Study of Some Problems of Cordial Graphs

Adel T. Diab
Faculty of Science, Department of Mathematics,
Ain Shams University
Abbassia, Cairo, Egypt.

Abstract. A graph is said to be cordial if it has a 0-1 labeling that satisfies certain properties. The purpose of this paper is to generalize some known theorems and results of cordial graphs. Specifically, we show that certain combinations of paths, cycles and stars are cordials.
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1 Introduction

It is well known that graph theory has applications in many other fields of study, including physics, chemistry, biology, communication, psychology, sociology, economics, engineering, operations research, and especially computer science.

One area of graph theory of considerable recent research is that of graph labeling. In a labeling of a particular type, the vertices are assigned values from a given set, the edges have a prescribed induced labeling, and the labelings must satisfy certain properties. An excellent reference on this subject is the survey by Gallian [3].

Two of the most important types of labelings are called graceful and harmonious. Graceful labelings were introduced independently by Rosa [6] in 1966 and Golomb [4] in 1972, while harmonious labelings were first studied by Graham and Sloane [5] in 1980. A third important type of labeling, which contains aspects of both of the other two, is called cordial and was introduced by Cahit [1] in 1990. Whereas the label of an edge vw for graceful and harmonious labeling is given respectively by $|f(v) - f(w)|$ and $f(v) + f(w)$ (modulo the number of edges), cordial labelings use only labels 0 and 1and the induced label $(f(v) + f(w))$ (mod2), which of course equals $|f(v) - f(w)|$. Because arithmetic modulo2 is an integral part of computer science, cordial labelings have close connections with that field.

More precisely, cordial graphs are defined as follows.
Let $G = (V, E)$ be a graph, let $f : V \rightarrow \{0, 1\}$ be a labeling of its vertices, and let $f^* : E \rightarrow \{0, 1\}$ is the extension of f to the edges of G by the formula $f^*(vw) = f(v) + f(w) \pmod 2$. (Thus, for any edge $e, f^*(e) = 0$ if its two vertices have the same label and $f^*(e) = 1$ if they have different labels). Let v_0 and v_1 be the numbers of vertices labeled 0 and 1 respectively, and let e_0 and e_1 be the corresponding numbers of edge. Such a labeling is

called cordial if both $|v_0 - v_1| \leq 1$ and $|e_0 - e_1| \leq 1$. A graph is called cordial if it has a cordial labeling.

Diab and Elsakhawi [2], determined that if m is not congruent to 3(mod 4) and $n \geq 3$, then the join of the cycle C_m and the path P_n is cordial. In section 3, we extend this result to show that the join of the cycle C_m and the path P_n is cordial for all m and all n if and only if $(m, n) \neq$ (3, 3), (3, 2) and (3,1). In section 4, we generalize the result due to Seoud, Diab and Elsakhawi [7], which state that the join of the path P_n and the star $K_{1,m}$ is cordial for all n and all odd m except $(n, m) =$ (2,1), to the join of the path P_n and the star $K_{1,m}$ is cordial for all n and all m if and only if $(n, m) \neq$ (2,1). Moreover, we show that the union of the path P_n and the star $K_{1,m}$ is cordial for all n and all m except $(n, m) =$ (2,1).

2 Terminology and notations

We introduce some notation and terminology for a graph with $4r$ vertices, we let L_{4r} denote the labeling 00110011...0011, S_{4r} denote the labeling 11001100...1100 and O_r denotes the labelling 0000...0000 (zero repeated r- times) , I_r denotes the labelling 111...1111 (one repeated r- times). In most cases, we then modify this by adding symbols at one end or the other (or both). Thus $01L_{4r}$ denotes the labeling 0100110011...0011 of either C_{4r+2} or P_{4r+2}. One exception to this is the labeling L'_{4r} obtained from L_{4r} by adding an initial 0 and deleting the last 1: that is, L'_{4r} is 000110011...11001. For specific labeling L and M of $G \cup H$, where G and H are paths or cycles or stars or null graphs, we let $[L; M]$ denote the joint labeling.

Additional notation that we use is the following.
For a given labeling of the join $G + H$, we let v_i and e_i (for i = 0,1) be the numbers of labels that are i as before, we let x_i and a_i be the corresponding quantities for G, and we let y_i and b_i be those for H. It follows that $v_0 = x_0 + y_0, v_1 = x_1 + y_1, e_0 = a_0 + b_0 + x_0y_0 + x_1y_1$ and $e_1 = a_1 + b_1 + x_0y_1 + x_1y_0$, thus, $v_0 - v_1 = (x_0 - x_1) + (y_0 - y_1)$ and $e_0 - e_1 = (a_0 - a_1) + (b_0 - b_1) + (x_0 - x_1)(y_0 - y_1)$. When it comes to the proof, we only need to show that, for each specified combination of labeling, $|v_0 - v_1| \leq 1$ and $|e_0 - e_1| \leq 1$.

3 Joins of Cycles and Paths

Diab and Elsakhawi [2] have shown that if m is not congurent to 3(mod 4) and $n \leq 3$, then $C_m + P_n$ is cordial. In this section, we introduce a new, short technique to re-prove the last result in order to make the paper self - contained and we extend this result to show that the join $C_m + P_n$ is cordial for all m and n if and only if $(m, n) \neq$ (3,3), (3,2) and (3,1).

The labeling that we use are given in Table 3.1, along with the corresponding values of x_i and a_i or y_i and b_i (for i = 0,1). We let $m = 4r + i$ (for i = 0,1,2) and $n = 4s + j$ (for j = 0,1,2,3).

$m = 4r + i$, $i = 0, 1, 2$	Labeling of C_m	x_0	x_1	a_0	a_1
$i = 0$	$A_0 = L_{4r}$	$2r$	$2r$	$2r$	$2r$
$i = 1$	$A_1 = 1L_{4r}$	$2r$	$2r+1$	$2r+1$	$2r$
$i = 2$	$A_2 = 11L_{4r}$	$2r$	$2r+2$	$2r+2$	$2r$
	$A_2' = 01L_{4r}$	$2r+1$	$2r+1$	$2r$	$2r+2$

$n = 4s + j$, $j = 0, 1, 2, 3$	Labeling of P_n	y_0	y_1	b_0	b_1
$j = 0$	$B_0 = L_{4s}$	$2s$	$2s$	$2s$	$2s-1$
	$B_0' = L_{4s}'$	$2s+1$	$2s-1$	$2s$	$2s-1$
$j = 1$	$B_1 = L_{4s}0$	$2s+1$	$2s$	$2s$	$2s$
$j = 2$	$B_2 = 0L_{4s}0$	$2s+2$	$2s$	$2s+1$	$2s$
	$B_2' = 01L_{4s}$	$2s+1$	$2s+1$	$2s+1$	$2s$
$j = 3$	$B_3 = L_{4s}001$	$2s+2$	$2s+1$	$2s+1$	$2s+1$

Table 3.1. Labelings of Cycles and Paths.

Theorem 3.1. If m is not congruent to 3 (mod 4) and $n \geq 3$, then $C_m + P_n$ is cordial.

Proof. For given values of i and j with $0 \leq i \leq 2$ and $0 \leq j \leq 3$, we use the labeling A_i or A_i' for the cycle C_m and B_j or B_j' for the path P_n as given in Table 3.1. Using Table 3.1 and the fact that $v_0 - v_1 = (x_0 - x_1) + (y_0 - y_1)$ and $e_0 - e_1 = (a_0 - a_1) + (b_0 - b_1) + (x_0 - x_1)(y_0 - y_1)$, we vcan compute the values shown in the last two columns of Table 3.2. Since these are all 0,1, or -1, the theorem follows.

$m=4r+i$, $i=0,1,2$	$n=4s+j$, $j=0,1,2,3$	C_m	P_n	v_0-v_1	e_0-e_1
0	0	A_0	B_0	0	1
0	1	A_0	B_1	1	0
0	2	A_0	B_2'	0	1
0	3	A_0	B_3	1	0
1	0	A_1	B_0'	1	0
1	1	A_1	B_1	0	0
1	2	A_1	B_2	1	0
1	3	A_1	B_3	0	0
2	0	A_2'	B_0	0	-1
2	1	A_2	B_1	-1	0
2	2	A_2'	B_2'	0	-1
2	3	A_2	B_3	-1	0

3.2. Combinations of labelings.

Theorem 3.2. If $m \equiv 3 \pmod 4$ and $m \geq 3$, then $C_m + P_n$ is cordial for all $n \geq 3$.
Proof. We use a labeling $A_3 = L_{4r}001$, i.e., $x_0 = 2r+2, x_1 = 2r+1$, $a_0 = 2r+1$ and $a_1 = 2r+1$ for the cycle C_m and B_j, where $j = 0$, 1, 2,3 for the path P_n as given in Table 3.3. Using Table 3.3 and the fact that $v_0 - v_1 = (x_0 - x_1) + (y_0 - y_1)$ and $e_0 - e_1 = (a_0 - a_1) + (b_0 - b_1) + (x_0 - x_1)(y_0 - y_1)$, we can compute the values shown in the last two columns of Table 3.4. Since these are all 0,1, or -1,the theorem follows.

$n=4s+j$, $j=0,1,2,3$	Labeling of P_n	y_0	y_1	b_0	b_1
$j=0$	$B_0 = L_{4s}$	$2s$	$2s$	$2s$	$2s-1$
$j=1$	$B_1 = L_{4s}1$	$2s$	$2s+1$	$2s+1$	$2s-1$
$j=2$	$B_2 = L_{4s}11$	$2s$	$2s+2$	$2s+2$	$2s-1$
$j=3$	$B_3 = L_{4s}011$	$2s+1$	$2s+2$	$2s+1$	$2s+1$

Table 3.3. Labelings of Paths.

$m=4r+i$,	$n=4s+j$, $j=0,1,2,3$	C_m	P_n	v_0-v_1	e_0-e_1
$i=3$	0	A_3	B_0	1	1
$i=3$	1	A_3	B_1	0	1
$i=3$	2	A_3	B_2	-1	1
$i=3$	3	A_3	B_3	0	-1

Table 3.4. Cambinations of Labelings.

Example 3.1. The following graphs are not cordials: $C_3 + P_1, C_3 + P_2$ and $C_3 + P_3$.
Solution. It is easy to verify that $C_3+P_1 = K_4$ and $C_3+P_2 = K_5$, but we known that K_n is cordial graph if and only if $n \leq 3$. Hence the graphs C_3+P_1 and C_3+P_2. Also, it is easy to see that if $n = 3$, then all labelings, which satisfy the condition that $v_1 = v_0 = 3$ are [000,111], [111,000], [001,011], [001,101], [001,110], [010,011], [010,101], [010,110], [011,001], [011,010], , [100,011], [100,101], [100,110]. The last labelings give us $e_0 = 5, e_1 = 9$ or $e_0 = 6, e_1 = 8$. Hence $|e_0 - e_1| \leq 1$, i.e., $C_3 + P_3$ is not a cordial graph.

From theorem 3.1, theorem 3.2 and example 3.1, we establish the following theorem.
Theorem 3.3. The join of the cycle C_m and the path P_n is cordial if and only if $(m, n) \neq$ (3, 3), (3, 2) and (3, 1).
Proof. The proof follows directly from theorem 3.1, theorem 3.2 and example 3.1.

4 Joins and Unions of Paths and Stars

Seoud, Diab and Elsakhawi [7] have proved that the join of the path P_n and the star $K_{1,m}$ is cordial for all n and odd m except $(n, m) =$ (2,1). In this section, we generalize this as follows:
$P_n + K_{1,m}$ is cordial for all n and all m if and only if $(n,m) \neq$ (2,1). Moreover, we prove that the union of the path P_n and the star $K_{1,m}$ is cordial for all n and all m except $(n, m) =$ (2,1).
Theorem 4.1. The join of the path P_n and the star $K_{1,m}$ is cordial for all n and all m except $(n, m) =$ (2,1).
Proof. The labelings that we use are given in Table 4.1, along with the corresponding values of x_i and a_i or y_i and b_i (for $i =$ 0,1). We let $n = 4r+i$ (for $i =$ 0,1,2,3) and $m = 2s + j$ (for $j =$ 0,1). For given values of i and j with $0 \leq i \leq 3$ and $0 \leq j \leq l$, we use the labeling A_i or A'_i for the path P_n and B_j for the star $K_{1,m}$ as given in Table 4.1. Using Table 4.1 and the fact that $v_0 - v_1 = (x_0 - x_1) + (y_0 - y_1)$ and $e_0 - e_1 = (a_0 - a_1) + (b_0 - b_1) + (x_0 - x_1)(y_0 - y_1)$, we can compute the values shown in the last two columns of Table 4..2. Since these are all 0,1, or -1, the theorem follows.

$n = 4r + i$, $i = 0, 1, 2, 3$	Labeling of P_n	x_0	x_1	a_0	a_1
$i = 0$	$A_0 = L_{4r}$	$2r$	$2r$	$2r$	$2r - 1$
$i = 1$	$A_1 = L_{4r}0$	$2r + 1$	$2r$	$2r$	$2r$
$i = 2$	$A_2 = L_{4r}01$	$2r + 1$	$2r + 1$	$2r$	$2r + 1$
	$A'_2 = L_{4r}10$	$2r + 1$	$2r + 1$	$2r + 1$	$2r$
$i = 3$	$A_3 = L_{4r}001$	$2r + 2$	$2r + 1$	$2r + 1$	$2r + 1$

$m = 2s + j$, $j = 0, 1$	Labeling of $K_{1,m}$	y_0	y_1	b_0	b_1
$j = 0$	$B_0 = 1M_{2s}$	s	$s + 1$	s	s
	$B'_0 = 0M_{2s}$	$s + 1$	s	s	s
	$B''_0 = 011M_{2s-2}$	s	$s + 1$	$s - 1$	$s + 1$
$j = 1$	$B_1 = 11M_{2s}$	s	$s + 2$	$s + 1$	s
	$B'_1 = 01M_{2s}$	$s + 1$	$s + 1$	s	$s + 1$

Table 4.1. Labelings of Paths and Stars.

$m = 2s + j$, $j = 0, 1$	Labeling of $K_{1,m}$	y_0	y_1	b_0	b_1
0	0	A_0	B_0	-1	1
1	0	A_1	B_0	0	-1
2	0	A_2	B_0	-1	-1
3	0	A_3	B_0	0	-1
0	1	A_0	B'_1	0	0
1	1	A_1	B_1	-1	-1
2	1	A'_2	B'_1	0	0
3	1	A_3	B_1	-1	-1

Table 4.2. Combinations of Labelings.

Example 4.1. The graph $P_2 + K_{1,1}$ is not cordial.
Solution. Since $P_n + K_{1,1} = P_2 + P_2 = K_4$ and K_4 is not cordial. Then $P_2 + K_{1,1}$ is not cordial.

From theorem 4.1 and example 4.1, we can establish the following theorem.

Theorem 4.2. The join of the path P_n and the star $K_{1,m}$ is cordial for all n and all m if and only if $(n, m) \neq$ (2,1).

Proof. The proof follows directly from theorem 4.1 and example 4.1.

Theorem 4.3. The union of the path P_n and the star $K_{1,m}$ is cordial for all n and all m except $(n, m) =$ (2,1).

Proof. By using the labelings of the path P_n and the star $_{1,m}$ in Table 4.1 from theorem 4.1 and the fact that $v_0 - v_1 = (x_0 - x_1) + (y_0 - y_1)$ and $e_0 - e_1 = (a_0 - a_1 + (b_0 - b_1)$, we can compute the values shown in the last two columns of Table 4.3. Since these are all 0,1, or -1, the theorem follows.

$n = 4r + i$, $i = 0,1,2,3$	$m = 2s + j$, $j = 0,1$	P_n	$K_{1,m}$	$v_0 - v_1$	$e_0 - e_1$
0	0	A_0	B_0	-1	1
1	0	A_1	B_0	0	0
2	0	A_2	B_0	-1	-1
3	0	A_3	B_0	0	0
0	1	A_0	B_1'	0	0
1	1	A_1	B_1	-1	1
2	2	A_2'	B_1'	0	0
3	3	A_3	B_1	-1	1

Table 4.3. Combinations of Labelings.

It well is known that $P_2 \cup K_{1,1} = P_2 \cup P_2$, which is not cordial.

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