SOME PARTITIONS WHERE EVEN PARTS APPEAR TWICE

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Abstract

We consider some partitions where even parts appear twice and some where evens do not repeat. Further, we offer a new partition theoretic interpretation of two mock theta functions of order 8.

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1. Introduction and Main Results

Recently [4] Andrews related the partition function

$$\Delta(n) = \sum_{m \geq 0} (-1)^m \delta(m, n),$$

where $\delta(m,n)$ is the number of DE-partitions (partitions with distinct evens) of n with DE-rank m, to the arithmetic of $\mathbb{Q}(\sqrt{2})$. In [11] we showed some partitions where evens do not repeat are in fact lacunary, and can be expressed in in terms of indefinite quadratic forms. Also, in [6] we find a q-series that generates a partition where evens do not repeat:

$$\sum_{n=0}^{\infty} \frac{q^{n+1}(-1)^n(1+q^2)(1+q^4)\cdots(1+q^{2n})}{(1-q)(1-q^3)\cdots(1-q^{2n+1})}, \tag{1}$$

which is similar to one of McIntosh's 2nd order mock theta functions with an additional weight.

In this paper we consider some partition functions where even parts appear twice, and that assume the value of 0 for almost all natural n. We also consider some partitions with negative parts that do not repeat. Furthermore, we offer a new partition theoretic interpretation of two mock theta functions of order 8, and show that the series

$$\sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n+1)}(1-q^2)(1-q^4)\cdots(1-q^{2n})}{(1+q)(1+q^3)\cdots(1+q^{2n+1})},$$

is related to the arithmetic of $\mathbb{Q}(\sqrt{2})$ by establishing a relation to (1).

Throughout we assume the notation [8]

$$(a;q)_n := (1-a)(1-aq)\cdots(1-aq^{n-1}),$$

$$\lim_{n\to\infty}(a;q)_n = (a;q)_\infty := \prod_{n=0}^{\infty}(1-aq^n).$$

Theorem 1.1. *Let $\Phi_e(n)$ (resp. $\Phi_o(n)$) be the number of partitions of n in which even parts appear either 0 or two times, largest part odd, and 2E-rank even (resp. odd). Then*

$$\Phi_e(n) - \Phi_o(n) = \sum_{\substack{j\geq 0 \\ j\leq 2r \\ n=3r^2+6r+1-j(j+3)/2}} 1. \tag{2}$$

Furthermore, we have that $\Phi_e(n) \geq \Phi_o(n)$ for all natural n.

Theorem 1.2. *Let $\Psi(n)$ be the number of partition of n where if $2k$ is the largest even part, then positive evens less than or equal to $2k$ appear exactly twice, odd parts may repeat with largest odd part $\leq 2k+1$, and negative parts are even, do not repeat, and $\leq 2k$. Let $\Psi_e(n)$ (resp. $\Psi_o(n)$) be the number of partitions counted by $\Psi(n)$ with an even (resp. odd) number of negative parts. Then $\Psi_e(n) - \Psi_o(n)$ is equal to the number of inequivalent solutions of $x^2-2y^2=k$ with norm $8k-1$, where $x+4y \equiv 3$ or $5 \pmod 8$.*

A natural corollary concerning the lacunarity of $\Psi_e(n) - \Psi_o(n)$ is given in the following:

Corollary 1.3. *For every positive integer n we have $\Psi_e(n) \geq \Psi_o(n)$. Furthermore, we have that $\Psi_e(n) = \Psi_o(n)$ infinitely often.*

The next theorem, while probably less elegant than our previous results, is a companion to the partition function given in Theorem 1.2.

Theorem 1.4. *Let $\psi(n)$ be the number of partitions of n with largest part $2k+2$ which appears once, all positive evens less than $2k+2$ appear exactly twice, odd parts may repeat with largest odd part $\leq 2k+1$, and negative parts are even, do not repeat, and $\leq 2k$. Let $\psi_e(n)$ (resp. $\psi_o(n)$) be the* *number of partitions counted by $\psi(n)$ with an even (resp. odd) number of negative parts. Then*

$$\psi_e(n) - \psi_o(n) = \sum_{\substack{j \geq 0 \\ j \leq 2r \\ n=4r^2+7r+2-j(j+3)/2}} 1 + \sum_{\substack{j \geq 0 \\ j \leq 2r \\ n=4r^2+9r+4-j(j+3)/2}} 1. \qquad (3)$$

Further, for all natural n we have $\psi_e(n) \geq \psi_o(n)$.

Of course, this theorem also has a relation to $\mathbb{Q}(\sqrt{2})$, which we do not mention here.

2. Proof of Theorems

First we recall [5] that a "signed" partition may be viewed as a partition pair $\sigma = (\mu, \lambda)$ where the partition μ consists of parts that appear with a "+," and λ consists of parts that appear with a "−." For a detailed account of these partitions we refer the reader to [5]. We shall also require the following lemma [1, 3, 12]:

Lemma 2.1. *If (α_n, β_n) form a Bailey pair with respect to a then*

$$\sum_{n=0}^{\infty} (z)_n (y)_n (aq/zy)^n \beta_n = \frac{(aq/z)_\infty (aq/y)_\infty}{(aq)_\infty (aq/zy)_\infty} \sum_{n=0}^{\infty} \frac{(z)_n (y)_n (aq/zy)^n \alpha_n}{(aq/z)_n (aq/y)_n}. \qquad (4)$$

Here we say a pair of sequences (α_n, β_n) forms a Bailey pair with respect to a if

$$\beta_n = \sum_{r=0}^{n} \frac{\alpha_n}{(aq;q)_{n+r}(q;q)_{n-r}}. \qquad (5)$$

Proof of Theorem 1.1: By [1, p.120, eq. (5.5)] with $a = q$, $b = q^{1/2}$, and then q replaced by q^2, we have that (α_n, β_n) forms a Bailey pair with respect to q^4 (where q has been replaced with q^2 in the definition) with

$$\alpha_n = \frac{(-1)^n q^{n(3n+4)}(1 - q^{4n+4})}{1 - q^4}\left(1 + \sum_{j=1}^{n} q^{-j(2j+3)}(1 + q^{2j+1})\right),$$

and

$$\beta_n = \frac{1}{(q;q^2)_{n+1}}.$$

Using the fact that

$$\sum_{j=1}^{n} q^{-j(2j+3)}(1+q^{2j+1}) = \sum_{j=1}^{2n} q^{-j(j+3)/2},$$

we obtain

$$\alpha_n = \frac{(-1)^n q^{n(3n+4)}(1-q^{4n+4})}{1-q^4} \sum_{j=0}^{2n} q^{-j(j+3)/2}.$$

Now inserting this pair in (4) with $z = q^2$, and $y = -q^2$ gives

$$\sum_{n=0}^{\infty} \frac{(q^4;q^4)_n(-q^2)^n}{(q;q^2)_{n+1}} = \sum_{n=0}^{\infty} q^{3n^2+6n} \sum_{j=0}^{2n} q^{-j(j+3)/2}. \tag{6}$$

Note that

$$\frac{q^{2n+1}(1+q^{2+2})(1+q^{4+4})\cdots(1+q^{2n+2n})}{(1-q)(1-q^3)\cdots(1-q^{2n+1})}$$

generates a partition where even parts appear either 0 or two times, largest part $\leq 2n+1$. For the weight in (6), we shall consider a modified form of Andrews' DE-rank [4], which we call the 2E-rank. The only difference being that we are keeping track of the number of evens that appear zero or two times and not distinct evens. The desired result follows after multiplying both sides of (6) by q, and noting that the right hand side ensures the partition function generated by the left sum is strictly positive.

Proof of Theorem 1.2: We shall require the pair (α_n, β_n) [1] with respect to q^2 (with q replaced by q^2 in the definition)

$$\alpha_n = \frac{(-1)^n q^{n(3n+2)}(1+q^{2n+1})}{1+q} \sum_{j=0}^{2n} q^{-j(j+1)/2},$$

and

$$\beta_n = \frac{1}{(q;q^2)_{n+1}}.$$

Insterting this pair in (4), letting $y = q^2$, and $z \to \infty$ gives

$$\sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n+1)}(q^2;q^2)_n}{(q;q^2)_{n+1}} = \sum_{n=0}^{\infty} q^{n(4n+3)}(1+q^{2n+1}) \sum_{j=0}^{2n} q^{-j(j+1)/2} \tag{7}$$

We now refer the reader to [6] where it was shown that

$$\sum_{n=0}^{\infty} \frac{(-q)_n(-q^{1/2})^n}{(q^{1/2};q)_{n+1}} = \sum_{n=0}^{\infty}(-1)^n q^{n^2+3n/2}(1+q^{n+1})\sum_{j=0}^{n} q^{-j(j+1)/2}. \quad (8)$$

Also, it was clearly outlined that

$$\sum_{\substack{n\geq 0 \\ -n\leq j\leq n-1}}^{\infty} (-1)^n q^{(4n-1)^2-2(2j+1)^2}(1+q^{16n})$$

is the excess of the number of the inequivalent solutions to $x^2 - 2y^2 = k$ with norm $8k-1$ in which $x+4y \equiv 3$ or $5 \pmod 8$, over the number in which $x+4y \equiv 1$ or $7 \pmod 8$. Comparing this with the right hand side of (7) shows that replacing q by q^{16}, and multiplying by $1/q$ in the series

$$\sum_{n=0}^{\infty} q^{n(4n+3)}(1+q^{2n+1})\sum_{j=0}^{2n} q^{-j(j+1)/2},$$

gives the generating function for the number of inequivalent solutions of $x^2 - 2y^2 = k$ with norm $8k-1$ in which $x+4y \equiv 3$ or $5 \pmod 8$.

Now consider the sum on the left hand side of (7). The component

$$\frac{q^{2+4+\cdots+2n}}{(q;q^2)_{n+1}},$$

generates a partition where if $2k$ is the largest even part, then all evens less than or equal to $2k$ appear exactly once, largest odd part is $\leq 2k+1$. Note that

$$\sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n+1)}(q^2;q^2)_n}{(q;q^2)_{n+1}} = \sum_{n=0}^{\infty} \frac{q^{2n(n+1)}(1-1/q^2)(1-1/q^4)\cdots(1-1/q^{2n})}{(q;q^2)_{n+1}}.$$

The sum on the right hand side now generates a partition of n where if $2k$ is the largest even part, then all positive evens less than or equal to $2k$ appear twice, odd parts may repeat with largest odd part $\leq 2k+1$, and negative parts are even, do not repeat, and $\leq 2k$. To see the "all evens less than $2k$ appear twice" part, note that $q^{2n(n+1)} = q^{2+2+4+4+\cdots+2n+2n}$. Lastly the negative distinct evens are weighted by -1 raised to the number of parts.

Proof of Theorem 1.4: First we employ the pair given in the proof of

Theorem 1.1 in Lemma 2.1 with $z = q^2$, $y \to \infty$, and multiplying by q^2 to get

$$\sum_{n=0}^{\infty} \frac{(-1)^n q^{(n+1)(n+2)}(q^2;q^2)_n}{(q;q^2)_{n+1}} = \sum_{n=0}^{\infty} q^{n(4n+7)+2}(1+q^{2n+2}) \sum_{j=0}^{2n} q^{-j(j+3)/2} \tag{9}$$

We can write the sum on the left hand side as

$$\sum_{n=0}^{\infty} \frac{(-1)^n q^{(n+1)(n+2)}(q^2;q^2)_n}{(q;q^2)_{n+1}} = \sum_{n=0}^{\infty} \frac{q^{2n(n+1)+2n+2}(1-1/q^2)(1-1/q^4)\cdots(1-1/q^{2n})}{(q;q^2)_{n+1}}.$$

This generates a partition similar to that of Theorem 1.2 but with a key difference because of the extra factor q^{2n+2}. The sum on the right hand side generates a partition of n with largest part $2k+2$ which appears once, all evens less than $2k+2$ appear twice, odd parts may repeat with largest odd part $\leq 2k+1$, and negative parts are even, do not repeat, and $\leq 2k$.

3. Related Partition Theorems

While seeking out partition theoretic interpretations of mock theta functions was not the main motivation of this paper, it is worth offering some related results. In this section we give some partition theorems that follow directly from our work in the previous section. In particular, the following proposition considers two mock theta functions of order 8 (with q replaced by $-q$) found by Gordon and McIntosh [9].

Proposition 3.1. *Let $\Psi(n)$, and $\psi(n)$ be as in Theorem 1.2 and 1.4. Then*

$$\sum_{n=0}^{\infty} \Psi(n) q^n = T_1(-q) = \sum_{n=0}^{\infty} \frac{q^{n(n+1)}(-q^2;q^2)_n}{(q;q^2)_{n+1}},$$

and

$$\sum_{n=0}^{\infty} \psi(n) q^n = T_0(-q) = \sum_{n=0}^{\infty} \frac{q^{(n+1)(n+2)}(-q^2;q^2)_n}{(q;q^2)_{n+1}}.$$

Since the proof is almost identical to the proofs given in Theorems 1.2 and 1.4 we omit the details. Lastly, we offer an interesting identity that follows naturally from this study.

Theorem 3.2. *We have*

$$\sum_{n=0}^{\infty} \frac{(-q^2;q^2)_n q^n}{(-q;q^2)_{n+1}} + \sum_{n=0}^{\infty} \frac{(-q^2;q^2)_n(-q)^n}{(q;q^2)_{n+1}} = 2\sum_{n=0}^{\infty} \frac{(-1)^n q^{2n(n+1)}(q^4;q^4)_n}{(q^2;q^4)_{n+1}}.$$

Proof. Replace q by q^2 in (8), and add the sum on the right hand side to itself when q is replaced by $-q$. That is,

$$\sum_{n=0}^{\infty}(-1)^n q^{2n^2+3n}(1+q^{2n+2})\sum_{j=0}^{n} q^{-j(j+1)}+\sum_{n=0}^{\infty} q^{2n^2+3n}(1+q^{2n+2})\sum_{j=0}^{n} q^{-j(j+1)}$$

$$=2\sum_{n=0}^{\infty} q^{8n^2+6n}(1+q^{4n+2})\sum_{j=0}^{2n} q^{-j(j+1)},$$

which is just the right side of (7) with q replaced by q^2 with an additional factor of 2. □

The series

$$\sum_{n=0}^{\infty}\frac{(-q^2;q^2)_n q^{n+1}}{(-q;q^2)_{n+1}},$$

has an interesting partition interpretation we mention in closing.

The component q^{n+1} generates ones, so we can say this series generates a partition of n where all parts are less than twice the number of ones, and evens do not repeat. Here the weight is -1 raised to the number of odd parts.

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