On the Arrangement of Cliques in Chordal Graphs with respect to the Cuts *

L. Sunil Chandran** N.S. Narayanaswamy * * *

Abstract. A cut (A, B) (where $B = V - A$) in a graph $G = (V, E)$ is called internal if and only if there exists a vertex x in A that is not adjacent to any vertex in B and there exists a vertex $y \in B$ such that it is not adjacent to any vertex in A. In this paper, we present a theorem regarding the arrangement of cliques in a chordal graph with respect to its internal cuts. Our main result is that given any internal cut (A, B) in a chordal graph G, there exists a clique with $\kappa(G) + 1$ vertices (where $\kappa(G)$ is the vertex connectivity of G) such that it is (approximately) bisected by the cut (A, B). In fact we give a stronger result: For any internal cut (A, B) of a chordal graph, and for each i, $0 \leq i \leq \kappa(G) + 1$, there exists a clique K_i such that $|K_i| = \kappa(G) + 1$, $|A \cap K_i| = i$ and $|B \cap K_i| = \kappa(G) + 1 - i$.

An immediate corollary of the above result is that the number of edges in any internal cut (of a chordal graph) should be $\Omega(k^2)$, where $\kappa(G) = k$. Prompted by this observation, we investigate the size of internal cuts in terms of the vertex connectivity of the chordal graphs. As a corollary, we show that in chordal graphs, if the edge connectivity is strictly less than the minimum degree, then the size of the mincut is at least $\frac{\kappa(G)(\kappa(G)+1)}{2}$, where $\kappa(G)$ denotes the vertex connectivity. In contrast, in a general graph the size of the mincut can be equal to $\kappa(G)$. This result is tight.

1 Introduction

Let C be a cycle in a graph G. A chord of C is an edge of G joining two vertices of C that are not consecutive. A graph G is called a chordal graph if and only if every cycle in G of length 4 or more has a chord.

Let $G = (V, E)$ be a simple connected undirected graph. Throughout this paper we will use V for the set of vertices of G and E for the set of edges. $|V|$ will be denoted by n. $N(v)$ will denote the set of neighbours of v, that is $N(v) = \{u \in$

* A preliminary version of this paper appeared in the proceedings of the tenth International Computing and Combinatorics Conference (COCOON) 2004

** (**Corresponding Author.**) Computer Science and Automation Department, Indian Institute of Science, Bangalore-560012, India. Email : sunil@csa.iisc.ernet.in Alternate email id: sunil.cl@gmail.com Phone: +91-802932368 Fax: +91-80-3600683

* * * Department of Computer Science and Engineering, Indian Institute of Technology, Chennai–600 036, India. Email: swamy@shiva.iitm.ernet.in

$V : (u,v) \in E\}$. For $A \subseteq V$, we use $N(A)$ to denote the set $\bigcup_{v \in A} N(v) - A$. The subgraph of G induced by the vertices in A will be denoted by $G[A]$.

The *vertex connectivity* $\kappa(G)$ of an undirected graph is defined to be the minimum number of vertices whose removal results in a disconnected graph or a trivial graph (i.e. single vertex). A subset $S \subset V$ is called a *separator* if $G[V - S]$ has at least two components. Then, $\kappa(G)$ is the cardinality of the minimum separator. Note that the complete graph K_n has no separator and its (vertex) connectivity is by definition $n - 1$.

A cut $(A, V - A)$ where $\emptyset \neq A \subset V$ is defined as the set of edges with exactly one end point in A. That is, $(A, V - A) = \{(u,v) \in E : u \in A \text{ and } v \in V - A\}$. In this paper we always use (A, B) instead of $(A, V - A)$ for notational ease, i.e. B always means $V - A$.

Definition 1. *A vertex $x \in A$ is called a hidden vertex with respect to B (or with respect to the cut (A, B)) if and only if $x \notin N(B)$.*

Definition 2. *A cut (A, B) is called an internal cut of G if and only if there exists a hidden vertex $x \in A$ and a hidden vertex $y \in B$. Equivalently, a cut (A, B) is internal if and only if $N(A) \neq B$ and $N(B) \neq A$, i.e. $N(A)$ is a proper subset of B and $N(B)$ is a proper subset of A.*

Note that if x is a hidden vertex in A, then $\{x\} \cup N(x) \subseteq A$. In fact, the only condition for (A, B) to be an internal cut is that there should exist a vertex $x \in A$ and $y \in B$ such that $N(x) \subseteq A$ and $N(y) \subseteq B$. Thus if we can find two non adjacent vertices x and y, such that $N(x) \cap N(y) = \emptyset$, then there exist internal cuts in the graph : we just have to include $\{x\} \cup N(x)$ in A and $\{y\} \cup N(y)$ in B and divide the remaining vertices arbitrarily. The reader can easily verify that this can be done in $2^{n - |\{x\} \cup N(x)| - |\{y\} \cup N(y)|}$ ways. Thus in general, the number of internal cuts in a graph can be exponential.

1.1 Our Main Result

We present a structural theorem about internal cuts in chordal graphs. Let (A, B) be an internal cut in a chordal graph G with connectivity $\kappa(G) = k$. Let i be an integer such that $0 \leq i \leq k + 1$. Then we prove that there exists a clique K_i in G, such that $|K_i| = k + 1$, $|K_i \cap A| = i$ and $|K_i \cap B| = k + 1 - i$. (Theorem 2).

An interesting special case of the above result is when $i = \lfloor \frac{k+1}{2} \rfloor$. Then, the theorem states that, irrespective of which internal cut (A, B) we consider, it is always possible to find a clique of size $k + 1$, such that it is bisected by (A, B).

As of now, we are unable to present any specific applications for our theorem. But we would like to point out that the chordal completions of graphs are of great interest from the point of view of two important optimisation problems: the treewidth and the fill–in problem. Given a graph G, the treewidth problem is to find a chordal completion of G, so as to minimise the clique number. The fill–in problem seeks to find a chordal completion of G, that minimises the number of edges added. Also note that, properties such as connectivity are monotonic, i.e. as new edges are added, the property never decreases. Thus, the value of the property in the chordal completion will be at least as much as in the original graph. Structural understanding of chordal graphs, with respect to such properties turn out to be useful, in making apriori inferences about the optimal chordal completions. (For example, see [3], where a lower bound for treewidth is derived in terms of a generalised connectivity property.)

Moreover, chordal graphs are well–studied from various perspectives. For example, the vertex separators in chordal graphs are well understood. But the corresponding concept of edge separator, namely the cut (due to some reason) is very little studied. In this paper, we attempt to understand chordal graphs, from the perspective of edge cuts.

1.2 Another Result inspired by the Main Result

An immediate consequence of the result (Theorem 2) is that the size of every internal cut has to be $\Omega(\kappa(G)^2)$. This is because for $i = \lfloor \frac{\kappa(G)+1}{2} \rfloor$ the internal cut would be approximately "bisecting" a clique with $\kappa(G) + 1$ vertices. After observing this, we investigated whether a better result is possible. In Theorem 3 we prove a lower bound for the size of internal cuts in chordal graphs, namely $\frac{\kappa(G)(\kappa(G)+1)}{2}$, which is better than what can be immediately infered from Theorem 2. We also show that this result is tight in the sense that there exist chordal graphs with vertex connectivity $\kappa(G)$ and having internal cuts with exactly $\frac{\kappa(G)(\kappa(G)+1)}{2}$ edges.

The *edge connectivity* $\lambda(G)$ of a graph G is defined to be the minimum number of edges whose removal results in a disconnected graph or a trivial graph. A *minimum cut* or *mincut* is a cut consisting of $\lambda(G)$ edges.

It is well known that the following inequality (due to Whitney) holds in a general undirected graph:

$$\kappa(G) \leq \lambda(G) \leq \delta(G)$$

where $\delta(G)$ is the minimum degree of the graph. For a discussion of this inequality and for related work see the chapter on connectivity in [6]. In [4], Chartrand and Harary show that for all integers a, b, c, such that $0 < a \leq b \leq c$, there exists a graph G with $\kappa(G) = a$, $\lambda(G) = b$ and $\delta(G) = c$.

We study this inequality when restricted to the class of chordal graphs. We observe that when $\lambda(G) < \delta(G)$, every mincut in G is also an internal cut. Thus, as a corollary of our result on internal cuts in chordal graphs, we show that there is a "gap" between $\lambda(G)$ and $\kappa(G)$ in chordal graphs provided $\lambda(G) \neq \delta(G)$. More specifically, if $\lambda(G) \neq \delta(G)$, then $\lambda(G) \geq \frac{\kappa(G)(\kappa(G)+1)}{2}$. This improves the result in [1] which shows that $\lambda(G) \geq 2\kappa(G) - 1$, when $\lambda(G) < \delta(G)$. And the lower bound we obtain here is tight.

1.3 Preliminaries

Theorem 1. *[See [6] for a proof.]* **Menger's theorem :** *The minimum number of vertices separating two nonadjacent vertices s and t is equal to the maximum number of internally vertex disjoint $s - t$ paths.*

A bijection $f : V \rightarrow \{1, 2, \cdots, n\}$ is called an ordering of the vertices of G. Then $f(v)$ is referred to as the number associated with the vertex v, or simply the *number of* v with respect to the ordering f. Given an ordering f of a graph G, we define the following terms: Let $A \subseteq V$. The $highest(A)$ is defined to be the vertex with the highest number in A. Also $lowest(A)$, second-lowest(A) etc. are defined in a similar way. A path $P = (w_1, w_2, \cdots, w_k)$ in G is called an *increasing path* if and only if $f(w_1) < f(w_2) < \cdots < f(w_k)$. It is called a *decreasing path* if and only if $f(w_1) > f(w_2) > \cdots > f(w_k)$. A single vertex can be considered as either increasing or decreasing. A vertex $u \in N(v)$ is called a *higher neighbour* of v if and only if $f(u) > f(v)$. The set of higher neighbours of v will be denoted by $N_h(v)$, i.e. $N_h(v) = \{u \in N(v) : f(u) > f(v)\}$.

An ordering f of G is called a *perfect elimination ordering* (PEO) if and only if for each $v \in V$, $G[\{v\} \cup N_h(v)]$ is a complete subgraph (clique) of G. Then $f(v)$ will be called the *PEO number of* v. It is well–known that an undirected graph G is chordal if and only if there exists a PEO for G. (See[5].) A vertex v is called *simplicial* if and only if $N(v)$ induces a clique. Dirac observed that in every chordal graph that is not a complete graph, there exist at least two non-adjacent simplicial vertices. (In a complete graph, every vertex is simplicial). From this observation, it is easy to infer the following: Suppose v is any vertex in a chordal graph G. Then there exists a PEO f for G such that $f(v) = n$. (See [5], page 84). In fact more can be inferred from Dirac's observation. (See also [7], page 8). Let K be a clique in a chordal graph G. Then if $V(G) - K \neq \emptyset$, there should be a simplicial vertex $x_1 \in V(G) - K$. Let $f(x_1) = 1$. Now remove x_1 from G. Clearly, the induced subgraph on the remaining vertices (say G') is also chordal and thus the observation of Dirac is applicable to G' also. Then we can find a vertex $x_2 \in V(G') - K$, that is simplicial (as long as $V(G') - K$ is nonempty). Let $f(x_2) = 2$. By repeating this procedure, one can get a PEO f for G such that the vertices of K get the highest numbers, namely the numbers from $n - |K| + 1$ to n. We state this useful observation as a lemma.

Lemma 1. *Let K be any clique in a chordal graph G. Then there exists a PEO f of G such that the highest $|K|$ numbers are assigned to the vertices of K. In other words, $n - |K| + 1 \leq f(x) \leq n$, for each vertex $x \in K$.*

Another well known characterisation of chordal graphs is in terms of minimal separators. (A separator S of two vertices x and y is said to be a minimal separator if no proper subset of S can separate x from y. Note that every separator of x and y will contain a minimal separator of x and y).

Lemma 2. *A graph G is a chordal graph if and only if every minimal separator of G induces a complete subgraph.*

A chordless path from u to v is defined to be a path from u to v in G such that no two non-consecutive vertices of the path are adjacent. The reader can easily verify that if there is a path between u and v then there is a chordless path also. For example, the shortest path between u and v has to be a chordless path. The following observations from [2] are useful tools.

Lemma 3. *[2] Let $P = (w_1, w_2, \cdots, w_k)$ be a chordless path in a chordal graph G and let $w_i = highest(P)$ with respect to a PEO f. Then $(w_1, w_2, \cdots, w_i)$ is an increasing path while $(w_i, w_{i+1}, \cdots, w_k)$ is a decreasing path with respect to f.*

Corollary 1. *Let $P = (w_1, w_2, \cdots, w_k)$ be a chordless path and $w_k = highest(P)$ with respect to a PEO f. Then P is an increasing path with respect to f.*

2 Internal Cuts in Chordal Graphs

Theorem 2. *Let (A, B) be an internal cut of a chordal graph G with $\kappa(G) = k$. Then for each i, $0 \leq i \leq (k+1)$, there exists a clique K_i such that $|K_i| = k+1$, $|A \cap K_i| = i$ and $|B \cap K_i| = k + 1 - i$.*

Proof: Let $x \in A$ and $y \in B$ be hidden vertices with respect to the cut (A, B). (Since (A, B) is an internal cut, such vertices exists in A and B). Clearly x and y are non-adjacent. Then by Menger's Theorem (Theorem 1), there exist $\kappa(G) = k$ internally vertex disjoint paths from x to y. If there are k internally vertex disjoint paths, then there are k internally vertex disjoint chordless paths also. Let these internally vertex disjoint chordless paths be $P_1', P_2', \cdots, P_k'$.

Clearly, $N(A)$ separates x from y. Thus there exists a minimal separator $M \subseteq N(A) \subset B$ that separates x from y. By Lemma 2, M is a clique. Then by Lemma 1, there exists a PEO such that the highest numbers are assigned to the vertices of M. Let f be such a PEO. Note that each path P_i', $1 \leq i \leq k$, should pass through M, since M is a separator of x and y. Let z_i be the first vertex at which P_i' meets M. Let P_i be the partial path from x to z_i of P_i'. Clearly, $z_i = highest(P_i)$ (with respect to the PEO f), since z_i is the only vertex in P_i, that belongs to M, and the PEO f was selected such that the vertices in M got the highest numbers. Thus by Corollary 1, P_i is an increasing path.

Now we define a series of $k+1$ element subsets of $V(G)$, namely $F_0, F_1, \cdots, F_r$ as follows. Let $F_0 = \{x, y_1, y_2, \cdots, y_k\}$ where y_i is the first vertex in P_i after x. Note that each $y_i \in A$, since x is a hidden vertex in A. Clearly each y_i has a higher PEO number than that of x, since P_i is an increasing path. Thus $\{y_1, y_2, \cdots, y_k\} \subseteq N_h(x)$. Thus $F_0 = \{x, y_1, y_2, \cdots, y_k\}$ induces a clique of size $k+1$, that is completely in A.

Now we will describe how to define F_{i+1} from F_i, for $0 \leq i < r$. Let $v_i = lowest(F_i)$. Define $F_i' = F_i - \{v_i\}$. Suppose that the subset F_i satisfies the following properties.

1. $|F_i| = k + 1$.
2. The vertices of F_i induces a clique.
3. $F_i \subset \bigcup_{j=1}^{j=k} P_j$. Moreover, $|F_i' \cap P_j| = 1$ for each path P_j, $1 \leq j \leq k$. (That is, in F_i', there is exactly one "representative" vertex from each path P_j).

Now let $u_i = lowest(F_i')$ = second-lowest(F_i). Suppose that u_i is on path P_j. Now we define F_{i+1} from F_i as follows.

If $u_i = z_j$, the last vertex on path P_j, then let $F_i = F_r$. That is, F_i will be the last subset in the series, in that case. Otherwise, consider the vertex w_i, that is placed just after u_i on the path P_j. Since P_j is an increasing path (with respect to f), $f(w_i) > f(u_i)$. Also note that $w_i \notin F_i$. To see this, note that $w_i \notin F_i'$, since $|F_i' \cap P_j| = 1$,and u_i is already there in $(F_i' \cap P_j)$. Also, $w_i \neq v_i = lowest(F_i)$, since $f(w_i) > f(u_i) > f(v_i)$. Thus w_i is a vertex on P_j that is not already in F_i. We define F_{i+1} as follows.

$$F_{i+1} = F_i - \{v_i\} \cup \{w_i\}$$

Note that F_{i+1} is obtained by replacing the vertex v_i, the lowest numbered vertex in F_i with a new vertex w_i. Thus $|F_{i+1}| = k + 1$. Clearly $lowest(F_{i+1}) = u_i$. Also it is clear that since F_i was assumed to induce a clique and w_i is a (higher) neighbour of u_i, $(F_{i+1} - \{u_i\}) \subseteq N_h(u_i)$, with respect to f. Thus F_{i+1} induces a clique also. Clearly $F_{i+1} \subset \bigcup_{j=1}^{j=k} P_j$. Note that $F_{i+1}' = F_{i+1} - \{u_i\} = F_i' - \{u_i\} \cup \{w_i\}$. In other words, F_{i+1}' is obtained by replacing u_i in F_i' by w_j. Remember that $|F_i' \cap P_j| = 1$, for each j. Since we are just replacing the "representative" vertex u_i of P_j (in F_i') by another vertex w_i (that is also on the same path P_j), to obtain the new set F_{i+1}', it follows that $|F_{i+1}' \cap P_j| = 1$, for each P_j, $1 \leq j \leq k$. Thus the three conditions satisfied by F_i are satisfied by F_{i+1} also.

From the inductive argument above, together with the fact that F_0 satisfies all the 3 properties stated above, it follows that each subset F_i in the sequence defined above satisfies those 3 properties.

Now we look at the last subset F_r in this sequence. Let $u_r = lowest(F_r')$. By definition of the sequence, $u_r = z_j$, (i.e. the last vertex of P_j), for some path P_j. We claim that $F_r' = \{z_1, z_2, \cdots, z_k\}$. Suppose not. Then, since F_r' should contain exactly k vertices, and $F_r' \subset \bigcup_{j=1}^{j=k} P_j$, there should be a vertex $x_r \in F_r'$, that is on some path P_l, but not an end vertex of P_l, i.e. $x_r \neq z_l$. Since we had selected z_l to be the first vertex at which P_l meets M, $x_l \notin M$. But the PEO f was selected such that the vertices of M got the highest numbers. It follows that $f(x_l) < f(z_j)$, contradicting the assumption that $z_j = lowest(F_r')$. Thus we infer that $F_r' = \{z_1, z_2, \cdots, z_k\}$.

Clearly $|F_0 \cap A| = k + 1$, and $|F_r \cap B| \geq k$, since $F'_r = \{z_1, z_2, \cdots, z_k\} \subseteq M \subseteq N(A) \subset B$. Noting that at each step, we were replacing one vertex in F_j with a new one to get F_{j+1}, it follows that for each value i, $1 \leq i \leq k + 1$, there should be a F_{j_i} such that $|F_{j_i} \cap A| = i$, and $|F_{j_i} \cap B| = k + 1 - i$. Let K_i be the clique induced by F_{j_i}. Clearly K_i has the desired property. Now, to see that there exists a clique K_0 such that $|K_0| = k+1$, $|K_0 \cap A| = 0$ and $|K_0 \cap B| = k+1$, just interchange the role of x and y and observe that just as there is a clique induced by F_0, comprising of x and its (higher) neigbours $\{y_1, y_2, \cdots, y_k\}$, that is completely in A, there exists a clique that is completely in B comprising of y and its higher neighbours. Hence the proof. ■

As our next result we show that the number of edges in an internal cut of a chordal graph is at least $\frac{\kappa(G)(\kappa(G)+1)}{2}$.

Theorem 3. *Let (A, B) be an internal cut of a chordal graph G with $\kappa(G) = k$. Then $|(A, B)| \geq \frac{k(k+1)}{2}$.*

Proof : For $\kappa(G) = 1$, the lemma is trivially true because the graph is connected and any cut will have at least one edge in it. So we consider the case when $\kappa(G) \geq 2$. Let (A, B) be an internal cut of G. By the definition of internal cuts, there exist hidden vertices $x \in A$ and $y \in B$. Since G is a k-connected graph, by Menger's theorem (Theorem 1) there are k internally vertex disjoint paths between x and y. Let $P'_1, \ldots, P'_k$ denote these paths. For each P'_i, $1 \leq i \leq k$, there is a chordless path between x and y whose vertices are a subset of the vertices of P'_i. As x and y are hidden vertices, it follows that each chordless path has at least 4 vertices in it. Let these chordless paths be $P_1, \ldots, P_k$. Clearly, these are internally vertex disjoint paths.

Let $(u_1, v_1), \ldots, (u_k, v_k)$ be the edges from $P_1, \ldots, P_k$, respectively, such that for $1 \leq i \leq k$, $u_i \in A$ and $v_i \in B$. An edge is said to cross a cut if its end points are not on the same side of the cut. The theorem is proved by showing that for every pair of paths $\{P_i, P_j\}, 1 \leq i \neq j \leq k$, there is an edge e_{ij} crossing the cut (A, B), one of whose end points is on $P_i - \{x, y\}$, and the other on $P_j - \{x, y\}$. As $P_1, \ldots, P_k$ are internally vertex disjoint paths between x and y, an edge cannot be associated with two distinct pairs of paths. This counts $\binom{k}{2}$ edges crossing the cut, which when considered along with the edges $\{u_i, v_i\}, 1 \leq i \leq k$, yields $|(A, B)| \geq \frac{k(k+1)}{2} = \frac{\kappa(G)(\kappa(G)+1)}{2}$.

Let P_i, P_j be two distinct paths from the set $\{P_1, \ldots, P_k\}$. We claim that there is an edge e_{ij} crossing the cut (A, B), one of whose end points is on $P_i - \{x, y\}$, and the other on $P_j - \{x, y\}$. Let us assume the contrary and prove our claim by arriving at a contradiction. Let w_1 be the last vertex in A on P_i that has an edge to a vertex on P_j. (Note that $w_1 \neq x$, since the vertex appearing just after x, on P_i, itself has a neighbour on P_j, namely x). By our assumption that no edge starting from a vertex on P_i and ending at a vertex on P_j crosses the cut, $N(w_1) \cap P_j \subseteq A$. In P_i, on the path from w_1 to y, let w_2 be the first vertex in B that has an edge to a vertex on P_j. Clearly, $w_2 \neq y$ and by our assumption, $N(w_2) \cap P_j \subseteq B$. Let w_4 and w_3 be the pair of vertices on P_j that are closest to each other such that $w_4 \in N(w_1) \cap P_j$ and $w_3 \in N(w_2) \cap P_j$. Clearly, the portion of the path P_j from w_3 to w_4 will not contain any other vertices of $N(w_1)$ or $N(w_2)$. Now, the path from w_1 to w_2, the edge (w_2, w_3), the path from w_3 to w_4, and the edge (w_4, w_1) form an induced cycle of length at least 4. (This easily follows from the construction and the assumption that P_i and P_j are chordless paths). This is clearly a contradiction since G is a chordal graph. Therefore, our assumption is wrong and there exists an edge e_{ij} that crosses the cut (A, B), with one end point on P_i and the other end point on P_j. Finally, such an edge cannot have x or y as one of its end points, since that would violate the assumption that P_i and P_j are chordless. Hence the theorem. ■

The above lower bound for the size of an internal cut of a chordal graph can be tight. Consider the following graph for instance.

Example 1: Construct a graph G as follows. Let A be a clique of N_1 vertices and B a clique of N_2 vertices. Let $u_1, u_2, \cdots, u_k$ be k vertices in A and $v_1, v_2, \cdots, v_k$ be k vertices in B. Add an edge from v_i $(1 \leq i \leq k)$, to each vertex in the set $\{u_i, u_{i+1}, \cdots, u_k\}$. Thus, v_1 is connected to $\{u_1, u_2, \cdots, u_k\}$, v_2 is connected to $\{u_2, u_3, \cdots, u_k\}$ etc.

To verify that the above graph is a chordal graph, we describe a PEO f for this graph. Let the last N_1 numbers (i.e., the numbers from $N_2 + 1$ to $N_2 + N_1$) be given to the vertices of A in some order. Now let $f(v_i) = N_2 - i + 1$, for $1 \leq i \leq k$. Let the remaining vertices in B get the numbers from 1 to $N_2 - k$ in any order. It is easy to verify that this ordering is a PEO. It follows that G is a chordal graph.

Suppose that $N_1, N_2 > k$. Then it is easy to verify that $\kappa(G) = k$. (A, B) is an internal cut since A contains vertices that are not adjacent to any vertex in B and B contains vertices that are not adjacent to any vertex in A. Clearly $|(A, B)| = k + k - 1 + \cdots + 1 = \frac{k(k+1)}{2}$. ■

3 Edge connectivity vs Vertex connectivity in Chordal graphs

In this section we show that if for a chordal graph G, $\lambda(G) < \delta(G)$, then $\lambda(G) \geq \frac{k(k+1)}{2}$, where $\kappa(G) = k$. The following fact is well–known.

Lemma 4. *Let (A, B) be a mincut of a connected undirected graph G. Then $G[A]$ and $G[B]$ are connected.*

Lemma 5. *Let (A, B) be a mincut of G, and let $\lambda(G) = |(A, B)| < \delta(G)$. Then there exists a vertex $x \in A$, such that $x \notin N(B)$. Similarly $\exists y \in B$ such that $y \notin N(A)$. That is (A, B) is an internal cut.*

Proof : Suppose that every vertex in A is adjacent to some vertex in B. Let u be a vertex in A. Let $F = (A, B)$, be the mininum cut. We have

$$d(u) = |N(u) \cap B| + |N(u) \cap A| \leq |N(u) \cap B| + |A| - 1$$

But

$$|F| = \sum_{x \in A} |N(x) \cap B| \geq |N(u) \cap B| + |A| - 1$$

since each term in the sum should be at least 1 by the assumption that every vertex in A is adjacent to at least one vertex in B. It follows that $d(u) \leq |F|$. But by assumption, $d(u) \geq \delta(G) > |F|$. Thus we have a contradiction and we conclude that there is a vertex $x \in A$ such that $x \notin N(B)$. By similar arguments, $\exists y \in B$ such that $y \notin N(A)$. ■

Theorem 4. *For a chordal graph G, if* $\lambda(G) \neq \delta(G)$, *then* $\lambda(G) \geq \frac{(\kappa(G))(\kappa(G)+1)}{2}$

Proof: By Lemma 5, when $\lambda(G) < \delta(G)$, the mincut will be an internal cut. Then the result follows from Theorem 3 ■

The above theorem is tight. Consider the graph G in Example 1. If $N_1, N_2 > \frac{k(k+1)}{2} + 1$, then it is clear that $\delta(G) > \frac{k(k+1)}{2} = |(A, B)| \geq \lambda(G)$. In this case it is easy to verify that (A, B) is in fact the mincut. Thus $\lambda(G) = \frac{\kappa(G)(\kappa(G)+1)}{2}$, for this graph.

References

1. L. Sunil Chandran. Edge connectivity vs vertex connectivity in chordal graphs. In *Proceedings of the 7th International Computing and Combinatorics conference, LNCS*, 2001.
2. L. Sunil Chandran. A linear time algorithm for enumerating all the minimum and minimal separators of a chordal graph. In *Proceedings of the 7th International Computing and Combinatorics Conference, LNCS 2108*, pages 308–317. Springer, Berlin, 2001.
3. L. Sunil Chandran and C. R. Subramanian. Girth and treewidth. *Journal of combinatorial theory, Series B*, 93(1):23–32, January 2005.
4. G. Chartrand and F.Harary. Graphs with prescribed connectivities. In P.Erdos and G.Katona, editors, *Theory of Graphs*, pages 61–63. Akademiai Kiado, Budapest, 1968.
5. Martin C Golumbic. *Algorithmic Graph Theory And Perfect Graphs*. Academic Press, New York, 1980.
6. F. Harary. *Graph Theory*. Addison-Wesley Reading, MA, 1969.
7. Ton Kloks. *Treewidth: Computations And Approximations*, volume 842 of *Lecture Notes In Computer Science*. Springer Verlag, Berlin, 1994.