work for all cases and this is easily proved in a manner similar to the last theorem.

Theorem 12. In a full m-ary tree having ℓ leaves, total path length t, and external path length x, we find that $t = \frac{m}{m-1}x - \frac{m}{(m-1)^2}\ell + \frac{m}{(m-1)^2}$.

Proof. We again induct on the number of nodes in the tree. If there is a single node, $t = 0$, $x = 0$, $\ell = 1$, and $0 = \frac{m}{m-1}(0) - \frac{m}{(m-1)^2}(1) + \frac{m}{(m-1)^2}$. If there are $m+1$ nodes ($K_{1,m}$), then $t = m, x = m$, $\ell = m$, and $\frac{m}{m-1}(m) - \frac{m}{(m-1)^2}(m) + \frac{m}{(m-1)^2} = m\frac{m^2-m-m+1}{(m-1)^2} = m$. Assuming the relation holds true for all full m-ary trees with n or fewer nodes, we consider a tree with $n+m$ nodes. This tree must have a set of m sibling nodes. Remove them and the edges back to their parent. Let this new tree have total path length t', external path length x', and ℓ' leaves. By our induction hypothesis we know $t' = \frac{m}{m-1}x' - \frac{m}{(m-1)^2}\ell' + \frac{m}{(m-1)^2}$. Removing the m leaf nodes converted their parent into a leaf so $\ell' = \ell - m + 1$. Assume the parent of the removed leaves was at level h. Then each leaf contributed $h+1$ to the total path length and to the external path length in the original tree. In the new tree, their parent makes the same contribution to the total path length but now contributes an additional h to the external path length. Thus, $t' = t - m(h+1) = t - mh - m$ and $x' = x - m(h+1) + h = x - (m-1)h - m$. Substituting these values in the equation from the induction hypothesis produces

$$
\begin{aligned}
t - mh - m &= \tfrac{m}{m-1}\Big(x - (m-1)h - m\Big) - \tfrac{m}{(m-1)^2}(\ell - m + 1) + \tfrac{m}{(m-1)^2} \\
&= \tfrac{m}{m-1}x - \tfrac{m}{(m-1)^2}\ell + \tfrac{m}{(m-1)^2} - mh - \tfrac{m^2}{m-1} + \tfrac{m}{m-1} \\
&= \tfrac{m}{m-1}x - \tfrac{m}{(m-1)^2}\ell + \tfrac{m}{(m-1)^2} - mh - m,
\end{aligned}
$$

so that $t = \frac{m}{m-1}x - \frac{m}{(m-1)^2}\ell + \frac{m}{(m-1)^2}$ as desired. □

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