On graph labeling problems and regular decompositions of complete graphs

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Abstract

In the first part of this paper we present a generalization of complete graph factorizations obtained by labeling the graph vertices by natural numbers. In this generalization the vertices are labeled by elements of an arbitrary group G, in order to achieve a G-transitive factorization of the graph.

In the second part of the paper we apply this generalization to obtain a G-transitive factorization of the regular graph on n vertices with degree $n-2$. We discuss also 'Ringel-type' problems on G-transitive factorization of this graph into trees.

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1 Introduction

Let $n \geq 3$ be an integer. By K_n, K_n^* we denote the complete graph on n vertices, and the complete directed graph on n vertices, respectively (every ordered pair of vertices in K_n^* is connected by one arc (directed edge)). For an even integer n, denote by $K_n - I$ the $(n-2)$-regular graph on n vertices. Observe that $K_n - I$ is isomorphic to the graph which results from K_n by deleting any complete matching. Similarly, we denote the $(n-2)$-regular directed graph on n vertices by $(K_n - I)^*$.

The problem of obtaining factorizations of the complete graph into a given subgraph by labeling the vertices of the subgraph with natural numbers is well known. The labeling of the vertices of the corresponding subgraph by elements of some abelian groups (especially cyclic) has also been treated. However, a comprehensive discussion of the above problem for arbitrary finite groups has not appeared in the literature. The objective of this paper is twofold: First, for any finite group G, to give a general treatment for the connection between G-transitive factorization of K_n and the problem of labeling the vertices of a subgraph by the elements of G. Second, to apply the above treatment in order to obtain some G-transitive facorizations.

At the beginning of this paper we study the connection between the labeling of the vertices of a directed graph F by the elements of a group G, and the decomposition of K_n^* into isomorphic copies of F on which G acts regularly. Such a labeling will be called a *G-graceful labeling* (see Definition 1.2 below). We will show that the existence of such a labeling ensures the existence of a corresponding decomposition and vice versa (Theorems 3.1 and 3.2). This extends results on the relation between cycle factorization of K_n and group sequencing treated in [12]. Then (in Section 4), it is shown that corresponding results for the undirected case follow from the directed case. In particular, our results generalize known results like a result of Rosa on β-valuations (see Corollary 4.3), etc. Furthermore, in Section 5 we apply this generalization to obtain G-regular factorizations of $K_n - I$. We discuss also a 'Ringel-type' conjecture on G-regular factorization of this graph into trees and finally consider the factorization of directed regular graphs.

The basic definitions are presented in the current section. In order to make the paper almost self contained, we devote Section 2 to a brief reminder from group theory contents which are relevant to our paper.

We present now some essential notation and definitions for this paper.

Notation 1.1 Let F be a directed graph on n vertices. The problem of determining whether or not the edge set of K_n^* can be partitioned into arc disjoint subgraphs isomorphic to F will be denoted by $P^*(F)$. If a solution to $P^*(F)$ exists, we shall say that there exists a *decomposition* of K_n^* into arc disjoint subgraphs isomorphic to F, and we shall denote this fact by $F|K_n^*$. Each isomorphic copy of F in the decomposition will be called a *factor* of the decomposition.

For the following definition, recall the definition of a regular action of a group (see Section 2).

Definition 1.1 Let F be a directed graph on n vertices. We shall say that K_n^* admits a *regular decomposition* (*G-regular decomposition*) into factors isomorphic to F if there exists a decomposition of K_n^* into subgraphs isomorphic to F, and a permutation group G of order $n-1$ acting on the vertices of K_n^*, such that G acts regularly on the factors of the decomposition. In this case we shall say that G acts regularly on the decomposition $F|K_n^*$.

Notation 1.2 The problem of determining whether or not K_n^* admits a regular decomposition into factors isomorphic to F will be denoted by $RP^*(F)$. The problem of determining whether or not K_n^* admits a regular decomposition into factors isomorphic to F with a corresponding group G will be denoted by $RP^*(G;F)$.

When dealing with the above regular decompositions, two fundamental problems arise:

1. For which subgraphs F of K_n^* does there exist a solution to $RP^*(F)$.
2. Given a graph H such that $|E(H)|$ divides $n(n-1)$, classify all the groups G for which $RP^*(G;F)$ has a solution.

Definition 1.2 Let G be a group of order n. Let H be a (not necessarily connected) directed graph such that $|E(H)| \leq n-1$. We shall say that H is *G-graceful* if there exists an injective function $f : V(H) \to G$ such that all the elements in the multiset $\{f(v)f(u)^{-1} \mid (u,v) \in E(H)\}$ are distinct.

Bloom and Hsu [4] dealt with the problem of graceful digraphs in which $G = Z_n$, while in [3] H was a cycle (or a directed cycle) and the connection to sequencing, in that case, was discussed. For other related papers one may see [1], [5], [10], [14], [17].

2 Preliminaries and notation

Throughout this paper all groups are finite. Let G be a group. In general, we shall consider G as a multiplicative group (with identity element 1). In some cases, where G is an abelian group, we shall consider G as an additive group (with identity element 0). The order of G will be denoted by $|G|$. For $x \in G$, we denote by $\langle x \rangle$ the (cyclic) group generated by x. An *elementary abelian group* is a group of order p^k, p a prime, $k \geq 1$, which is the direct product of k subgroups of order p. The centre of a group G is denoted by $Z(G)$. An *involution* in a group G is an element of order 2.

Let G be a group which acts on a set S. We shall always assume that G acts faithfully on S, i.e., that only the identity of G fixes all the elements of S. This means that G is in fact a permutation group on S. We shall also assume that G acts on S on the right. G is *transitive* on S if for every $x, y \in S$ there is $g \in G$ such that $xg = y$. G is *regular* on S if for every $x, y \in S$, there is a unique element $g \in G$ such that $xg = y$. In particular, G is regular on S if and only if G is transitive on S and only the identity fixes an element of S. Clearly, if G is regular on S, then $|G| = |S|$.

Let G act transitively on S, $S = \{x_1, x_2, \dots, x_n\}$, and let $x \in S$. Then the action of G on S is equivalent to the action of G on the right cosets of the stabilizer $H = St_G(x) = \{g \in G | xg = x\}$ of x, i.e., there is one to one mapping $x_i \leftrightarrow g_i$ between S and a set $\{g_1, g_2, \dots, g_n\}$ of right coset representatives of H in G such that $x_i g = x_j$ if and only if $Hg_i g = Hg_j$. In particular, if G is regular on S, then the action of G on S is equivalent to the action of G on itself by right multiplication, i.e., we may order the elements of G, $G = \{g_1, g_2, \ldots, g_n\}$, such that for every $g \in G$, $x_i g = x_j$ if and only if $g_i g = g_j$.

All graphs in this paper will be simple, i.e., without loops or multiple edges (arcs). The graphs considered are not necessarily connected, but we assume that all graphs have no isolated vertices. Given vertices u, v of an undirected graph, the corresponding edge will be denoted by $\{u, v\}$, and for the directed case, the corresponding arc will be denoted by (u, v). Given a graph (directed graph) H, $V(H)$ denotes the set of vertices of H, and $E(H)$ denotes the edge (arc) set of H. Let H be a graph (directed graph) and let A be a subset of $V(H)$. We shall denote by $H - A$ the graph (directed graph) which results from H by the deletion of all the vertices of A and all the edges (arcs) which are adjacent to a vertex from A. An isolated vertex of a graph (directed graph) is a vertex with no adjacent edges (arcs). A cycle of length k in a directed graph, which contains the arcs (v_1, v_2), $(v_2, v_3), \ldots, (v_{k-1}, v_k)$, (v_k, v_1) will be denoted by $(v_1, v_2, \ldots, v_k)$. This notation will be used also for a cycle in an undirected graph.

3 Regular decomposition of K_n^*

In this section we show that the existence of a solution to $RP^*(G; F)$ is equivalent to the existence of a corresponding G-graceful labeling of F (if $|G| = n - 1$) or of $F - \{v_0\}$ (if $|G| = n$) for an appropriate vertex v_0 (see Theorem 3.1 and 3.2 below). The results of this section extend results on cycle factorizations treated in [12] and [13]. First, we need some preliminary results.

Proposition 3.1 *Let C_1 be a subgraph of K_n^* and let G be a permutation group on the vertices of K_n^*, with $G = \{g_1 = 1, g_2, \ldots, g_k\}$, $C = \{C_1, C_1 g_2, \ldots, C_1 g_k\} =_{def} \{C_1, C_2, \ldots, C_k\}$. Suppose that k di-*

vides $n(n-1)$. Then C is a decomposition of K_n^* on which G acts regularly if and only if for every $1 \neq g \in G$ we have $E(C_1) \cap E(C_1 g) = \phi$.

Proof. Assume first that $E(C_1) \cap E(C_1 g) = \phi$ for every $1 \neq g \in G$, and let i, j be any integers satisfying $1 \leq i, j \leq n$. We have to show that $E(C_i) \cap E(C_j) = \phi$ for $i \neq j$ and that there is a unique $g \in G$ such that $C_i g = C_j$. Since $C_i = C_1 g_i$ and $C_j = C_1 g_j$, we have that $C_i g_i^{-1} g_j = C_j$. Furthermore, since $|G| = k = |C|$, there is a unique $g \in G$ such that $C_i g = C_j$. Thus we have $E(C_i) \cap E(C_j g_i^{-1}) = \phi$ whenever $i \neq j$. Hence, by acting with g_i on both sides of the latter equality, we have $E(C_i) \cap E(C_j) = \phi$, as required. Since the converse direction of the proposition is obvious, the proof is completed. □

Proposition 3.2 *Let F be a directed graph such that $|E(F)| = n$, and let G be a group of order $n-1$ which acts regularly on a decomposition $F|K_n^*$. Then G fixes one vertex of K_n^* and acts regularly on the remaining $n-1$ vertices.*

Proof. The proof is similar to that of Lemma 3.1 in [12]. □

Proposition 3.3 *Let F be a directed graph such that $|E(F)| = n-1$, and let G be a group of order n which acts regularly on a decomposition $F|K_n^*$. Then G acts regularly on $V(K_n^*)$.*

Proof. Assume that there is a vertex v_0 which is fixed by some non-identity element of G, say g_0. Then the action of G on $V(K_n^*)$ induces an action of $H = St_G(v_0)$ on the $n-1$ arcs, say, $e_1, e_2, \ldots, e_{n-1}$, which are adjacent to v_0 and which are directed into v_0. For every $1 \leq i \leq n-1$ the arc e_i is not fixed by any element $1 \neq g \in H$. For otherwise, if C is the (unique) factor which contains e_i, and if $g \neq 1$ fixes e_i, we have $e_i \in E(C) \cap E(Cg)$, a contradiction. Hence the action of H on the set $\{e_1, e_2, \ldots, e_{n-1}\}$ is fixed point free, which is impossible since $|H|$ does not divide $n-1$. Hence we conclude that the action of G on $V(K_n^*)$ is fixed point free, and since $n = |V(K_n^*))| = |G|$, it is regular. □

Theorem 3.1 *Let F be a directed graph such that $|E(F)| = n$ and let G be a group of order $n-1$. Then $RP^*(G;F)$ has a solution if and only if there is a vertex $v_0 \in V(F)$ such that $d_+(v_0) = d_-(v_0) = 1$ (where $d_+(v_0)$ and $d_-(v_0)$ are the indegree and outdegree of v_0, respectively), and $F - \{v_0\}$ is G-graceful.*

Proof. Assume first that $RP^*(G;F)$ has a solution. This means that there is a regular decomposition $F|K_n^*$ with a corresponding group G. Then, by Proposition 3.2, there is a vertex, say v_0, which is fixed by all the elements of G. Since G acts regularly on $V(K_n^*) - \{v_0\}$ by Proposition 3.2, G acts regularly on $E_+(v_0)$ and on $E_-(v_0)$, where $E_+(v_0)$ ($E_-(v_0)$) denotes the $n-1$ arcs which are adjacent to v_0 and directed inward (outward, respectively) v_0. Then, since G fixes v_0 and transforms any arc of F to an arc outside F, we have that for the factor F, $d_+(v_0) \leq 1$ and $d_-(v_0) \leq 1$. However, if $d_+(v_0) = 0$ in F, then no arc of F is transformed by G to an arc of K_n^* which is incident to v_0 and directed inward v_0, which is impossible. Hence $d_+(v_0) = 1$ in F. By similar arguments we conclude also that $d_-(v_0) = 1$ in F.

We shall show now that $F - \{v_0\}$ is G-graceful. Since the action of G on $V(F) - \{v_0\}$ is regular, we can label the vertices in $V(F) - \{v_0\}$ by the elements of G, such that for any two vertices $a, b \in V(F) - \{v_0\}$ and any $g \in G$, a is sent to b by g if and only if $ag = b$ in G. Assume on the contrary that there are different arcs (a, b) and (c, d) of F such that $ba^{-1} = dc^{-1}$. Denote $x = a^{-1}c$, and observe that by the previous identity, $x = b^{-1}d$. If $x = 1$, then $a = c$ and $b = d$, which is impossible since the arcs (a, b) and (c, d) are different. Hence $x \neq 1$. We have $ax = c$ and $bx = d$, and so the non-identity element x of G moves the arc (a, b) of F to a different arc (c, d) of F. This contradicts the fact that the decomposition $F|K_n^*$ is regular with a corresponding group G. Hence $F - \{v_0\}$ is G-graceful.

Assume now that there is a vertex v_0 of F such that $d_+(v_0) = d_-(v_0) = 1$ and $F - \{v_0\}$ is G-graceful. We shall show that $RP^*(G;F)$ has a solution. Given the graceful labeling of F, we claim that $\{Fg \mid g \in G\}$ is a regular decomposition of K_n^* (note that in this case, v_0 is fixed by all the elements of G). By proposition 3.1, it suffices to show that the factors F and Fx have no common arc for every non-identity element $x \in G$. Assume, on the contrary, that there is a non-identity element $x \in G$ such that F and Fx have a common arc, say (c, d). Then, since G fixes v_0 and does not fix any vertex of $V(F) - \{v_0\}$, and since $d_+(v_0) = d_-(v_0) = 1$ in F, we have that $v_0 \neq c$ and $v_0 \neq d$. Thus there is an arc (a, b) of F such that $ax = c$ and $bx = d$ (where clearly, $a \neq v_0$ and $b \neq v_0$). Then we have $ba^{-1} = dc^{-1}$, which contradicts the fact that $F - \{v_0\}$ is G-graceful. Hence $\{Fg \mid g \in G\}$ is a regular decomposition of K_n^*, as required. □

Theorem 3.2 below is the analog of Theorem 3.1 in the case $|G| = n$ and $E(F) = n - 1$. The proof of this theorem is based on considerations similar to those used in the proof of Theorem 3.1. Thus, we omit the proof.

Theorem 3.2 *Let F be a directed graph such that $|E(F)| = n - 1$, and let G be a group of order n. Then $RP^*(G; F)$ has a solution if and only if F is G-graceful.* □

4 Some connections to other combinatorial problems

In this section we derive some corollaries of the theorems of the previous section. These corollaries show that the results of section 3 are generalizations of some known results on the decomposition of the undirected complete graph. We also note on the connection of these results to some well known combinatorial problems, such as problems on (v, k, λ)-quotient sets, group sequencing and Oberwolfach factorization.

First, we introduce some notation. The graph $2K_n$ is an undirected graph on n vertices in which each pair of vertices is connected by two edges. The notions of decomposition and labeling presented in the previous sections are naturally extended to the undirected graphs K_n and $2K_n$. A decomposition $F|K_n$ (or $F|2K_n$) is *G-regular* if G is a permutation group on $V(K_n)$ ($V(2K_n)$) which permutes the factors of the decomposition regularly.

The first two corollaries are straightforward. They follow from Theorems 3.1 and 3.2.

Corollary 4.1 *Let F be an undirected graph with $|E(F)| = n$, and let G be a group of order $n-1$. Then there is a G-regular decomposition $F|2K_n$ if and only if there is a vertex v_0 of degree 2 in F and an orientation of the subgraph $F - \{v_0\}$, such that $d_+(v_0) = d_-(v_0) = 1$ in the above orientation, and such that the resulting directed graph is G-graceful.* □

Corollary 4.2 *Let F be an undirected graph such that $|E(F)| = n - 1$, and let G be a group of oder n. Then there is a G-regular decomposition $F|2K_n$ if and only if there is an orientation of F such that the resulting directed graph is G-graceful.* □

In Corollary 4.3 below we show that the result of Theorem 3.2 is actually a generalization of a result of Rosa on β-valuations. The notion of *β-valuation* was established by Rosa in [16], and was called *graceful labeling* by Golomb [7]. Its definition is as follows.

Definition 4.1 Let H be an undirected graph. An injective function $f : V(H) \to \{0, 1, \ldots, |E(H)|\}$ is called a *β-valuation* (*graceful labeling*) if $|f(x) - f(y)|$ are distinct for all $\{x, y\} \in E(H)$. A graph which admits a β-valuation will be called a *graceful* graph.

A comprehensive review of the subject may be found in [6]. Rosa introduced this notion as a tool for decomposing the complete graph into isomorphic subgraphs. In particular, β-valuation originated as a tool for attacking a conjecture of Ringel [18] which says:

Conjecture 4.1 *Let T be any tree on $n+1$ vertices. Then $T|K_{2n+1}$.*

It holds that if T is a tree on $n + 1$ vertices which admits a β-valuation, then $T|K_{2n+1}$. In the next corollary we show that the above result is a consequence of Theorem 3.2. First, we introduce the following notation and definition.

Notation 4.1 Let F be a graph on n vertices and let G be a group of order m. The problem of determining whether or not K_m admits a regular decomposition $F|K_m$ with a corresponding group G is denoted by $RP(G; F)$.

Definition 4.2 Let H be a graph with n edges and let G be a group of order $2n + 1$. Then H admits a *G-extended β-valuation* if there exists an injective function $f : V(H) \to G$ such that $G - \{1_G\} = \{f(v)f(u)^{-1} \mid \{u, v\} \in E(H)\}$

Corollary 4.3 *Let H be a graph with n edges and let G be a group of order $2n + 1$. Then*

(i) $RP(G; H)$ has a solution if and only if H admits a G-extended β-valuation. In particular, we have:

(ii) if H admits a β-valuation, then $H|K_{2n+1}$.

Proof. Let H_1 be a directed graph with the same edge set as H, and arc set $E(H_1) = \{(u, v) \mid \{u, v\} \in E(H)\}$ (actually, H_1 results from H by replacing every edge by two arcs of opposite direction). Then, by Theorem 3.2, $RP^*(G; K)$ has a solution if and only if K is G-graceful. Since for any two distinct vertices $u, v \in V(K)$, with corresponding labels $g_u, g_v \in G$, it holds that $g_v g_u^{-1} = (g_u g_v^{-1})^{-1}$, it follows that K is G-graceful if and only if H admits a G-extended β-valuation. It holds also that $RP^*(G; K)$ has a solution if and only if $RP(G; H)$ has a solution. Hence (i) follows. Furthermore, if H admits a β-valuation, then it is easily observed that H admits a C_{2n+1}-extended β-valuation, where C_{2n+1} is the cyclic group of order $2n + 1$. Indeed, given a bijective function $f : V(H) \to \{0, 1, \ldots, n = |E(H)|\}$, then $|f(x) - f(y)|$ are distinct for all $\{x, y\} \in E(H)$ if and only if the set $\{f(x) - f(y) \mid \{x, y\} \in E(H)\}$ is equal to the set $\{-n, -(n-1), \ldots -1, 1, \ldots n\}$, and one easily observes that the above is equivalent to the existence of C_{2n+1}-extended β-valuation for H. It follows that if H admits a β-valuation, then $H|K_{2n+1}$, completing the proof. □

An unpublished result of Erdős says that most graphs are not graceful (cf. [8]). Sheppard [19] has shown that there are exactly

$e!$ graceful graphs with e edges. In particular, Rosa [16] has shown that if every vertex has an even degree and the number of edges is congruent to 1 or 2 (mod 4) then the graph is not graceful. A natural question which arise is whether the above non-graceful graphs have a G-extended β-valuation (a positive answer will show that such graphs, though not graceful, decompose K_{2n+1}). The discussion of this problem is beyond the scope of this paper. We only pose the following problem:

Problem 4.1 *Find examples of non-graceful graphs which admit a G-extended β-valuation for some group G.*

A (v,k,λ)*-quotient set* in a group G of order v is a k-subset $D \subsetneq G$ such that each $g \in G$ occurs exactly λ times in the list $(xy^{-1} \mid x, y \in D)$. When G is abelian the quotient set is called a *difference set* (see [20], pp. 330-331). It is well known that the existence of a (v,k,λ)-quotient set in G is equivalent to the existence of a symmetric (v,k,λ)-design that admits G as a regular group of automorphisms ([20], Theorem 27.2). We claim now that for $\lambda = 1$, the above result is a particular case of Theorem 3.2. Indeed, Let a $(v,k,1)$-symmetric design D be given, i.e., $D = (V,B)$, where $V = \{v_1, v_2, \ldots, v_n\}$ is a set of points, $B = \{B_1, B_2, \ldots, B_n\}$ is a set of n k-subsets of V called blocks, such that every pair $\{v_i, v_j\}$ of points is contained in exactly one block. It holds that $v - 1 = k(k-1)$ ([20], p. 196). Furthermore, assume that there exists a regular permutation group G on D. Then one may easily verify that the existence of such a symmetric design D (with the permutation group G) is equivalent to the existence of a G-regular decomposition $K_k|K_v$, and the existence of this decomposition is equivalent to the existence of a G-regular decomposition $K_k^*|K_v^*$. However, the existence of the later decomposition is equivalent, by Theorem 3.2, to the existence of a G-graceful labeling of K_k^*. Then, since the existence of a G-graceful labeling of K_k^* is equivalent to the existence of a $(v,k,1)$-quotient set in G, our claim follows.

We conclude this section by pointing out the connection between

G-graceful labeling of cycles and group sequencing. Given a group G of order n, a *sequencing* of G is a sequence $a_1, a_2, \ldots, a_n$ of all the (distinct) elements of G such that the partial products $a_1, a_1a_2, a_1a_2a_3, \ldots, a_1a_2 \cdots a_n$ are all distinct. In this case we say that the group G is *sequenceable* . The classification of all the sequenceable groups is a well known problem which is still unsolved (see [11] for a survey). However, various infinite families of sequenceable groups are known. It was shown in [12] that a group G of order n is sequenceable if and only if there is a G-regular factorization of K_n^* into hamiltonian (directed) cycles. It follows that there is a G-graceful labeling of the directed cycle of size n, C_n^*, if and only if the group G is sequenceable. A discussion on the connection between various generalizations of group sequencings and Oberwolfach factorizations of K_n^* and K_n is given in [12]. The existence of those sequencings (and factorizations) is equivalent to the existence of corresponding G-graceful labeling of factors which are a union of distinct cycles. We also mention the connection between the existence of R-sequencing of a group G of order n and the factorization of K_n^* into $(n-1)$-cycles, as discussed in [13] (an *R-sequencing* of a group G of order n is a sequence $a_0 = 1, a_1, a_2, \ldots, a_{n-1}$ of all the (distinct) elements of G such that $a_0a_1 \cdots a_{n-1} = 1$ and so that the partial products $a_0, a_0a_1, a_0a_1a_2, \ldots a_0a_1 \cdots a_{n-1}$ are all distinct). The existence of an R-sequencing of a group G of order n is equivalent to the existence of a G-graceful labeling of the cycle C_{n-1}^*.

5 Regular decomposition of $K_n - I$

Let n be an even integer. In this section we show that results similar to those of the previous sections hold for the graphs $(K_n - I)^*$, $2(K_n - I)$ and $K_n - I$ (where $(K_n - I)^*$ and $2(K_n - I)$ are the analogs of K_n^* and $2K_n$ for the graph $K_n - I$). The corresponding regular decomposition problem for $(K_n - I)^*$, $2(K_n - I)$ and $K_n - I$ will be denoted by $RP_I^*(G; F)$, $RP_{2I}(G; F)$ and $RP_I(G; F)$, respectively. Similarly to the previous sections, we shall use the notation

$F|(K_n - I)^*$, etc. We shall discuss some analogs to the Ringel conjecture for these graphs. Then we shall apply our results to obtain solutions to some decomposition problems which are related to the above conjectures.

The following results are proved by arguments similar to those used in the proofs in Section 3. Thus, we omit their proofs.

Proposition 5.1 *Let n be an even integer, let F be a directed graph such that $|E(F)| = n-2$, and let G be a group of order n which acts regularly on a decomposition $F|(K_n - I)^*$. Then G acts regularly on $V((K_n - I)^*)$.*

Theorem 5.1 *Let n be an even integer, and let F be a directed graph such that $|E(F)| = n - 2$, and let G be a group of order n. Then $RP_I^*(G; F)$ has a solution if and only if F is G-graceful.*

Remark 5.1 We note that in proving the "if" direction of the theorem, we construct a G-graceful labeling of $V(F)$. Then the orbit of $E(F)$ under the action of G results in an $(n-2)$-regular graph on n vertices which is decomposed by F. Since every $(n-2)$-regular graph on n vertices is isomorphic to $K_n - I$, the result follows.

Corollary 5.1 *Let F be an undirected graph such that $|E(F)| = n - 2$ and let G be a group of even order n. Then $RP_{2I}(G; F)$ has a solution if and only if there is an orientation of F such that the resulting graph is G-graceful.*

We apply now our results to the decomposition of the (undirected) graph $K_n - I$. First, we introduce the following extension of Definition 4.2.

Definition 5.1 *Let H be a graph with $n-1$ edges and let G be a group of order $2n$ having a unique involution u. Then H admits a G-extended β-valuation if there exists an injective function $f: V(H) \to G$ such that $G - \{1_G, u\} = \{f(v)f(w)^{-1} | \{w, v\} \in E(H)\}$.*

The proof of the following corollary is similar to that of Corollary 4.3.

Corollary 5.2 *Let H be a graph with $n-1$ edges and let G be a group of order $2n$ having a unique involution. Then there is a G-regular decomposition $H|(K_{2n} - I)$ if and only if H admits a G-extended β-valuation.*

The above results lead us to the following 'Ringel-type' conjecture.

Conjecture 5.1 *Let T be a tree on $n-1$ vertices, n even. Then $T|2(K_n - I)$.*

We pose also the following conjecture, which is a special case of a conjecture of Häggkvist [9] :

Conjecture 5.2 *Let T be a tree on n vertices. Then $T|(K_{2n} - I)$.*

In the following, we use our results to solve particular cases of the above conjectures. First we apply Corollary 5.1 to solve Conjecture 5.1.1 for a special family of trees.

Theorem 5.2 *Let T be the binary balanced tree on $n-1 = 2^k - 1$ vertices, $k \geq 2$. Then $T|2(K_n - I)$.*

Proof. Let T^* be the directed tree which results by orienting each edge of T in the direction from the root to the leaves. By Corollary 5.1, it suffices to show that $T^*|K_n^*$. Let G be the elementary abelian group of order $n = 2^k$. By Theorem 5.1, it suffices to show that T^* is G-graceful. Denote the root of T by v_0, and denote $G = \{0, u_1, \ldots, u_{n-1}\}$ (Since G is abelian, we shall use additive notation in G). We shall describe now how to obtain, recursively, a G-graceful labeling of T^*, such that $\{f(v)|v \in V(T)\} = G - \{0\}$.

Assume first that $k = 2$, and denote $V(T^*) = \{v_0, v_1, v_2\}$. Labeling T^* by $f(v_0) = u_1$, $f(v_1) = u_2$, $f(v_2) = u_3$ results in the required labeling. We note that in this labeling, $\{f(v_i) - f(v_j) | (v_i, v_j) \in E(T^*\} = G - \{0, u_1\}$.

Assume now that $k = 3$. Then we may write $V(T^*) = \{v_0, v_1, \ldots, v_6\}$, where the subtrees T_1,T_2 on the vertices $\{v_1, v_2, v_3\}$ and $\{v_4, v_5, v_6\}$, respectively, are binary directed balanced subtrees on $2^{k-1} - 1 = 3$ vertices, with roots v_1 and v_4, respectively. Let $H = \{0, u_1, u_2, u_3\}$ be a subgroup of G of index 2. Then we have a disjoint union $G = H \cup (H + x)$ for some $x \in G - H$. Now, we use the labeling of the case $k = 2$ in order to obtain a labeling of T^*. For each level l_i of the tree on 3 vertices ($i = 1, 2$), we label the vertices of level $i + 1$ in T^* by corresponding elements from H and $H + x$ as follows. In level l_1 of T^* let $f(v_1) = u_1$ and $f(v_4) = u_1 + x$, and in level l_2 of T^* let $f(v_2) = u_2$, $f(v_3) = u_2 + x$, $f(v_5) = u_3$ and $f(v_6) = u_3 + x$. Then we have: $\{f(v_i) | v_i \in V(T^*)\} = G - \{0, x\}$ and $\{f(v_i) - f(v_j) | (v_i, v_j) \in E(T^*), i, j \neq 0\} = G - \{0, u_1, u_1 + x\}$. Then, setting $f(v_0) = 0$, results in a G-graceful labeling of T^*. Since the element x of G was not used in the above labeling, we may add x to all the labels used for T^* to obtain a G-graceful labeling for T^* such that $\{f(v_i) | v_i \in V(T^*)\} = G - \{0\}$, as required.

For $k > 3$ we use the procedure described in the previous paragraph to obtain the required G-graceful labeling using the labeling for the binary balanced tree on $2^{k-1} - 1$ vertices. □

In the following, we deal with the 'simplest case' of Conjecture 5.1, namely, when T is a simple path. On the other hand, we shall deal with the 'harder' problem of this case: the classification of all the groups G for which the corresponding G-regular decomposition exists (see the second basic problem in the introduction). Partial solutions to this problem are given in the following propositions. Recall the notions of sequenceable and R-sequenceable groups mentioned at the end of Section 4.

Proposition 5.2 *Let n be an even integer, and let T be a simple path on $n - 1$ vertices. Let G be a group of order n which is either sequenceable or R-sequenceable. Then there is a G-regular decomposition $T|2(K_n - I)$. In particular, there is such a decomposition for* *the following groups:*

(i) All solvable groups with a unique involution, except the quaternion group Q_4. In particular, all cyclic groups of even order.

(ii) All the Dihedral groups, except D_3.

(iii) All the dicyclic groups, except the quaternion group.

Proof. Suppose first that G is sequenceable. Then it is easy to show (see, for example, [12], Corollary 3.1) that there exists a sequence $g_1, g_2, \ldots, g_n$ of all the elements of G, such that the differences of the sequence $g_2 g_1^{-1}, g_3 g_2^{-1}, \ldots, g_n g_{n-1}^{-1}$ are all distinct. Now, by considering the given path T as a directed path, and by labeling the vertices $g_1, g_2, \ldots, g_{n-1}$ according to this direction, we obtain a G-graceful labeling. Hence there is a G-regular decomposition $T|2(K_n - I)$ by Corollary 5.1.

Next, suppose that G is R-sequenceable. Then it is easy to show (see [13], Lemma 4.1) that there exists a sequence $g_1, g_2, \ldots, g_{n-1}$ of distinct elements of G, such that the differences of the sequence are distinct. Similar to the above, we obtain the result also in this case.

The families (i),(ii),(iii) satisfy the stated property, since all the groups mentioned are sequenceable by [11], except D_4, which is R-sequenceable. □

Proposition 5.3 *Let T be a simple path on n vertices, and let G be a solvable group of order $2n$ with a unique involution. Assume G is not the quaternion group Q_4. Then there is a G-regular decomposition $T|(K_{2n} - I)$.*

Proof. By [2], G has a symmetric sequencing, i.e., a sequencing $a_0 = 1, a_1, a_2, \ldots, a_{2n-1}$, such that $a_n = u$, the unique involution of G, and $(a_{n+i})^{-1} = a_{n-i}$ for $1 \leq i \leq n-1$. Define $g_0 = (a_0)^{-1} = 1, g_1 = (a_0 a_1)^{-1}, g_2 = (a_0 a_1 a_2)^{-1}, \ldots, g_{n-1} = (a_0 a_1 \ldots a_{n-1})^{-1}$. We obtain a sequence $g_0, g_1, \ldots, g_{n-1}$ such that the differences of the sequence are all distinct. Furthermore, for each $h \in G - \{1, u\}$, exactly one element from the set $\{h, h^{-1}\}$ is a difference of the sequence. It follows that by considering the given path T as a directed path, and by labeling the vertices $g_0, g_1, \ldots, g_{n-1}$ according to this direction, we obtain that T admits a G-extended β-valuation. This completes the proof. □

Remark 5.2 By the odd order theorem of Feit and Thompson, it follows that for an odd n, Proposition 5.3 holds for *any* group of order $2n$ with a unique involution.

We end this section by considering the factorization of k-regular directed graphs for $k < n - 2$ (a k-regular directed graph results from a k-regular graph by replacing each edge by two opposite directed arcs). Since in such cases, the family of k-regular graphs $(2 < k < n-2)$ contains various non-isomorphic graphs (where some of them may not be vertex transitive), the result of Theorem 5.1 (and its corollaries) cannot be directly extended to every k-regular graph (see also Remark 5.1). However, the existence of a k-regular graph with the required factorization may be obtained, as stated in Proposition 5.4 below (the proof of this proposition is omitted since it uses arguments which have been used previously). We include first a definition.

Definition 5.2 Let F be a directed graph with $|E(F)| = k$, let H be a k-regular directed graph and let G be a group of order n. We shall say that H admits a G-regular decomposition into factors isomorphic to F if there exists a decomposition of H into subgraphs isomorphic to F such that G acts regularly on the factors of the decomposition. In this case we shall say that the problem $RP^*(F, H, G)$ has a solution

Proposition 5.4 *Let G be a group of order n and let F be a directed graph such that $|E(F)| = k$, where $1 \leq k < n$. Then the following holds:*

1. *If F is G-graceful, then there exists a k-regular directed graph H such that $RP^*(F, H, G)$ has a solution.*

2. *If H is a k-regular directed graph such that $RP^*(F, H, G)$ has a solution, then F is G-graceful.*

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