On Weak Magic Graphs

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ABSTRACT An *edge-magic total labeling* on a graph with p vertices and q edges is defined as a one-to-one map taking the vertices and edges onto the integers 1, 2, ..., $p + q$ with the property that the sum of the labels on an edge and of its endpoints is constant, independent of the choice of edge. The *magic strength* of a graph G, denoted by $emt(G)$ is defined as the minimum of all constants over all edge-magic total labelings of G. The *maximum magic strength* of a graph G, denoted by $eMt(G)$ is defined as the maximum constant over all edge-magic total labelings of G. A graph G is called *weak magic* if $eMt(G) - emt(G) > p$. In this paper we study some classes of weak magic graphs.

AMS Mathematics Subject Classification : 05C78

Keywords : Edge-magic total labeling, Maximum magic strength, Weak magic graphs

1. Introduction

All graphs in this paper are finite and have no loops or multiple edges. In general we follow the graph theoretic-terminology and notation of [2] unless otherwise specified.

The subject of edge-magic labelings of graphs had its origin in the work of Kotzig and Rosa [4, 5] on what they called magic valuations of graphs. They define a magic labeling to be a total labeling in which the labels are the integers from 1, 2, 3, ..., $p + q$. The sum of the labels on an edge and its two endpoints is constant. These labelings are currently referred to as either edge-magic total labelings or edge-magic labelings. These terms were coined by Ringel and Llado[6], and Wallis[7] respectively.

Let G be a graph with p vertices and q edges. A bijection f from $V(G) \cup E(G)$ to $\{1, 2, 3, \ldots, p+q\}$ is called an *edge-magic total labeling* of G if there exists a constant $k(f)$ (called the *magic constant* or *magic number* or *magic sum of* f) such that $f(u) + f(v) + f(uv) = k(f)$ for any edge uv of G. In such a case, G is said to be *edge-magic total.*

Avadayappan et al. [1] introduced the notion of magic strength of a graph. We know that for any edge-magic total labeling f of G, there is a constant $k(f)$ such that $f(u)+f(v)+f(uv)=k(f)$ for any edge uv of G. The *magic strength* of G, denoted by $emt(G)$ is defined as the minimum of all $k(f)$ where the minimum is taken over all edge-magic total labelings of G.

Hegde and Shetty [3] introduced the concept of maximum magic strength of a graph. The *maximum magic strength* of G, denoted by $eMt(G)$ is defined as the maximum of all $k(f)$ where the maximum is taken over all edge-magic total labelings of G. They also called a graph G strong magic if $emt(G)=eMt(G)$; ideal magic if $1\le eMt(G)-emt(G)\le p$ and weak magic if $eMt(G)-emt(G)>p$. They proved that P_n is ideal magic for $n>2$; $K_{1,2}$, $K_{1,3}$ and cycles are ideal magic; $K_{1,1}$ and $(2n+1)P_2$ are strong magic and $K_{1,n}$ is weak magic for $n>3$.

We note the following fact also. Let f be an edge-magic total labeling of a graph G with magic constant $k(f)$. Then $f(u)+f(v)+f(uv)=k(f)$ for every edge uv of G. Now adding all constants obtained at each edge of G, we get

$$qk(f)=\sum_{u\in V(G)} f(u)d(u)+\sum_{uv\in E(G)} f(uv) \qquad (1)$$

We will find the following result useful ; see[3].

Lemma 1 For an edge-magic total graph G with p vertices and q edges, $eMt(G)=3(p+q+1)-emt(G)$.

2. On edge-magic total graphs

If G_1 and G_2 are graphs and G_1 has n vertices, then the Corona of G_1 and G_2, denoted by G_1/G_2, is the graph obtained by taking one copy of G_1 and n copies of G_2, and then joining the ith vertex of G_1 with an edge to every vertex in the ith copy of G_2. In this section we consider the graphs C_n/P_2 and C_n/P_3 for all odd $n\ge 3$.

Theorem 2.1 *For all odd $n\ge 3$, the graph C_n/P_2 has an edge-magic total labeling with magic constant* $k(f)=\dfrac{27n+3}{2}$

Proof. Let n be an odd integer and $n=2m+1\ge 3$. Let $v_1, v_2, \dots, v_n$ be the vertices of the cycle C_n. Now C_n/P_2 is the graph obtained by attaching P_2 to each vertex of C_n. Let a_i, b_i, $1\le i\le n$ be the vertices adjacent to the rim vertices v_i of C_n in C_n/P_2. The graph C_n/P_2 has $3n$ vertices and $4n$ edges. Define a labeling

$f: V(C_n/P_2)\cup E(C_n/P_2)\rightarrow \{1,2,3,\dots,7n\}$ such that

$$f(v_i)=\begin{cases}\dfrac{14n+1-i}{2} & \text{if } i \text{ is odd}\\[2mm] \dfrac{13n+1-i}{2} & \text{if } i \text{ is even}\end{cases}$$

$$f(a_i)=5n+i,\ 1\le i\le n, \qquad f(b_1)=\frac{9n+1}{2}$$

$$f(b_i)=\begin{cases}5n+1-\dfrac{i}{2} & \text{if } i \text{ is even}\\[2mm] \dfrac{9n+2-i}{2} & \text{if } i \text{ is odd, } i\ne 1\end{cases}$$

$$f(v_iv_{i+1})=i+1 \text{ for } 1\le i\le (n-1), \qquad f(v_nv_1)=1$$

$$f(v_ia_i)=\begin{cases}\dfrac{3n+2-i}{2} & \text{if } i \text{ is odd}\\[2mm] 2n+1-\dfrac{i}{2} & \text{if } i \text{ is even}\end{cases}$$

$$f(v_ib_i)=2n+i \quad \text{for } 1\le i\le n$$

$$f(a_ib_i)=\begin{cases}\dfrac{8n+1-i}{2} & \text{if } i \text{ is odd}\\[2mm] \dfrac{7n+1-i}{2} & \text{if } i \text{ is even}\end{cases}$$

It is easily verified that for odd $n\ge 3$, f is an edge-magic total labeling of C_n/P_2 with magic constant $k(f)=\dfrac{27n+3}{2}$.

Example. Figure 1 shows the edge-magic total labeling of C_5/P_2 with $k(f)=69$.

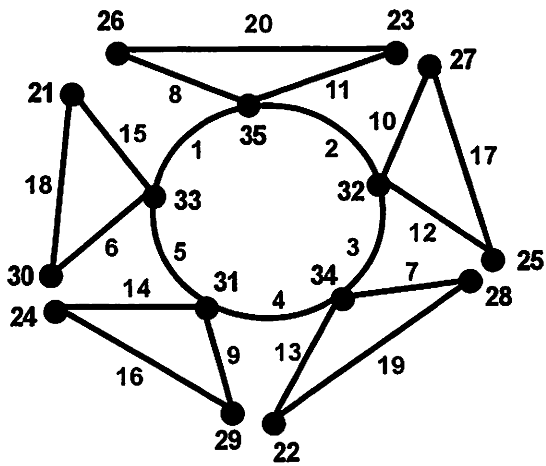

Figure 1

Theorem 2.2 *For all odd* $n \geq 3$, *the graph* C_n / P_3 *has an edge-magic total labeling with magic constant* $k(f) = \dfrac{39n+3}{2}$.

Proof. Let C_n be an odd cycle with $n = 2m+1 \geq 3$ vertices. Let $v_1, v_2, \ldots, v_n$ be the vertices of the cycle C_n. Let P_3 be a path on three vertices. Now C_n / P_3 is the graph obtained by attaching P_3 to each vertex of C_n and it has $4n$ vertices and $6n$ edges. Define a labeling f: $V(C_n / P_3) \cup E(C_n / P_3) \to \{1, 2, 3, \ldots, 10n\}$ such that

$$f(v_i) = \begin{cases} \dfrac{20n+1-i}{2} & \text{if } i \text{ is odd} \\ \dfrac{19n+1-i}{2} & \text{if } i \text{ is even} \end{cases}$$

Let us label the $3n$ vertices outside the rim of C_n in C_n / P_3 as follows. Let $u_1, u_2, \ldots, u_m$ be the vertices of degree two outside the rim, adjacent to the rim vertices whose f-values are $\dfrac{19n-3}{2}, \dfrac{19n+1}{2}, \dfrac{19n+5}{2}, \ldots, (10n-1)$

respectively. Again let $u_{n+1}, u_{n+2}, \ldots, u_{n+m}$ be the remaining vertices of degree two, adjacent to the rim vertices whose f-values are $\frac{19n-3}{2}, \frac{19n+1}{2}, \frac{19n+5}{2}, \ldots, (10n-1)$ respectively. Let $u_{m+1}, u_{m+2}, \ldots, u_n$ be the vertices of degree two outside the rim, adjacent to the rim vertices whose f-values are $9n+1, 9n+3, 9n+5, \ldots, 10n$ respectively. Also let $u_{n+m+1}, u_{n+m+2}, \ldots, u_{2n}$ be the remaining vertices of degree two outside the rim, adjacent to the rim vertices whose f-values are $9n+1, 9n+3, 9n+5, \ldots, 10n$ respectively. Let $u_{2n+1}, u_{2n+2}, \ldots, u_{2n+m+1}$ be the vertices of degree three outside the rim adjacent to the rim vertices whose f-values are $9n+1, 9n+3, 9n+5, \ldots, 10n$ respectively. Finally, let $u_{2n+m+2}, u_{2n+m+3}, \ldots, u_{3n}$ be the vertices of degree three adjacent to the rim vertices whose f-values are $9n+2,\ 9n+4, 9n+6, \ldots, 10n-1$ respectively.

Now define $f(u_i) = 9n + 1 - i$ for $1 \le i \le 3n$.

$f(v_i v_{i+1}) = i + 1$ for $1 \le i \le (n-1)$, $f(v_n v_1) = 1$

$$f(u_i u_j) = \frac{3n-1}{2} + (i+j),\ 1 \le i \le 3n,$$

$$f(v_i u_j) = \begin{cases} \dfrac{n+i+2j}{2} & \text{if } i \text{ is odd} \\ \dfrac{2n+i+2j}{2} & \text{if } i \text{ is even} \end{cases},\ 1 \le i \le n, 1 \le j \le 3n$$

It is easy to verify that C_n / P_3 is an edge-magic total graph with magic constant $k(f) = \frac{39n+3}{2}$, when $n \ge 3$ is odd.

Example. Figure 2 shows the edge-magic total labeling of C_7 / P_3 with $k(f) = 138$.

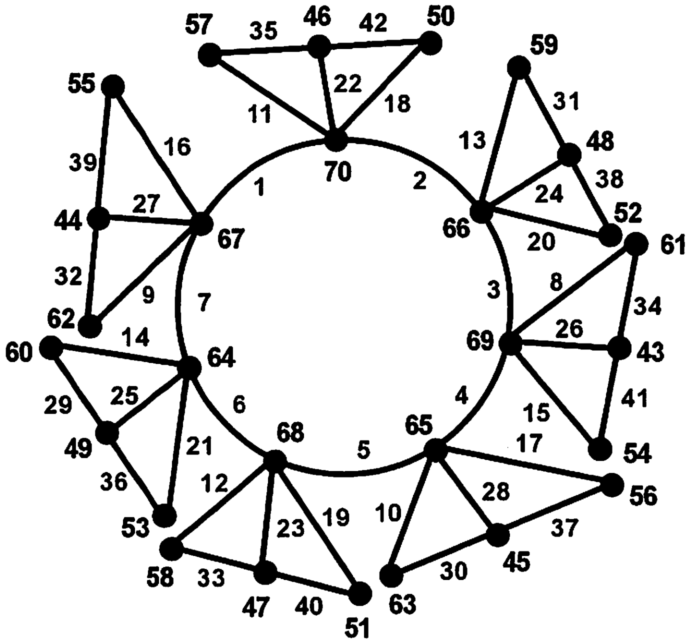

Figure 2

3. Maximum magic strength of graphs

In this section we find the maximum magic strength of some graphs.

Theorem 3.1. $eMt(C_n/P_2) = \frac{27n+3}{2}$ *for all odd* $n \geq 3$.

Proof. By Theorem 2.1, $eMt(C_n/P_2) \geq \frac{27n+3}{2}$ for all odd $n \geq 3$. Let f be an edge-magic total labeling of the graph C_n/P_2 with magic constant $k(f)$. Then using (1) with $p = 3n$ and $q = 4n$ we get

$$4nk(f) = 2\sum_{i=1}^{n} f(a_i) + 2\sum_{i=1}^{n} f(b_i) + 4\sum_{i=1}^{n} f(v_i) + \sum_{e \in E} f(e),$$

where $a_i, b_i, (1 \leq i \leq n)$ are the vertices adjacent to the rim vertices v_i of C_n in C_n/P_2.

Again,

$$4nk(f) = \left[\sum_{i=1}^{n} f(a_i) + \sum_{i=1}^{n} f(b_i) + \sum_{i=1}^{n} f(v_i) + \sum_{e\in E} f(e) \right] +$$

$$\left[\sum_{i=1}^{n} f(a_i) + \sum_{i=1}^{n} f(b_i) + \sum_{i=1}^{n} f(v_i) \right] + 2\sum_{i=1}^{n} f(v_i)$$

$$= (1+2+\cdots\cdots+7n) + \left[\sum_{i=1}^{n} f(a_i) + \sum_{i=1}^{n} f(b_i) + \sum_{i=1}^{n} f(v_i) \right] +$$

$$2\sum_{i=1}^{n} f(v_i)$$

$$= \frac{7n(7n+1)}{2} + \left[\sum_{i=1}^{n} f(a_i) + \sum_{i=1}^{n} f(b_i) + \sum_{i=1}^{n} f(v_i) \right] + 2\sum_{i=1}^{n} f(v_i).$$

Hence,

$$k(f) \le \frac{1}{4n}\left[\frac{7n(7n+1)}{2} + \{(4n+1)+(4n+2)+\cdots\cdots+7n\} + 2\,\{(6n+1)+(6n+2)+\cdots\cdots+7n\}\right]$$

$$= \frac{1}{4n}\left[\frac{7n(7n+1)}{2} + \frac{3n}{2}\{4n+1+7n\} + 2\,\frac{n}{2}\{6n+1+7n\}\right]$$

$$= \frac{1}{4}\left[\frac{7(7n+1)}{2} + \frac{3}{2}\{11n+1\} + \frac{2}{2}\{13n+1\}\right]$$

$$= \frac{1}{4}\left[\frac{108n+12}{2}\right]$$

$$= \frac{27n+3}{2}$$

Thus, $eMt(C_n/P_2) \le \dfrac{27n+3}{2}$ for all odd $n \ge 3$.

Therefore, $eMt(C_n/P_2) = \dfrac{27n+3}{2}$ for all odd $n \ge 3$.

Theorem 3.2 $\dfrac{39n+3}{2} \le eMt(C_n/P_3) \le \dfrac{40n+3}{2}$ *for all odd* $n \ge 3$.

Proof. By Theorem 2.2, $eMt(C_n/P_3) \geq \frac{39n+3}{2}$ for all odd $n \geq 3$.

Let $a_i (1 \leq i \leq 2n)$ denote the vertices of degree two, $b_i (1 \leq i \leq n)$ denote the vertices of degree three and $v_i (1 \leq i \leq n)$ denote the vertices of degree five in C_n/P_3. Let f be an edge-magic total labeling of the graph C_n/P_3 with magic constant $k(f)$. Then using (1) with $p = 4n$ and $q = 6n$ we get

$$6nk(f) = 2\sum_{i=1}^{2n} f(a_i) + 3\sum_{i=1}^{n} f(b_i) + 5\sum_{i=1}^{n} f(v_i) + \sum_{e \in E} f(e)$$

$$= \left[\sum_{i=1}^{2n} f(a_i) + \sum_{i=1}^{n} f(b_i) + \sum_{i=1}^{n} f(v_i) + \sum_{e \in E} f(e) \right] +$$

$$\left[\sum_{i=1}^{2n} f(a_i) + \sum_{i=1}^{n} f(b_i) + \sum_{i=1}^{n} f(v_i) \right] + \sum_{i=1}^{n} f(b_i) + 3\sum_{i=1}^{n} f(v_i)$$

$$= (1 + 2 + \cdots + 10n) +$$

$$\left[\sum_{i=1}^{2n} f(a_i) + \sum_{i=1}^{n} f(b_i) + \sum_{i=1}^{n} f(v_i) \right] + \sum_{i=1}^{n} f(b_i) + 3\sum_{i=1}^{n} f(v_i)$$

Thus,

$$k(f) = \frac{1}{6n}\left[\frac{10n(10n+1)}{2} + \left\{\sum_{i=1}^{2n} f(a_i) + \sum_{i=1}^{n} f(b_i) + \sum_{i=1}^{n} f(v_i)\right\} + \sum_{i=1}^{n} f(b_i) + 3\sum_{i=1}^{n} f(v_i)\right]$$

$$\leq \frac{1}{6n}\left[\frac{10n(10n+1)}{2} + \{(6n+1) + (6n+2) + \cdots + 10n\} + 4\{(9n+1) + (9n+2) + \cdots + 10n\}\right]$$

$$= \frac{1}{6n}\left[5n(10n+1) + \frac{4n}{2}\{6n+1+10n\} + 4\,\frac{n}{2}\{9n+1+10n\}\right]$$

$= \frac{1}{6}[5(10n+1)+2(16n+1)+2(19n+1)]$

$= \frac{1}{6}[120n+9]$

$= \frac{40n+3}{2}$

Hence, $eMt(C_n/P_3) \le \frac{40n+3}{2}$ for all odd $n \ge 3$.

Therefore, $\frac{39n+3}{2} \le eMt(C_n/P_3) \le \frac{40n+3}{2}$ for all odd $n \ge 3$.

4. New classes of weak magic graphs

Theorem 4.1 *The graph* C_n/P_2 *is weak magic for all odd* $n \ge 3$.

Proof. By Theorem 3.1, $eMt(C_n/P_2) = \frac{27n+3}{2}$ for all odd $n \ge 3$. Using Lemma 1 we get, $emt(C_n/P_2) = 3(7n+1) - \frac{27n+3}{2} = \frac{15n+3}{2}$ for all odd $n \ge 3$. Hence, $eMt(C_n/P_2) - emt(C_n/P_2) = \frac{27n+3}{2} - \frac{15n+3}{2} = 6n > 3n$ for all odd $n \ge 3$. Therefore, C_n/P_2 is weak magic for all odd $n \ge 3$.

Theorem 4.2 *The graph* C_n/P_3 *is weak magic for all odd* $n \ge 3$.

Proof. By Theorem 3.2, $\frac{39n+3}{2} \le eMt(C_n/P_3) \le \frac{40n+3}{2}$ for all odd $n \ge 3$.

Using Lemma 1 we get,

$3(10n+1) - \frac{40n+3}{2} \le emt(C_n/P_3) \le 3(10n+1) - \frac{39n+3}{2}$ for all odd $n \ge 3$.

i.e., $\frac{20n+3}{2} \le emt(C_n/P_3) \le \frac{21n+3}{2}$ for all odd $n \ge 3$.

Hence, for all odd $n \ge 3$,

$$\frac{39n+3}{2} - \frac{21n+3}{2} \leq eMt(C_n/P_3) - emt(C_n/P_3) \leq \frac{40n+3}{2} - \frac{20n+3}{2}$$

i.e., $9n \leq eMt(C_n/P_3) - emt(C_n/P_3) \leq 10n$.

Thus, $eMt(C_n/P_3) - emt(C_n/P_3) > 4n$ for all odd $n \geq 3$.

Therefore, C_n/P_3 is weak magic for all odd $n \geq 3$.

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