Logarithmic upper bound for the upper chromatic number of S$(t, t+1, v)$ systems

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Dedicated to the memory of Lucia Gionfriddo (1973–2008)

Abstract

Vertex colorings of Steiner systems S$(t, t+1, v)$ are considered in which each block contains at least two vertices of the same color. Necessary conditions for the existence of such colorings with given parameters are determined, and an upper bound of the order $O(\ln v)$ is found for the maximum number of colors. This bound remains valid for nearly complete *partial* Steiner systems, too. In striking contrast, systems S(t, k, v) with $k \geq t+2$ always admit colorings with at least $c \cdot v^{\alpha}$ colors, for some positive constants c and α, as $v \to \infty$.

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1 Introduction

In 1993, V. Voloshin introduced a new kind of vertex coloring for hypergraphs where the edges of one specified type contain at least two vertices of the same color and the edges of the other type contain at least two vertices of different colors. The new concepts of *strict coloring*, *upper* and

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lower chromatic number, *chromatic spectrum* and *uncolorable* hypergraph [11, 12] were introduced for these colorings.

In 1997, the present authors [6] studied these vertex colorings in Steiner Triple and Quadruple Systems: for STSs a general upper bound for the upper chromatic number was established and proved to be best possible. Further results were subsequently obtained by different authors; these have been surveyed in [8]. Some more restrictive types of colorings for Steiner systems have been studied in [9].

The aim of the present note is to extend the earlier estimates to Steiner systems with general parameters, and also to partial Steiner systems when possible. As it turns out by comparing the two types of our new results, substantial change occurs in the behavior of the maximum number of colors when block size exceeds intersection size by more than 2.

1.1 Mixed hypergraphs

A *mixed hypergraph* $\mathcal{H}$ is defined by the 3-tuple (X, $\mathcal{C}$, $\mathcal{D}$) where X is a finite set of vertices — its cardinality will be denoted by v throughout this paper — and $\mathcal{C}$ and $\mathcal{D}$ are two families of subsets of X whose elements are called $\mathcal{C}$-edges and $\mathcal{D}$-edges, respectively.

A *strict coloring* $\mathcal{P}$ of $\mathcal{H}$, which uses precisely h colors, is defined by partitioning the vertex set X into h nonempty subsets $X_i \subseteq X$, called *color classes*, under the following condition: each element of $\mathcal{C}$ must contain at least two vertices colored with the same color, while each element of $\mathcal{D}$ must contain at least two vertices colored with two different colors. We shall use the notation $n_i = |X_i|$ for $1 \leq i \leq h$. The set $\{1, 2, \ldots, h\}$ is usually called the *color set*.

Assuming that a mixed hypergraph $\mathcal{H}$ admits at least one coloring, the *upper (lower) chromatic number*, denoted by $\bar{\chi}(\mathcal{H})$ ($\chi(\mathcal{H})$) is the maximum (minimum) h for which there exists a strict coloring of $\mathcal{H}$. The *feasible set* of $\mathcal{H}$ is the set $\Omega(\mathcal{H}) = \{i \in \mathbb{N} \mid \exists$ *strict coloring of* $\mathcal{H}$ *with* i *colors*$\}$, and the *chromatic spectrum* is the sequence $(r_i)_{1 \leq i \leq v}$ where r_i denotes the number of strict i-colorings for $i = 1, \ldots, v$ [11, 12].

The existence of $\mathcal{C}$-edges and $\mathcal{D}$-edges and "interaction" between them mean that some mixed hypergraphs admit no colorings at all [12], while others may have a *broken spectrum*; i.e., the integers $i - 1$ and j for some $j > i$ belong to $\Omega(\mathcal{H})$ but the integer i does not [5].

It is easy to see that the chromatic spectrum of a mixed hypergraph containing only $\mathcal{C}$-edges cannot be broken. The first hypergraph with broken spectrum was found in [5], where also the smallest possible number of vertices in such examples was determined under various conditions. On the other hand, it is still a open problem to find a Steiner system with broken spectrum. In fact, the only block designs which are known to have a broken spectrum are P_3-designs, see [2].

1.2 Steiner systems and strict colorings

For three integers $v \geq k > t \geq 2$, a *Steiner system* $\mathrm{S}(t,k,v)$ is defined by the pair $(X, \mathcal{B})$, where X is a finite set of vertices, with $|X| = v$, and $\mathcal{B}$ is a family of subsets of X called *blocks* respecting two conditions:

1. any t distinct vertices are contained in one and only one block;

2. each block contains exactly k vertices.

If $t = 2$ and $k = 3$, such a system is called a Steiner Triple System and is denoted by $\mathrm{STS}(v)$; and if $t = 3$ and $k = 4$, it is called a Steiner Quadruple System and is denoted by $\mathrm{SQS}(v)$. A Steiner system with $|X| = v$ is said to be of *order* v.

In this paper we mostly deal with Steiner systems where $k = t+1$, that is $\mathrm{S}(t, t+1, v)$. As in [6], we will treat these systems as two distinct types of mixed hypergraph:

1. $\mathcal{C}$-$\mathrm{S}(t, t+1, v)$, where each block is a $\mathcal{C}$-edge — in this case, strict coloring means that each block contains at least two vertices of the same color. Since all the vertices are allowed to be colored with a single color, such systems always are colorable, and in their colorings some *monochromatic blocks* may occur.

2. $\mathcal{B}$-$\mathrm{S}(t, t+1, v)$, where each block is at the same time a $\mathcal{C}$-edge and a $\mathcal{D}$-edge — so it must contain at least two vertices of the same color, and also a pair of vertices colored with different colors. Therefore, in these strict colorings there cannot occur any monochromatic blocks.

In [6], the following result was proved for $\mathcal{C}$-$\mathrm{S}(2,3,v)$ and $\mathcal{B}$-$\mathrm{S}(2,3,v)$ systems.

Theorem 1 ([6]) *If $\mathcal{P}$ is a strict coloring of a $\mathcal{C}$-$S(2,3,v)$ or a $\mathcal{B}$-$S(2,3,v)$ system, using h colors, then $n_i \geq 2^{i-1}$ for all $1 \leq i \leq h$. As a consequence, if $v \leq 2^h - 1$, then $\bar{\chi} \leq h$. This upper bound is tight for all $h \geq 2$.* □

Theorem 1 provides necessary conditions for the existence of strict colorings and for the cardinalities of color classes in $\mathcal{C}$-$\mathrm{S}(2,3,v)$ and $\mathcal{B}$-$\mathrm{S}(2,3,v)$, as well as an upper bound for their upper chromatic number. There are also some known necessary conditions for the existence of strict colorings of $\mathcal{C}$-$\mathrm{S}(3,4,v)$ and $\mathcal{B}$-$\mathrm{S}(3,4,v)$ systems [8], but an upper bound for their upper chromatic number has not been determined.

In Section 2 we will obtain conditions for the existence of strict colorings for $\mathcal{C}$-$\mathrm{S}(t, t+1, v)$ and $\mathcal{B}$-$\mathrm{S}(t, t+1, v)$ systems and necessary conditions regarding the cardinalities of their color classes. In Section 3 we derive an upper bound of logarithmic growth for the upper chromatic number, and also some necessary conditions for the lower chromatic number.

If the block size k exceeds $t+1$, then the logarithmic upper bound is not valid any longer. This fact will be proved in the last section. We note that the contrast between $k = t+1$ and $k \geq t+2$ is even larger when the *maximum* value of $\bar{\chi}$ is considered over the classes of those Steiner systems. For some $k > t+1$ there are constructions where $\bar{\chi}$ grows as a linear function of the order v. More explicitly, such $\mathrm{S}(2,4,v)$ systems whose largest arcs attain the extremal size can be found in [4] for every $v \equiv 4 \pmod{12}$. Viewing all blocks as $\mathcal{C}$-edges, their upper chromatic number is linear in v. But currently no detailed information is known about the chromatic spectrum of general systems of the type $\mathrm{S}(t,k,v)$.

2 Strict colorings for $\mathrm{S}(t, t+1, v)$

Let $\mathcal{P}$ be a strict coloring of a $\mathcal{C}$-$\mathrm{S}(t,t+1,v)$ or $\mathcal{B}$-$\mathrm{S}(t,t+1,v)$ system, which uses h colors; $X_i \subseteq X$ is the set of vertices colored with color i in $\mathcal{P}$, while n_i is its cardinality. For the sake of convenience we will order the labels of X_i and n_i, with $1 \leq i \leq h$, in increasing order of size; i.e., we shall assume

$$n_1 \leq n_2 \leq \cdots \leq n_h .$$

The letter I will stand for an arbitrary subset of the color set $\{1, 2, \ldots, h\}$. The following inequality generalizes those in [6, 7].

Proposition 1 *Let $\mathcal{P}$ be a strict coloring of a $\mathcal{C}$-$S(t,t+1,v)$ or a $\mathcal{B}$-$S(t,t+1,v)$ system which uses h colors, and let $S_k = \bigcup_{j \in I} X_j$ be the union of $k = |I|$ color classes of $\mathcal{P}$ with respective cardinalities $|X_j| = n_j$ for every $j \in I$. Denoting $s_k = |S_k|$, the inequality*

$$s_k(s_k - 1) \leq \binom{t+1}{2} \sum_{j \in I} n_j(n_j - 1) \qquad (a)$$

holds and, as a consequence,

$$s_k^2 \leq \binom{t+1}{2} \sum_{j \in I} n_j^2 \qquad (b)$$

is valid, too, for every $2 \leq |I| \leq h$.

Proof. We consider the family $\mathcal{T}_k$ of t-element subset of S_k, and the family $\mathcal{B}_k$ of blocks meeting S_k in t or $t+1$ vertices. Clearly,

$$|\mathcal{T}_k| = \binom{s_k}{t} = \frac{s_k(s_k-1)}{t(t-1)} \binom{s_k - 2}{t-2}. \qquad (1)$$

Each $T \in \mathcal{T}_k$ is contained in exactly one block $B_i \in \mathcal{B}_k$, therefore $\mathcal{T}_k$ can be partitioned into singletons and $(t+1)$-tuples, depending on whether $|B_i \cap S_k| = t$ or $B_i \subseteq S_k$. Let us write $\mathcal{T}_k = \mathcal{T}^1 \cup \cdots \cup \mathcal{T}^m$, where $m = |\mathcal{B}_k|$ and the $\mathcal{T}^i$ are indexed with the blocks $B_i \in \mathcal{B}_k$.

Each B_i $(1 \leq i \leq m)$ must contain at least two vertices colored with the same color contained in I: if this were not so, there would be a block multicolored inside S_k, and its single possible vertex outside S_k would have a color not in I, hence $\mathcal{P}$ would not be a strict coloring. We can choose the monochromatic pair inside S_k in $\sum\binom{n_j}{2}$ ways, and each pair can be completed to a t-tuple in $\binom{s_k-2}{t-2}$ ways. If $B_i \subseteq S_k$ then the monochromatic pair in B_i has been counted for $t-1$ distinct elements of $\mathcal{T}^i$, and if $|B_i \cap S_k| = t$ then the pair has been counted for the single t-tuple of $\mathcal{T}^i$. It follows that in each $\mathcal{T}^i$ at least $\frac{t-1}{t+1}$ proportion of t-tuples has been counted for the monochromatic pairs. (In fact, it may be even more, since a t-tuple may contain more than one monochromatic pair.) Thus,

$$\sum_{j \in I} \binom{n_j}{2} \binom{s_k - 2}{t-2} \geq \frac{t-1}{t+1} \, |\mathcal{T}_k| \, . \tag{2}$$

The comparison of (1) and (2) yields the inequalities (a) for every k and every set I of k colors. Moreover, considering that $s_k = \sum_{j \in I} n_j$, the inequalities (a) yield the inequalities (b), thus proving the proposition. □

Over the entire range of $k = 1, 2, \ldots, h$ we shall use the following notational convenience from now on. Assuming that I represents the k *smallest* color classes, that is $S_k = X_1 \cup \cdots \cup X_k$, the inequalities of type (b) will be referred to as

$$s_k^2 \leq K \sum_{j=1}^{k} n_j^2 \tag{k}$$

where $K = K(t)$ is meant to be a constant independent of k. The numbering (k) of the inequality represents the number of terms in the summation. For this type of inequalities, we prove the following lemma that will be very useful concerning estimates on the upper chromatic number.

Lemma 1 *Let $n_1, n_2, \ldots, n_h$ be a non-decreasing sequence of positive integers, and write $s_k = \sum_{j=1}^{k} n_j$ for all $1 \leq k \leq h$. If there is a constant K and a threshold value k_0 such that the inequalities (k) are valid for all $k \geq k_0$, then $n_{i+4K} \geq 2n_i$ holds for all $k_0 \leq i \leq h - 4K$.*

Proof. Let us assume for a contradiction that $n_{i+4K} < 2n_i$ for some value i. Introduce the notation $x = n_i$, $y = s_i$, $L = s_{i+4K}^2$ and $R = K\sum_{j=1}^{i+4K} n_j^2$. We are going to prove that $R < L$, and this will contradict the inequality (k) with $k = i + 4K$.

Since $s_{i+4K} \geq y + 4Kx$, we have

$$L \geq y^2 + 8Kxy + 16K^2x^2.$$

Moreover, $n_\ell^2 < 4x^2$ holds for all $i \leq \ell \leq i + 4K$, and $\sum_{j=1}^{i} n_j^2 \leq xy$, so that

$$R < Kxy + (4K)^2x^2 < y^2 + 8Kxy + 16K^2x^2 \leq L.$$

This contradiction proves $n_{i+4K} \geq 2n_i$. □

Theorem 2 *If $\mathcal{P}$ is a strict coloring for a $\mathcal{C}$-$S(t, t+1, v)$ or a $\mathcal{B}$-$S(t, t+1, v)$ system which uses h colors, then, with the notation $K = \binom{t+1}{2}$, we have $n_{i+4K} \geq 2n_i$ and $n_i \geq ab^i$ for all $1 \leq i \leq h$, for some $a \geq 1/2$ and $b > 1$. Here b is independent of $\mathcal{P}$, it only depends on t.*

Proof. We have seen in Proposition 1 that the conditions of Lemma 1 are satisfied for all $k \geq 1$, and consequently $n_{i+4K} \geq 2n_i$ holds for all feasible values of i. In order to derive an explicit lower bound on the n_i, let us choose

$$b = 2^{1/4K} \qquad \text{and} \qquad a = \min_{1 \leq i \leq 4K} \left(\frac{n_i}{2^{i/4K}} \right).$$

We estimate n_i as follows. If $i \leq 4K$, then $n_i \geq ab^i$ is true by the choice of a and b. For i larger, let us write i in the form $i = j + 4nK$, where $1 \leq j \leq 4K$. Let us assume by induction on n that $n_{i'} \geq ab^{i'}$ for all $1 \leq i' \leq j + 4(n-1)K$. Then

$$n_i = n_{j+4nK} \geq 2n_{j+4(n-1)K} \geq 2ab^{j+4(n-1)K} = ab^{j+4nK} = ab^i.$$

Finally, the lower bound $a \geq 1/2$ follows from the fact that $n_{4K} \geq 1$ trivially holds. □

Theorem 2 describes some necessary conditions, involving the cardinalities of the color classes, for the existence of a strict coloring of a $\mathcal{C}$-S$(t, t+1, v)$ or a $\mathcal{B}$-S$(t, t+1, v)$ system. The first condition, $n_{i+4K} \geq 2n_i$, is independent of the type of coloring $\mathcal{P}$, while the second one, $n_i \geq ab^i$, depends on the type of $\mathcal{P}$. Nevertheless, a universal lower bound has been established for the value of a, in fact $a \geq 1/2$ always holds. Moreover, since the choice of $b = 2^{1/4K}$ is independent of the actual coloring, we always obtain $n_i \geq 2^{\frac{i}{4K}-1}$ for every i in every coloring $\mathcal{P}$.

In Theorem 1, similar — and even stronger — conditions have been found for S$(2,3,v)$ systems: in them the cardinality of each color class has to meet the condition $n_i \geq 2^{i-1}$ for $1 \leq i \leq h$, and these conditions are independent of the coloring $\mathcal{P}$. The inequalities (a) and (b) in Proposition 1 hold with $K = 3$ in S$(2,3,v)$, whereas an explicit lower bound (other than the one in Theorem 2) has not yet been determined for the cardinality of the color classes of $S(t, t+1, v)$ with values of $t \geq 4$. Some estimates for $S(3, 4, v)$ are available by means of particular constructions (*doubling constructions*) [8].

Let us recall that in $\mathcal{C}$-$S(t, t+1, v)$ a strict coloring $\mathcal{P}$ is allowed to have monochromatic blocks; i.e., blocks colored with a single color. The next observation shows that such blocks are unavoidable under some conditions.

Proposition 2 *If $\mathcal{P}$ is a strict coloring for a $\mathcal{C}$-$S(t, t+1, v)$ system, where t is odd, and there exists a color class X_i with $n_i \geq \frac{v+2}{2}$, then there are monochromatic blocks in $\mathcal{P}$.*

Proof. Let us denote by $(X, \mathcal{B})$ the $\mathcal{C}$-$S(t, t+1, v)$ system under consideration, with a strict coloring $\mathcal{P}$ and a color class X_i of carcinality $|X_i| = n_i \geq \frac{v+2}{2}$. Then one can define the following two subfamilies of blocks:

$$T(X_i) = \{b \in \mathcal{B} \mid b \subseteq X_i\},$$

and

$$T(X \setminus X_i) = \{b \in \mathcal{B} \mid b \subseteq X \setminus X_i\}.$$

It has been proved in Lemma 3.3 of [3] that the quantity

$$|T(X_i)| - |T(X \setminus X_i)| = f(t, v, n_i)$$

is greater than zero for $n_i > v/2$. So $T(X_i) \neq \emptyset$, what means that there are monochromatic blocks colored with color i. □

A related result in [1, p. 341, Theorem 18.28] states that if a $\mathcal{B}$-$S(2, 3, v)$ system has independence number at most $v/3$, then it is uncolorable. From this, we see that every $\mathcal{C}$-$S(2, 3, v)$ with so small independence number necessarily contains some monochromatic blocks in every strict coloring.

3 Upper chromatic number for $S(t, t+1, v)$

In this section we will prove a logarithmic upper bound for the upper chromatic number, and obtain a result on the lower chromatic number for both $\mathcal{C}$-$S(t, t+1, v)$ and $\mathcal{B}$-$S(t, t+1, v)$ systems. Theorem 1 involves systems just with $t = 2$. When $t = 3$, similar results have only been achieved in particular systems of a certain order v: no general estimates have yet been obtained with $t > 2$.

For the proof of the next result, let us recall the definition of $b = 2^{1/4K}$ from the proof of Theorem 2.

Theorem 3 *For every $t \geq 2$ there exists a constant $C = C(t)$ with the following property. If a $\mathcal{C}$-$S(t, t+1, v)$ or a $\mathcal{B}$-$S(t, t+1, v)$ system $\mathcal{H}$ is colorable, then*

$$\bar{\chi}(\mathcal{H}) \leq C \ln v .$$

Proof. Let us consider any coloring with the maximum number of colors. Then the largest color class has $n_{\bar{\chi}}$ vertices, and by the results of the previous section we have

$$v > n_{\bar{\chi}} \geq ab^{\bar{\chi}} \geq 2^{\bar{\chi}/4K}/2 .$$

Rearrangement yields

$$\bar{\chi} < \frac{4K}{\ln 2} \ln 2v .$$

Here K depends on t only. This fact completes the proof of the theorem. □

As regards the lower chromatic number of any $\mathcal{C}$-S$(t, t+1, v)$ system, it is obvious that $\chi = 1$, hence these systems are all colorable. The uncolorability of $\mathcal{B}$-S$(t, t+1, v)$ systems with $t \geq 3$ remains an open issue. For these systems the following result is obtained.

Proposition 3 *If a $\mathcal{B}$-$S(t, t+1, v)$ system has $\chi = 2$, then t must be odd. If t is even, then $\chi \geq 3$.*

Proof. The assertion follows from the fact that if a blocking set exists, then t is odd; see Theorem 3.4 in [3]. □

4 Partial systems PS$(t, t+1, v)$

In *partial* Steiner systems PS$(t, t+1, v)$, the first condition defining S$(t, t+1, v)$ is relaxed to the following:

1′. any t distinct vertices are contained in *at most one* block.

The existence of asymptotically optimal PS$(t, t+1, v)$ systems, as $v \to \infty$, follows from the more general result of Rödl [10] where the analogous assertion is proved for any fixed block size k.

In order to state our results on partial Steiner systems, we need to introduce some notions.

Definition 1 Let $\mathcal{H}$ be any PS$(t, t+1, v)$ system. Let us call a t-element subset of the vertex set an *uncovered t-tuple* if it is not contained in any block of $\mathcal{H}$; and call it *covered* otherwise.

We first observe here that a large number of uncovered t-tuples may result in an upper chromatic number that grows faster than $O(\ln v)$.

Remark 1 *If there exists an $S(t,t+1,v)$ system with an embedded $S(t,t+1,u)$ subsystem, then there exists a $PS(t,t+1,v)$ system with precisely $\binom{u}{t}$ uncovered t-tuples and upper chromatic number greater than u. Indeed, omitting the blocks of the subsystem we can assign u different colors to its vertices and a new distinct color to all of the remaining $v-u$ vertices.*

This observation yields that if we wish to ensure $\bar{\chi} = O(\ln v)$ then in general we cannot allow more than $O(\binom{\ln v}{t})$ uncovered t-tuples, what means that the system should have at least $\left(\frac{1}{t+1} - O\left(\left(\frac{\ln v}{v}\right)^t\right)\right)\binom{v}{t}$ blocks. We will prove that this estimate is asymptotically tight, namely this number of blocks is already sufficient to ensure $\bar{\chi} = O(\ln v)$. Some further definitions will be needed.

Definition 2 Let $\mathcal{H}$ be a PS$(t,t+1,v)$ system, and c any fixed real number in the range $0 < c < 1$. We call a vertex subset Y in a PS$(t,t+1,v)$ system *c-dense* if at least $c\binom{|Y|}{t}$ of its t-subsets are covered.

Definition 3 With the notation $\mathcal{H}$ and c of the previous definition, and for a given coloring of $\mathcal{H}$, we consider the largest s_k such that the inequality (k) is not valid with $K = \frac{1}{c}\binom{t+1}{2}$. The maximum of this value s_k taken over all strict colorings is called the *c-density threshold* of $\mathcal{H}$ and will be denoted by $\tau(\mathcal{H},c)$.

Remark 2 *The proof of Proposition 1 implies that if every $Y \subseteq X$ with $|Y| \geq u$ is c-dense, then $\tau(\mathcal{H},c) \leq u$.*

Theorem 4 *Let $\mathfrak{H}$ be an infinite family of $PS(t,t+1,v)$ systems, and c any fixed real number, $0 < c < 1$. If $\tau(\mathcal{H},c) = O(\ln v)$ for all $\mathcal{H} \in \mathfrak{H}$ as $v \to \infty$, then $\bar{\chi}(\mathcal{H}) = O(\ln v)$ for all $\mathcal{H} \in \mathfrak{H}$.*

Proof. On applying Lemma 1, we see that if $i > k$ where k is the index associated with the c-density threshold, then the inequality $n_{i+4K} \geq 2n_i$ is valid. If the inequality holds for $i = k$, too, then this implies

$$\bar{\chi}(\mathcal{H}) \leq c'(\ln v - \ln s_k) + s_k$$

for some constant c', what means $\bar{\chi} = O(\ln v)$ by the assumption on k. Otherwise, if $n_{k+4K} < 2n_k$, then s_k and s_{k+1} are at most a small multiplicative constant apart, and therefore the analogous inequality $\bar{\chi}(\mathcal{H}) \leq c'(\ln v - \ln s_{k+1}) + s_{k+1}$ yields $\bar{\chi} = O(\ln v)$. □

From this result we can derive that sufficiently many blocks always guarantee $\bar{\chi} = O(\ln v)$.

Theorem 5 *Assume that $t \geq 2$ is fixed and $v \to \infty$. If $\mathcal{H}$ is a $PS(t, t+1, v)$ system with as many as*

$$\left(\frac{1}{t+1} - O\left(\left(\frac{\ln v}{v}\right)^t\right)\right)\binom{v}{t}$$

blocks, then $\bar{\chi}(\mathcal{H}) = O(\ln v)$.

Proof. Let $f = f(v)$ denote the number of uncovered t-tuples. We write f in the form $f = \frac{1}{2}\binom{x}{t}$. Note that $x = O(\ln v)$, by the assumption on $|\mathcal{H}|$. Moreover, if $s_k > x$, then at least half of the t-tuples must be covered in the union of the first k color classes. It follows that the inequality (k) is valid with $K \leq t^2 + t$. Therefore, the $\frac{1}{2}$-density threshold of $\mathcal{H}$ is as small as $O(\ln v)$, and hence $\bar{\chi}(\mathcal{H}) = O(\ln v)$ follows by Theorem 4. □

Remark 1 indicates that the number of blocks given in the theorem cannot be expected to be smaller, since the bound is tight whenever an embedded Steiner system exists.

5 $S(t, k, v)$ with $k \geq t + 2$

Finally, we observe that the logarithmic upper bound on $\bar{\chi}$ does not remain valid in Steiner systems $S(t, k, v)$ with $k \geq t+2$. Such a result can be proved by various methods; here we do not aim at optimizing the lower bound, just show that the growth speed is strictly larger than in the case of $S(t, t+1, v)$ systems.

Theorem 6 *If $\mathcal{H}$ is an $S(t, k, v)$ system with $k \geq t + 2$ and $t \geq 2$, then $\bar{\chi}(\mathcal{H}) \geq (1 - o(1))\,(k - t - 1)\left(\frac{v}{2}\right)^{1/t}$ as $v \to \infty$.*

Proof. Let Y be any maximal set meeting each block of $\mathcal{H}$ in at most $k - 2$ vertices. Then $\bar{\chi}(\mathcal{H}) > |Y|$ because a strict coloring is obtained if we assign color 1 to all vertices of $X \setminus Y$ and a dedicated color to each vertex of Y. Hence, it will suffice to show that Y is fairly large.

By the maximality of Y, every $x \in X \setminus Y$ is incident with a block B such that $|B \cap Y| = k - 2 \geq t$. The number of such blocks cannot exceed $\binom{|Y|}{t} / \binom{k-2}{t}$, and each of them covers only two vertices outside Y. Thus, $2\binom{|Y|}{t} / \binom{k-2}{t} \geq v - |Y|$ is valid, and from this inequality the lower bound on $\bar{\chi}(\mathcal{H})$ follows. □

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