ON OPTIMAL ORIENTATIONS OF TENSOR PRODUCT OF GRAPHS AND CIRCULANT GRAPHS

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Abstract. For a graph G, let $\mathcal{D}(G)$ be the set of all strong orientations of G. Define the *orientation number* of G, $\vec{d}(G) = \min\{d(D) \mid D \in \mathcal{D}(G)\}$, where $d(D)$ denotes the diameter of the digraph D. In this paper, it has been shown that $\vec{d}(G \times H) = d(G)$, where $\times$ denotes the tensor product of graphs, H is a special type of circulant graph and the diameter, $d(G)$, of G is at least 4. Some interesting results have been obtained using this result. Further, it is shown that $\vec{d}(P_r \times K_s) = d(P_r)$ for suitable r and s. Moreover, it is proved that $\vec{d}(C_r \times K_s) = d(C_r)$ for appropriate r and s.

1 Introduction

Let G be a simple graph with vertex set $V(G)$ and edge set $E(G)$. For $v \in V(G)$, the *eccentricity*, denoted by $e_G(v)$, of v is defined as $e_G(v) = \max\{d_G(v,x) \mid x \in V(G)\}$, where $d_G(v,x)$ denotes the distance from v to x in G. The *diameter* of G, denoted by $d(G)$, is defined as $d(G) = \max\{e_G(v) \mid v \in V(G)\}$.

Let D be a digraph with vertex set $V(D)$ and arc set $A(D)$ which has neither loops nor multiple arcs (that is, arcs with same tail and same head). For $v \in V(D)$, the notions $e_D(v)$ and $d(D)$ are defined as in the undirected graph. For $v \in V(D)$, $N_D^+(v)$ and $N_D^-(v)$ denote the set of out-neighbours and in-neighbours of v, respectively, in D. We call a digraph D to be k-*regular* if $d_D^+(v) = d_D^-(v) = k$ for every $v \in V(D)$. For $x, y \in V(D)$, we write $x \to y$ or $y \leftarrow x$ if $(x,y) \in A(D)$. For sets $X, Y \subseteq V(D)$, $X \to Y$ denotes $\{(x,y) \in A(D) : x \in X \text{ and } y \in Y\}$. For distinct vertices

$v_1, v_2, \ldots, v_k$, $v_1 \to v_2 \to \ldots \to v_k$ represents the directed path in D with arcs $v_1 \to v_2, v_2 \to v_3, \ldots, v_{k-1} \to v_k$. For subsets $V_1, V_2, \ldots, V_k$ of V, we write $V_1 \to V_2 \to \ldots \to V_k$ for the set of all directed paths of length $k-1$ whose ith vertex is in V_i, $1 \le i \le k$. For $x \in V(D)$ and $V' \subseteq V(D)$, by $d_D(x, V') \le k$, we mean $d_D(x, v') \le k$, for all $v' \in V'$.

For graphs G and H, the *tensor product*, $G \times H$, of G and H is the graph with vertex set $V(G) \times V(H)$ and $E(G \times H) = \{(u,v)(x,y) : ux \in E(G) \text{ and } vy \in E(H)\}$. If G and H are connected and nontrivial, then $G \times H$ is connected if and only if at least one of G and H is nonbipartite. Clearly, the tensor product is commutative. For $x \in V(G)$, the *H-layer*, denoted by H_x, is the subset $\{(x,y) : y \in V(H)\}$ of vertices of $G \times H$, and similarly, for $y \in V(H)$, the *G-layer*, denoted by G_y, of $G \times H$ is $\{(x,y) : x \in V(G)\}$.

Let P_n, C_n, K_n denote the path, cycle and complete graph of order n, respectively. Let $V(P_n) = V(C_n) = V(K_n) = \{0, 1, \ldots, n-1\}$ and the edge sets of P_n and C_n are $E(P_n) = \{\{i, i+1\} : i \in \{0, 1, \ldots, n-2\}\}$ and $E(C_n) = E(P_n) \cup \{\{n-1, 0\}\}$.

An *orientation* of a graph G is a digraph D obtained from G by assigning a direction to each of its edge. By abuse of notation, by D we mean an orientation of G and also the digraph arising out of an orientation of G.

A vertex v is *reachable* from a vertex u of a digraph D if there is a directed path in D from u to v. An orientation D of G is *strong* if any pair of vertices in D are mutually reachable in D. Robbins' celebrated one-way street theorem [7] states that a connected graph G has a strong orientation if and only if G is 2-edge-connected. Throughout this paper, whenever an orientation of a graph G is considered, we assume that G is 2-edge connected. For a 2-edge-connected graph G, let $\mathcal{D}(G)$ denote the set of all strong orientations of G. The *orientation number* of G is defined to be $\vec{d}(G) = \min \{d(D) \mid D \in \mathcal{D}(G)\}$. In [5], $\vec{d}(G) - d(G)$ is defined as $\rho(G)$.

Any orientation D in $\mathcal{D}(G)$ with $d(D) = \vec{d}(G)$ is called an *optimal orientation* of G. The problem of evaluating the orientation number of an arbitrary connected graph is very difficult as Chvátal and Thomassen [2] have shown that the problem of deciding whether a graph admits an orientation of diameter 2 is NP-hard. Further, among other results, they have shown that $\vec{d}(G) \le d(2d+1)$ if $d \ge 3$ and $\vec{d}(G) \le 6$ if $d = 2$, where d is the diameter of the 2-edge-connected graph G.

Goldberg [3] evaluated the extreme value of the diameter of a strong digraph with n vertices and $n+m$ arcs; he proved that if G is a 2-edge-connected graph with n vertices and $n+m$ edges, where $n \ge 4$ and $m \ge 1$, then $\vec{d}(G) \ge \left\lceil \frac{2(n-1)}{m+1} \right\rceil$. It is easy to see that if G is of girth g, then $\vec{d}(G) \ge g-1$; in particular, for C_g, $\vec{d}(C_g) = g-1$. Again, it is easy to see that if H is a spanning subgraph of G, then $\vec{d}(H) \geq \vec{d}(G)$. But this kind of property is not true for ρ, as $\rho(C_4) = 1 < 2 = \rho(K_4)$ and $\rho(C_5) = 2 > 1 = \rho(K_5)$. The parameter $\vec{d}(G)$ has also been studied in various particular classes of graphs including complete n-partite graphs and cartesian product of graphs (see the references in [5]). In [6], we have determined the exact value of $\rho(K_r \times K_s)$ for $(r,s) \notin \{(3,5),(3,6),(4,4),(5,3),(6,3)\}$. It is shown that for $r \leq s$ and $(r,s) \notin \{(3,5),(3,6),(4,4)\}$,

$$\rho(K_r \times K_s) = \begin{cases} 2 & \text{if } (r,s) \in \{(2,3),(2,4)\}, \\ 1 & \text{if } (r,s) \in \{(3,3),(3,4)\}, \\ 0 & \text{otherwise,} \end{cases}$$

and for the exceptional values, $(r,s) \in \{(3,5),(3,6),(4,4)\}$, $\rho(K_r \times K_s) \leq 1$. Optimal orientations have variety of applications, see [5]. For further results on orientations of graphs, see a recent survey by Koh and Tay [5].

Notations and terminology not defined here can be seen in [1] or [4].

Let n be a positive integer and let L be a subset of $\{1,2,\ldots,\lfloor\frac{n}{2}\rfloor\}$. A *circulant* $X(n;L)$ is a simple graph with vertex set $V(X(n;L)) = \mathbb{Z}_n$ and edge set $E(X(n;L)) = \{\{i,i+\ell\} : i \in \mathbb{Z}_n, \ell \in L\}$, where $\mathbb{Z}_n$ is the set of integers *modulon*.

If $L = \{1,2,\ldots,\lfloor\frac{n}{2}\rfloor\}$, then $X(n;L)$ is K_n. If n is even and if $L = \{1,2,\ldots,\frac{n}{2}-1\}$, then $X(n;L)$ is isomorphic to $K_n - F'$, where $F' = \{\{0,\frac{n}{2}\},\{1,\frac{n}{2}+1\},\ldots,\{\frac{n}{2}-1,n-1\}\}$ is a 1-factor of K_n.

In Sections 2, 3 and 4, we focus on the orientation numbers of $G \times X(n;L)$, for some $L \subseteq \{1,2,\ldots,\lfloor\frac{n}{2}\rfloor\}$, $P_r \times K_s$ and $C_r \times K_s$, respectively. As $P_2 \times K_s = K_2 \times K_s$ and $P_r \times K_2$ is disconnected, we assume in Section 3 that $r \geq 3$ and $s \geq 3$; again as $C_3 \times K_s = K_3 \times K_s$ and as $C_r \times K_2$ is either disconnected (if r is even) or isomorphic to C_{2r} (if r is odd), we assume in Section 4 that $r \geq 4$ and $s \geq 3$. One of the consequences of the main theorem of Section 2 is: if $d(G) \geq 4$, then $\rho(G \times K_r) = 0$, for every $r = 7$ or $r \geq 9$.

The results of Sections 3 and 4 are, respectively:

Theorem A.

1. $\rho(P_r \times K_s) = 0$ when $r \geq 5$ and $s \geq 5$, or $r \in \{3,4\}$ and $s \geq 9$, or $r \geq 8$ and $s = 4$.
2. $\rho(P_r \times K_s) \leq 1$ when $r \in \{3,4\}$ and $s \in \{5,6,7,8\}$, and $\rho(P_7 \times K_4) \leq 1$.
3. $\rho(P_3 \times K_3) = 1 = \rho(P_3 \times K_4)$.
4. $\rho(P_r \times K_3) \leq 2$ when $r \geq 5$, $\rho(P_5 \times K_4) \leq 2$ and $\rho(P_6 \times K_4) \leq 2$.
5. $1 \leq \rho(P_4 \times K_4) \leq 2$ and $\rho(P_4 \times K_3) = 2$.

Theorem B.

1. $\rho(C_r \times K_s) = 0$ when $r \geq 4$ and $s \geq 9$, or $r \geq 8$ and $s \in \{5,6,7,8\}$, or $r \geq 16$ and $s = 4$, or $r \in \{4,5\}$ and $s \in \{5,6,7,8\}$, or $(r,s) \in \{(4,4),(14,4),(6,7),(6,8)\}$.

2. $\rho(C_r \times K_s) \leq 1$ when $(r, s) \in \{(5,3),(5,4),(6,5),(6,6),(7,5),(7,6),(7,7),(7,8),(8,4),(9,4),(11,4),(12,4),(13,4),(15,4)\}$.
3. $\rho(C_4 \times K_3) = 1$.
4. $\rho(C_r \times K_3) \leq 2$ when $r \geq 7$ and $\rho(C_r \times K_4) \leq 2$ when $r \in \{6,7,10\}$.
5. $\rho(C_6 \times K_3) = 2$.

2 Optimal orientations of $G \times X(n; L)$

In this section, we find some sufficient conditions for $\rho(G \times X(n;L)) = 0$.

For $t \geq 3$, a subset $\hat{L}$ of $\mathbb{Z}_n$ is called an $\mathbb{Z}_n^t$*-set*, if $j \in \hat{L}$, then $(n-j)$ *mod* $n \notin \hat{L}$ and every element $i \in \mathbb{Z}_n$ can be written as $(a_1 + a_2 + \ldots + a_t) \bmod n$ for some $a_1, a_2, \ldots, a_t \in \hat{L}$; for $t = 2$, $\hat{L}$ is an $\mathbb{Z}_n^2$*-set*, if $j \in \hat{L}$, then $(n-j)$ *mod* $n \notin \hat{L}$ and every element $i \in \mathbb{Z}_n \backslash \{0\}$ can be written as $(a_1 + a_2) \bmod n$ for some $a_1, a_2 \in \hat{L}$. For an $\mathbb{Z}_n^t$-set $\hat{L}$, define $L = \{i \in \hat{L} : 0 < i < \frac{n}{2}\} \cup \{n - i : i \in \hat{L} \text{ and } \frac{n}{2} < i < n\}$.

For $x \in \mathbb{Z}_n$ and $A \subseteq \mathbb{Z}_n$, we define $x + A = \{(x+a) \bmod n : a \in A\}$. For $a, b \in \mathbb{Z}_n$, $[a,b] = \{a, (a+1) \bmod n, (a+2) \bmod n, \ldots, b\}$.

Lemma 2.1. *If $n \geq 12$ is even, then $\{2, 3, \ldots, \lceil \frac{n}{4} \rceil + 1, \frac{n}{2} + 1\}$ is an $\mathbb{Z}_n^3$-set and $\{2,3,6,9\}$ is an $\mathbb{Z}_{10}^3$-set.*

Proof. Let $n \geq 12$ be even. Observe that $(\frac{n}{2}+1) + (\frac{n}{2}+1) + [2,3] = [4,5]$, $2+2+[2, \lceil \frac{n}{4} \rceil + 1] = [6, \lceil \frac{n}{4} \rceil + 5]$, $3 + (\lceil \frac{n}{4} \rceil + 1) + [2, \lceil \frac{n}{4} \rceil + 1] \supseteq [\lceil \frac{n}{4} \rceil + 6, \frac{n}{2} + 5]$, $3 + (\frac{n}{2}+1) + [2, \lceil \frac{n}{4} \rceil + 1] = [\frac{n}{2}+6, \lceil \frac{3n}{4} \rceil + 5]$ and $(\lceil \frac{n}{4} \rceil + 1) + (\frac{n}{2}+1) + [2, \lceil \frac{n}{4} \rceil + 1] \supseteq [\lceil \frac{3n}{4} \rceil + 4, 3]$. As $[4,5] \cup [6, \lceil \frac{n}{4} \rceil + 5] \cup [\lceil \frac{n}{4} \rceil + 6, \frac{n}{2} + 5] \cup [\frac{n}{2}+6, \lceil \frac{3n}{4} \rceil + 5] \cup [\lceil \frac{3n}{4} \rceil + 4, 3] = \mathbb{Z}_n$ and each element in $[4,5] \cup [6, \lceil \frac{n}{4} \rceil + 5] \cup [\lceil \frac{n}{4} \rceil + 6, \frac{n}{2} + 5] \cup [\frac{n}{2} + 6, \lceil \frac{3n}{4} \rceil + 5] \cup [\lceil \frac{3n}{4} \rceil + 4, 3]$ is the sum of three elements, the result follows.

For $n = 10$, the verification is similar. ■

Lemma 2.2. *If $n = 7$ or $n \geq 11$ is odd, then $\{1, 2, \ldots, \lfloor \frac{n+1}{4} \rfloor, \frac{n-1}{2}\}$ is an $\mathbb{Z}_n^3$-set and $\{2,3,4,8\}$ is an $\mathbb{Z}_9^3$-set.*

Proof. Let $n = 7$ or $n \geq 11$ be odd. Now the proof follows from $\frac{n-1}{2} + \frac{n-1}{2} + [1,3] = [0,2]$, $1 + 1 + [1, \lfloor \frac{n+1}{4} \rfloor] = [3, \lfloor \frac{n+9}{4} \rfloor]$, $2 + \lfloor \frac{n+1}{4} \rfloor + [1, \lfloor \frac{n+1}{4} \rfloor] \supseteq [\lfloor \frac{n+13}{4} \rfloor, \frac{n+3}{2}]$, $1 + 2 + \frac{n-1}{2} = \frac{n+5}{2}$, $2 + \frac{n-1}{2} + [2, \lfloor \frac{n+1}{4} \rfloor] = [\frac{n+7}{2}, \lfloor \frac{3n+7}{4} \rfloor]$ and $\lfloor \frac{n+1}{4} \rfloor + \frac{n-1}{2} + [3, \lfloor \frac{n+1}{4} \rfloor] \supseteq [\lfloor \frac{3n+11}{4} \rfloor, n-1]$ (if $n \geq 11$).

For $n = 9$, the verification is similar. ■

Lemma 2.3. *(1). If $n \geq 12$ is even, then $\{2, 3, \ldots, \frac{n}{2} - 2, \frac{n}{2} + 1, n - 1\}$ is both an $\mathbb{Z}_n^2$-set and an $\mathbb{Z}_n^3$-set.*

(2). If $n \geq 9$ is odd, then $\{2, 3, \ldots, \frac{n-1}{2}, n - 1\}$ is both an $\mathbb{Z}_n^2$-set and an $\mathbb{Z}_n^3$-set.

Proof of (1). As $\{2,3,\ldots,\lceil\frac{n}{4}\rceil+1,\frac{n}{2}+1\}\subseteq\{2,3,\ldots,\frac{n}{2}-2,\frac{n}{2}+1,n-1\}$, $\{2,3,\ldots,\frac{n}{2}-2,\frac{n}{2}+1,n-1\}$ is an $\mathbb{Z}_n^3$ - set, by Lemma 2.1. $\{2,3,\ldots,\frac{n}{2}-2,\frac{n}{2}+1,n-1\}$ is an $\mathbb{Z}_n^2$ - set, since $(n-1)+[2,4]=[1,3]$, $2+[2,\frac{n}{2}-2]=[4,\frac{n}{2}]$, $(\frac{n}{2}-2)+[3,\frac{n}{2}-2]=[\frac{n}{2}+1,n-4]$ and $(\frac{n}{2}+1)+[\frac{n}{2}-4,\frac{n}{2}-2]=[n-3,n-1]$.

Proof of (2). $\{2,3,\ldots,\frac{n-1}{2},n-1\}$ is an $\mathbb{Z}_n^3$ - set, since $\frac{n-1}{2}+\frac{n-3}{2}+[2,4]=[0,2]$, $(n-1)+2+[2,4]=[3,5]$, $2+2+[2,\frac{n-1}{2}]=[6,\frac{n+7}{2}]$ and $\frac{n-1}{2}+3+[2,\frac{n-1}{2}]\supseteq[\frac{n+9}{2},n-1]$ (if $n>9$). Clearly, $\{2,3,\ldots,\frac{n-1}{2},n-1\}$ is an $\mathbb{Z}_n^2$ - set, since $(n-1)+[2,4]=[1,3]$, $2+[2,\frac{n-1}{2}]=[4,\frac{n+3}{2}]$ and $\frac{n-1}{2}+[3,\frac{n-1}{2}]=[\frac{n+5}{2},n-1]$. ∎

Theorem 2.1. *If $d(G)\geq 4$ and $\hat{L}$ is an $\mathbb{Z}_n^3$ - set, then $\rho(G\overset{\bullet}{\times}X(n;L))=0$.*

Proof. Let $d(G)=d'$ and let $H=X(n;L)$. Orient $G\times H$ so that for every edge xy of G, $(x,i)\to\{(y,i+\ell):\ell\in\hat{L}\}$. Let D be the resulting digraph. We shall show that $d(D)\leq d'$. This together with $\vec{d}(G\times H)\geq d(G\times H)\geq d(G)=d'$ imply that $\vec{d}(G\times H)=d'$. To show that $d(D)\leq d'$, it is enough to show that the eccentricity $e_D((x,i))\leq d'$ for each (x,i) in $V(D)$. By the nature of the orientation, we consider $(x,0)$ instead of (x,i).

Claim 1. For $z\in V(G)$, $z\neq x$, and $i\in\mathbb{Z}_n$, $d_D((x,0),(z,i))\leq d'$.

As $d(G)=d'$, there exists an (x,z)-path $x=v_0,v_1,\ldots,v_{k-1},v_k=z$ of length k $(\leq d')$ in G. Since $\hat{L}$ is an $\mathbb{Z}_n^3$ - set, $i=(a_1+a_2+a_3)\,mod\,n$ for some $a_1,a_2,a_3\in\hat{L}$.

Case 1. $k\geq 3$.

The existence of the path $(x,0)\to(v_1,a_1)\to(v_2,a_1+a_2)\to(v_3,i)$ in D proves that $d_D((x,0),H_{v_3})\leq 3$. Observe that, for $p<k$, if $d_D((x,0),H_{v_p})\leq p$, then $d_D((x,0),H_{v_{p+1}})\leq p+1$. Hence $d_D((x,0),H_z)\leq k$, since $k\geq 3$.

Case 2. $k=1$.

The existence of the path $(x,0)\to(v_1,a_1)\to(x,a_1+a_2)\to(v_1,i)$ in D proves that $d_D((x,0),H_{v_1})\leq 3$.

Case 3. $k=2$.

By Case 2, $d_D((x,0),H_{v_1})\leq 3$. Hence $d_D((x,0),H_{v_2})\leq 4$.

This completes the proof of Claim 1.

Claim 2. For $i\in\mathbb{Z}_n\backslash\{0\}$, $d_D((x,0),(x,i))\leq 4$.

Let $xy\in E(G)$. By Case 2 of Claim 1, $d_D((x,0),H_y)\leq 3$, and hence $d_D((x,0),H_x)\leq 4$. This proves Claim 2.

Hence $\vec{d}(G\times H)=d'$ and therefore $\rho(G\times H)=0$. ∎

Remark. Suppose that $\rho(G\times X(n;L))=0$ for a graph G with $d(G)\geq 4$

and a subset L of $\{1,2,\ldots,\lfloor\frac{n}{2}\rfloor\}$, then, clearly, every element $i \in \mathbb{Z}_n$ can be written as $(a_1+a_2+a_3) \bmod n$ for some a_1, a_2, a_3 with either $a_i \in L$ or $n-a_i \in L$, $i \in \{1,2,3\}$. ■

Let $K_r(s)$ denote the complete r-partite graph in which each partite set has s vertices. Note that $K_r(s)$ is isomorphic to the circulant graph $X(rs;L)$, where $L = \{1,2,\ldots,\lfloor\frac{rs}{2}\rfloor\}\setminus\{kr \mid 1 \le k \le \lfloor\frac{s}{2}\rfloor\}$. In particular, $K_r(2) \cong K_{2r} - F'$, where F' is a 1-factor of K_{2r}.

Corollary 2.1. *Let G be a graph with $d(G) \ge 4$.*
1. If $r=7$ or $r \ge 9$, then $\rho(G \times K_r) = 0$.
2. If $r \ge 5$, then $\rho(G \times K_r(2)) = 0$.
3. If $r \ge 5$, then $\rho(G \times K_r(3)) = 0$.

Proof of (1). If $r = 7$ or $r \ge 9$, then there is an $\mathbb{Z}_r^3$-set $\hat{L}$, by Lemmas 2.1 and 2.2. Consequently, $\rho(G \times X(r;L)) = 0$, by Theorem 2.1, and hence $\rho(G \times K_r) = 0$.

Proof of (2). If $r \ge 5$, then $\hat{L} = \{1,2,\ldots,r-1\} = L$ is an $\mathbb{Z}_{2r}^3$-set. Hence $\rho(G \times K_r(2)) = 0$, by Theorem 2.1.

Proof of (3). If $r \ge 5$ is odd, then $\hat{L} = \{1,2,\ldots,r-1,r+1,r+2,\ldots,\frac{3r-1}{2}\} = L$ is an $\mathbb{Z}_{3r}^3$-set, and hence $\rho(G \times K_r(3)) = 0$, by Theorem 2.1. If $r \ge 6$ is even, then $\hat{L} = \{1,2,\ldots,r-1,r+1,r+2,\ldots,\frac{3r}{2}-1\} = L$ is an $\mathbb{Z}_{3r}^3$-set, and hence $\rho(G \times X(3r;L)) = 0$, by Theorem 2.1; as $X(3r;L)$ is isomorphic to $K_r(3) - F$, where F is a 1-factor of $K_r(3)$, $\rho(G \times K_r(3)) = 0$. ■

The next theorem assures that even if the diameter of G is less than 4 still we may get optimal orientation in some special classes of graphs.

Let $\mathcal{G}^T$ denote the set of graphs G such that every vertex of G is in a cycle of length 3 (also called a triangle) in G and let $\mathcal{G}_2^3$ denote the set of graphs G such that for any pair of vertices a and b in G of distance 2, there also exists an (a,b)-path of length 3 in G.

Theorem 2.2. *Let $\hat{L}$ be an $\mathbb{Z}_n^3$-set and $G \in \mathcal{G}^T \cap \mathcal{G}_2^3$.*
1. If $d(G) = 3$, then $\rho(G \times X(n;L)) = 0$.
2. If $d(G) = 2$, then $\rho(G \times X(n;L)) \le 1$.
3. If $d(G) = 2$ and there is an edge of G which is not in a triangle of G, then $\rho(G \times X(n;L)) = 0$.

Proof. Let $H = X(n;L)$ and let D be the digraph obtained from $G \times H$ by the orientation described in Theorem 2.1. The proof technique of this theorem is similar to that of Theorem 2.1.

Claim 1. For $z \in V(G)$, $z \ne x$, and $i \in \mathbb{Z}_n$, $d_D((x,0),(z,i)) \le 3$.

Clearly, there exists an (x,z)-path $x = v_0, v_1, \ldots, v_{k-1}, v_k = z$ of length k $(\le d(G))$ in G.

Case 1. $k \in \{1,3\}$.

The paths of D as in Cases 1 and 2 of Theorem 2.1 prove, respectively, that $d_D((x,0), H_{v_3}) \leq 3$ and $d_D((x,0), H_{v_1}) \leq 3$.

Case 2. $k = 2$.

As $G \in \mathcal{G}_2^3$, there exists an (x,z)-path of length 3 in G, and hence the proof follows by Case 1.

Claim 2. For $i \in \mathbb{Z}_n \backslash \{0\}$, $d_D((x,0),(x,i)) \leq 3$.

As $G \in \mathcal{G}^T$, there is a triangle, say, $xyzx$ containing x in G. The existence of the path $(x,0) \to (y,a_1) \to (z,a_1+a_2) \to (x,i)$ in D proves that $d_D((x,0), H_x) \leq 3$.

By Claims 1 and 2, we have $d(D) \leq 3$ and hence $\rho(G \times H) = 0$ if $d(G) = 3$ and $\rho(G \times H) \leq 1$ if $d(G) = 2$. If $d(G) = 2$ and there is an edge of G which is not in a triangle of G, then $d(G \times H) = 3$ and hence $\rho(G \times H) = 0$. ■

There are many graphs in the class $\mathcal{G}^T \cap \mathcal{G}_2^3$. For example consider the graph G with $V(G) = \{w_1, \ldots, w_p, x_1, \ldots, x_q, y_1, \ldots, y_r, z_1, \ldots, z_s\}$, p,q,r and s are at least 2 and $E(G) = \{w_i x_j : i \in \{1,\ldots,p\}, j \in \{1,\ldots,q\}\} \cup \{x_i y_j : i \in \{1,\ldots,q\}, j \in \{1,\ldots,r\}\} \cup \{y_i z_j : i \in \{1,\ldots,r\}, j \in \{1,\ldots,s\}\} \cup \{w_i w_j : i,j \in \{1,\ldots,p\}, i \neq j\} \cup \{y_i y_j : i,j \in \{1,\ldots,r\}, i \neq j\}$. Then $G \in \mathcal{G}^T \cap \mathcal{G}_2^3$ and $d(G) = 3$.

For $n \geq 3$, the graph H is defined as follows: $V(H) = \{u_1, \ldots, u_n, v_1, \ldots, v_n\}$ and $E(H) = \{u_i v_i : i \in \{1,\ldots,n\}\} \cup \{u_i u_j, v_i v_j : i,j \in \{1,\ldots,n\}, i \neq j\}$. In otherwords, H is the cartesian product [4] of K_n and K_2. Then $H \in \mathcal{G}^T \cap \mathcal{G}_2^3$, $d(H) = 2$ and for any $i \in \{1,\ldots,n\}$, the edge $u_i v_i$ is not in a triangle of H.

Theorem 2.3. *Let $\hat{L}$ be both $\mathbb{Z}_n^2$ - set and $\mathbb{Z}_n^3$ - set and let $G \in \mathcal{G}_2^3$.*
1. If $d(G) = 3$, then $\rho(G \times X(n;L)) = 0$.
2. If $d(G) = 2$, then $\rho(G \times X(n;L)) \leq 1$.
3. If $d(G) = 2$ and there is an edge of G which is not in a triangle of G, then $\rho(G \times X(n;L)) = 0$.

Proof. Let $H = X(n;L)$ and let D be the digraph obtained from $G \times H$ by the orientation described in Theorem 2.1.

Claim 1. For $z \in V(G)$, $z \neq x$, and $i \in \mathbb{Z}_n$, $d_D((x,0),(z,i)) \leq 3$.

If $d_G(x,z) = 2$, then by hypothesis, there exists an (x,z)-path of length 3 in G. Now the proof of this claim is similar to the proof of Claim 1 of Theorem 2.2.

Claim 2. For $i \in \mathbb{Z}_n \backslash \{0\}$, $d_D((x,0),(x,i)) \leq 2$.

Let $xy \in E(G)$. Since $\hat{L}$ is an $\mathbb{Z}_n^2$ - set, $i = (a_1 + a_2) \, mod \, n$ for some

$a_1, a_2 \in \hat{L}$. The existence of the path $(x,0) \to (y,a_1) \to (x,i)$ in D, guaranteed by the orientation, proves that $d_D((x,0),H_x) \leq 2$. This proves Claim 2.

Claims 1 and 2 complete the proof of $d(D) \leq 3$ and hence $d(D) = 3$. ■

Note that, it can be verified that for $n \geq 8$, the set $\{1,2,\ldots,\lceil \frac{n}{3} \rceil\}$ is an $\mathbb{Z}_n^4$ - set and for $t \geq 4$, the set $\{1,2\}$ is an $\mathbb{Z}_{t+1}^t$ - set.

Theorem 2.4. *If $d(G) \geq t+1$, $t \geq 4$, and $\hat{L}$ is an $\mathbb{Z}_n^t$ - set, then* $\rho(G \times X(n;L)) = 0$.

Proof. Similar to the proof of Theorem 2.1. ■

As $K_8 - F_1 \cong X(8;\{1,2,3\})$ and $K_6 - F_2 \cong X(6;\{1,2\})$, where F_1 and F_2 are 1-factors of K_8 and K_6, respectively, we have the following

Corollary 2.2. *If $d(G) \geq 5$, then $\rho(G \times K_8) = 0$ and $\rho(G \times K_5) = 0$. If $d(G) \geq 6$, then $\rho(G \times K_6) = 0$.* ■

Let G be a graph and let n be a positive integer. Let $\ell(G;n) = \{\, L : L \subseteq \{1,2,\ldots,\lfloor \frac{n}{2} \rfloor\}$ and $\rho(G \times X(n;L)) = 0\}$. If L is in $\ell(G;n)$, then any superset of L contained in $\{1,2,\ldots,\lfloor \frac{n}{2} \rfloor\}$ is also in $\ell(G;n)$. Let $\ell_m(G;n)$ denote the set of minimal elements of $\ell(G;n)$. In view of the results of this section, we raise the following:

Problem. Given a graph G and a positive integer n, determine $\ell_m(G;n)$.

3 Optimal orientations of $P_r \times K_s$

As the subgraph induced by any two consecutive K_s-layers of $P_r \times K_s$ are isomorphic to $K_{s,s} - F_0$, where $F_0 = \{(i,k)(i+1,k) : k \in \mathbb{Z}_s\}$ is a 1-factor of $K_{s,s}$, we have

Lemma 3.1.

$$d(P_r \times K_s) = \begin{cases} 3, & \text{if } r \in \{2,3\} \text{ and } s \geq 3, \\ r-1, & \text{if } r \geq 4 \text{ and } s \geq 3. \end{cases}$$

■

In this section, we determine $\rho(P_r \times K_s)$ for $r \geq 3$ and $s \geq 3$. By Corollaries 2.1 and 2.2, $\rho(P_r \times K_s) = 0$ when $r \geq 5$ and $s \geq 9$, or $r \geq 6$ and $s = 8$, or $r \geq 5$ and $s = 7$, or $r \geq 7$ and $s = 6$, or $r \geq 6$ and $s = 5$. Next we shall look at the left over cases.

Lemma 3.2. *If $d(G) \in \{2,3\}$ and G has a vertex of degree 1, then both $\vec{d}(G \times K_3)$ and $\vec{d}(G \times K_4)$ are at least 4.*

Proof. Clearly, $d(G \times K_3) = 3 = d(G \times K_4)$. If possible assume that there is an orientation D of $G \times K_n$, $n \in \{3,4\}$, so that $d(D) = 3$. Let x be a vertex of degree 1 in G and let $N_G(x) = \{y\}$. If (x,j) has exactly one out-neighbour, say, (y,k), $k \in \mathbb{Z}_n \backslash \{j\}$, then $d_D((x,j),(x,k)) \geq 4$, a contradiction. Hence $d_D^+((x,j)) \neq 1$. Similarly, $d_D^-((x,j)) \neq 1$ (can be obtained by considering the converse digraph of D). This implies that $n \notin \{3,4\}$, a contradiction. ■

Corollary 3.1. $\rho(P_3 \times K_3) = 1 = \rho(P_3 \times K_4)$ *and* $1 \leq \rho(P_4 \times K_4) \leq 2$.

Proof. As $d(P_3 \times K_3) = d(P_3 \times K_4) = d(P_4 \times K_4) = 3$, it is enough to show that $\vec{d}(P_3 \times K_3) = 4$, $\vec{d}(P_3 \times K_4) = 4$ and $4 \leq \vec{d}(P_4 \times K_4) \leq 5$.

By Lemma 3.2, $\vec{d}(P_3 \times K_3) \geq 4$. The digraph $D_{3,3}$ obtained by the orientation $(0,j) \to (1,j+1)$, $(1,j) \to \{(0,j+1),(2,j+2)\}$ and $(2,j) \to (1,j+2)$, where $j \in \mathbb{Z}_3$, shows that $\vec{d}(P_3 \times K_3) \leq 4$.

By Lemma 3.2, $\vec{d}(P_3 \times K_4) \geq 4$. The digraph $D_{3,4}$ obtained by the orientation $(0,j) \to \{(1,j+1),(1,j+2)\}$, $(1,j) \to \{(0,j+1),(2,j+1),(2,j+3)\}$ and $(2,j) \to (1,j+2)$, where $j \in \mathbb{Z}_4$, shows that $\vec{d}(P_3 \times K_4) \leq 4$.

By Lemma 3.2, $\vec{d}(P_4 \times K_4) \geq 4$. The digraph $D_{4,4}$ obtained by the orientation $(0,j) \to \{(1,j+1),(1,j+2)\}$, $(1,j) \to \{(0,j+1),(2,j+2)\}$, $(2,j) \to \{(1,j+1),(1,j+3),(3,j+1),(3,j+3)\}$ and $(3,j) \to \{(2,j+2)\}$, where $j \in \mathbb{Z}_4$, shows that $\vec{d}(P_4 \times K_4) \leq 5$. ■

Lemma 3.3. $\rho(P_4 \times K_3) = 2$.

Proof. As $d(P_4 \times K_3) = 3$, to complete the proof, it is enough to show that $\vec{d}(P_4 \times K_3) = 5$. The digraph $D_{4,3}$ obtained by the orientation $(0,j) \to (1,j+1)$, $(1,j) \to \{(0,j+1),(2,j+2)\}$, $(2,j) \to \{(1,j+2),(3,j+1)\}$, $(3,j) \to (2,j+1)$, for $j \in \mathbb{Z}_3$, shows that $\vec{d}(P_4 \times K_3) \leq 5$. Next we show that $\vec{d}(P_4 \times K_3) \geq 5$. If possible assume that there is an orientation D of $P_4 \times K_3$ so that $d(D) \leq 4$. Clearly, for $i \in \{0,3\}$ and $j \in \mathbb{Z}_3$, $d_D^+((i,j)) = 1 = d_D^-((i,j))$.

Claim 1. For $i \in \{1,2\}$ and $j \in \mathbb{Z}_3$, $d_D^+((i,j)) = 2 = d_D^-((i,j))$.

By the symmetric nature of the graph $P_4 \times K_3$, it is enough to verify Claim 1 for the vertex $(1,0)$. First we suppose that $d_D^+((1,0)) = 1$. If $N_D^+((1,0))$ is $\{(0,1)\}$ or $\{(2,1)\}$, then $d_D((1,0),(0,2)) > 4$, a contradiction, and if $N_D^+((1,0))$ is $\{(0,2)\}$ or $\{(2,2)\}$, then $d_D((1,0),(0,1)) > 4$, again a contradiction. Hence $d_D^+((1,0)) \neq 1$. Similarly, $d_D^-((1,0)) \neq 1$ (can be obtained by considering the converse digraph of D) and therefore $d_D^+((1,0)) = 2 = d_D^-((1,0))$.

Claim 2. For $i \in \{1,2\}$ and $j \in \mathbb{Z}_3$, the two out-neighbours of (i,j) are in different P_4-layers of D.

Again, by the symmetric nature of $P_4 \times K_3$, we prove Claim 2 only for the vertex $(1,0)$. By Claim 1, $|N_D^+((1,0))| = 2$. If $N_D^+((1,0))$ is contained in a P_4-layer, then $N_D^+((1,0))$ is $\{(0,1),(2,1)\}$ or $\{(0,2),(2,2)\}$. Without loss of generality, assume that $N_D^+((1,0)) = \{(0,1),(2,1)\}$. Consequently, $N_D^-((1,0)) = \{(0,2),(2,2)\}$. We now have, $(0,1) \to (1,2)$. Hence $d_D((1,0),(0,2)) > 4$, a contradiction. This contradiction establishes the claim.

Claim 3. For $i \in \{1,2\}$ and $j \in \mathbb{Z}_3$, the two out-neighbours of (i,j) are in different K_3-layers of D.

Once again, by the symmetric nature of $P_4 \times K_3$, we prove Claim 3 only for the vertex $(1,0)$. If $N_D^+((1,0))$ is contained in a K_3-layer, then $N_D^+((1,0))$ is $\{(0,1),(0,2)\}$ or $\{(2,1),(2,2)\}$.

Case 1. If $N_D^+((1,0)) = \{(0,1),(0,2)\}$, then $N_D^-((1,0)) = \{(2,1),(2,2)\}$. We now have, $(0,1) \to (1,2)$ and $(0,2) \to (1,1)$. By Claims 1 and 2, $(1,2) \to (2,1)$, $(1,1) \to (2,2)$, $(2,1) \leftarrow (3,0)$ and $(2,2) \leftarrow (3,0)$. Now $d_D^+((3,0)) = 2$, a contradiction.

Case 2. If $N_D^+((1,0)) = \{(2,1),(2,2)\}$, then $N_D^-((1,0)) = \{(0,1),(0,2)\}$. For the converse digraph of D, Case 1 happens and therefore again a contradiction. Thus both the cases lead to contradictions, and hence the proof of Claim 3 is complete.

If any one of the edges of $P_4 \times K_3$ is oriented, then any strong orientation arising out of it satisfying Claims 1, 2 and 3 is isomorphic to $D_{4,3}$ or its converse digraph. Note that $D_{4,3}$ and its converse digraph are isomorphic (the required isomorphism f is $f((i,0)) = (i,0)$, $f((i,1)) = (i,2)$ and $f((i,2)) = (i,1)$; $i \in \{0,1,2,3\}$). But $d(D_{4,3}) = 5$ and this yields the required contradiction. Hence $\vec{d}(P_4 \times K_3) \geq 5$ and this proves $\vec{d}(P_4 \times K_3) = 5$. ■

Lemma 3.4. *If $s \geq 9$, then $\rho(P_3 \times K_s) = 0$. Furthermore, $\rho(P_3 \times K_s) \leq 1$ if $s \in \{5,6,7,8\}$.*

Proof. Orient $P_3 \times K_s$ so that for any $j \in \mathbb{Z}_s$, $(0,j) \to \{(1,j+1),(1,j+2),\ldots,(1,j+\lfloor\frac{s-3}{2}\rfloor),(1,j+\lceil\frac{s+1}{2}\rceil)\}$, $(1,j) \to \{(0,j+1),(0,j+2),\ldots,(0,j+\lfloor\frac{s-3}{2}\rfloor),(0,j+\lceil\frac{s+1}{2}\rceil),(2,j+2),(2,j+3),\ldots,(2,j+\lfloor\frac{s-1}{2}\rfloor),(2,j-1)\}$ and $(2,j) \to \{(1,j+2),(1,j+3),\ldots,(1,j+\lfloor\frac{s-1}{2}\rfloor),(1,j-1)\}$. In addition, if s is even, orient $(0,j) \to (1,j+\frac{s}{2})$ and $(2,j) \to (1,j+\frac{s}{2})$. Let D be the resulting digraph.

We assert that if $s \geq 9$, then $d(D) = 3$. To show this, by the nature of the orientation, it is enough to show that the eccentricities of the vertices $(0,0)$, $(1,0)$ and $(2,0)$ are all equal to 3. As it is a routine verification, we leave it to the reader. Also, for $s \in \{5,6,7,8\}$, it can be verified that $d(D) = 4$. ■

Lemma 3.5. *If $s \geq 9$, then $\rho(P_4 \times K_s) = 0$.*

Proof. Orient $P_4 \times K_s$ so that for any $j \in \mathbb{Z}_s$, $(p,j) \to \{(q,j+1), (q,j+2), \ldots, (q,j+\lfloor\frac{s-3}{2}\rfloor), (q,j+\lceil\frac{s+1}{2}\rceil)\}$ whenever $(p,q) \in \{(0,1),(1,0),(2,3),(3,2)\}$ and $(p,j) \to \{(q,j+2), (q,j+3), \ldots, (q,j+\lfloor\frac{s-1}{2}\rfloor), (q,j-1)\}$ whenever $(p,q) \in \{(1,2),(2,1)\}$. In addition, if s is even, orient $(0,j) \to (1,j+\frac{s}{2})$ and $(2,j) \to \{(1,j+\frac{s}{2}),(3,j+\frac{s}{2})\}$. Let D be the resulting digraph. It is easy to verify that the eccentricities of the vertices $(0,0)$, $(1,0)$, $(2,0)$ and $(3,0)$ are all equal to 3, and hence $d(D) = 3$. ■

Lemma 3.6. *If $s \in \{5,6,7,8\}$, then $\rho(P_4 \times K_s) \leq 1$.*

Proof. Orient $P_4 \times K_s$ so that for any $j \in \mathbb{Z}_s$, $(p,j) \to \{(q,j+1), (q,j+2), \ldots, (q,j+\lfloor\frac{s-3}{2}\rfloor), (q,j+\lceil\frac{s+1}{2}\rceil)\}$ whenever $(p,q) \in \{(0,1),(1,0)\}$, $(p,j) \to \{(q,j+2), (q,j+3), \ldots, (q,j+\lfloor\frac{s-1}{2}\rfloor), (q,j-1)\}$ whenever $(p,q) \in \{(1,2),(2,1)\}$ and $(p,j) \to \{(q,j+\lceil\frac{s+1}{2}\rceil), (q,j+\lceil\frac{s+3}{2}\rceil), \ldots, (q,j-1)\}$ whenever $(p,q) \in \{(2,3),(3,2)\}$. In addition, if s is even, orient $(0,j) \to (1,j+\frac{s}{2})$ and $(2,j) \to \{(1,j+\frac{s}{2}),(3,j+\frac{s}{2})\}$. Let D be the resulting digraph. It is easy to verify that $d(D) = 4$. ■

Lemma 3.7. *If $s \in \{5,6,8\}$, then $\rho(P_5 \times K_s) = 0$.*

Proof. Orient $P_5 \times K_s$, $s \in \{5,6,8\}$, so that for any $j \in \mathbb{Z}_s$, $(p,j) \to \{(q,j+1), \ldots, (q,j+\lfloor\frac{s-1}{2}\rfloor)\}$ whenever $(p,q) \in \{(0,1),(1,0),(2,3),(3,2)\}$ and $(p,j) \to \{(q,j+2), \ldots, (q,j+\lfloor\frac{s-1}{2}\rfloor), (q,j-1)\}$ whenever $(p,q) \in \{(1,2),(2,1),(3,4),(4,3)\}$. In addition, if $s \in \{6,8\}$, orient $(0,j) \to (1,j+\frac{s}{2})$, $(2,j) \to \{(1,j+\frac{s}{2}),(3,j+\frac{s}{2})\}$ and $(3,j) \to (4,j+\frac{s}{2})$. It can be verified that the resulting digraph D is of diameter 4. ■

Lemma 3.8. $\rho(P_6 \times K_6) = 0$.

Proof. Orient $P_6 \times K_6$ so that for any $j \in \mathbb{Z}_6$, $(p,j) \to \{(q,j+1), (q,j+2)\}$ whenever $(p,q) \in \{(0,1),(1,0),(2,3),(3,2),(4,5),(5,4)\}$ and $(p,j) \to \{(q,j+2), (q,j+5)\}$ whenever $(p,q) \in \{(1,2),(2,1),(3,4),(4,3)\}$. In addition, orient $(0,j) \to (1,j+3)$, $(2,j) \to \{(1,j+3),(3,j+3)\}$, $(3,j) \to (4,j+3)$ and $(4,j) \to (5,j+3)$. It can be verified that the resulting digraph D is of diameter 5. ■

Lemma 3.9. *If $r \geq 8$, then $\rho(P_r \times K_4) = 0$. Furthermore, $\rho(P_7 \times K_4) \leq 1$, $\rho(P_6 \times K_4) \leq 2$ and $\rho(P_5 \times K_4) \leq 2$.*

Proof. Orient $P_r \times K_4$ so that for any $i \in \{1,2,\ldots,r-2\}$ and $j \in \mathbb{Z}_4$, $(i,j) \to (i\pm 1,j+1)$, $(0,j) \to (1,j+1)$, $(r-1,j) \to (r-2,j+1)$. In addition, (i,j)

$\to (i\pm 1, j+2)$ whenever i is even, $(0,j) \to (1, j+2)$ and $(r-1,j) \to (r-2,j+2)$ whenever r is odd. Let D be the resulting digraph. We leave the routine verification of $d(D) = r-1$ if $r \geq 8$, $d(D) = 7$ if $r \in \{6,7\}$ and $d(D) = 6$ if $r = 5$. ■

Lemma 3.10. *If* $r \geq 5$, *then* $\rho(P_r \times K_3) \leq 2$.

Proof. Let $(a,b) \in \{(0,1),(1,0),(2,3),(3,2),(4,5),(5,4),\ldots,(r-2,r-1),(r-1,r-2)\}$ if r is even and $(a,b) \in \{(0,1),(1,0),(2,3),(3,2),(4,5),(5,4),\ldots,(r-3,r-2),(r-2,r-3)\}$ if r is odd, and let $(c,d) \in \{(1,2),(2,1),(3,4),(4,3),(5,6),(6,5),\ldots,(r-3,r-2),(r-2,r-3)\}$ if r is even and $(c,d) \in \{(1,2),(2,1),(3,4),(4,3),(5,6),(6,5),\ldots,(r-2,r-1),(r-1,r-2)\}$ if r is odd. Orient $P_r \times K_3$ so that for any $j \in \mathbb{Z}_3$, $(a,j) \to (b,j+1)$ and $(c,j) \to (d,j+2)$. Let D be the resulting digraph. It can be verified that $d(D) = r+1$. ■

Combining all the results of this section, we have the proof of Theorem A.

4 Optimal orientations of $C_r \times K_s$

As the subgraph induced by any two consecutive K_s-layers of $C_r \times K_s$ are isomorphic to $K_{s,s} - F_0$, where $F_0 = \{(i,k)(i+1,k) : k \in \mathbb{Z}_s\}$ is a 1-factor of $K_{s,s}$, we have

Lemma 4.1.

$$d(C_r \times K_s) = \begin{cases} 2, & \text{if } r = 3 \text{ and } s \geq 3, \\ 3, & \text{if } (r,s) = (3,2) \text{ or } r \in \{4,5\} \text{ and } s \geq 3, \\ \lfloor \frac{r}{2} \rfloor, & \text{if } r \geq 6 \text{ and } s \geq 3, \\ r, & \text{if } r \text{ is odd and } s = 2. \end{cases}$$

■

In this section, we consider $\rho(C_r \times K_s)$, $r \geq 4$ and $s \geq 3$. By Corollaries 2.1 and 2.2, $\rho(C_r \times K_s) = 0$ when $r \geq 8$ and $s \geq 9$, or $r \geq 10$ and $s = 8$, or $r \geq 8$ and $s = 7$, or $r \geq 12$ and $s = 6$, or $r \geq 10$ and $s = 5$. By Lemma 2.3 and Theorem 2.3 (3), $\rho(C_5 \times K_s) = 0$ when $s \geq 11$ or $s = 9$. Next we shall consider small values of r and s.

Lemma 4.2. $\vec{d}(C_{2k} \times K_3) \leq k+2$.

Proof. Orient $C_{2k} \times K_3$ as follows: for any $i \in \mathbb{Z}_{2k}$ and $j \in \mathbb{Z}_3$, $(i,j) \to \{(i+1,j+1),(i-1,j+2)\}$ if i is even and $(i,j) \to \{(i+1,j+2),(i-1,j+1)\}$ if i is odd. Let the resulting digraph be $D'_{2k,3}$. It can be verified that $d(D'_{2k,3}) \leq k+2$. ■

For $k \geq 3$, $d(C_{2k} \times K_3) = k$, so we have

Corollary 4.1. *If* $k \geq 3$, *then* $\rho(C_{2k} \times K_3) \leq 2$. ■

Lemma 4.3. $\rho(C_4 \times K_3) = 1$.

Proof. As $d(C_4 \times K_3) = 3$, to complete the proof, it is enough to show that $\vec{d}(C_4 \times K_3) = 4$. By Lemma 4.2, $\vec{d}(C_4 \times K_3) \leq 4$.

Next we show that $\vec{d}(C_4 \times K_3) \geq 4$. If possible assume that there is an orientation D of $C_4 \times K_3$ so that $d(D) = 3$.

Claim 1. $d_D^+((i,j)) = 2 = d_D^-((i,j))$ for all $(i,j) \in V(C_4 \times K_3)$.

If there exists a vertex (i,j) such that (i,j) has exactly one out-neighbour, say, $(i+1,j+1)$, then $d_D((i,j),(i,j+1)) > 3$, a contradiction. Hence for any (i,j), $d_D^+((i,j)) \neq 1$. Similarly, $d_D^-((i,j)) \neq 1$ (can be obtained by considering the converse digraph of D) and therefore $d_D^+((i,j)) = 2 = d_D^-((i,j))$.

Claim 2. For any vertex (i,j), the two out-neighbours of (i,j) are in different C_4-layers of D.

If there exists a vertex (i,j) such that $N_D^+((i,j))$ is contained in a single C_4-layer, then $N_D^+((i,j))$ is $\{(i+1,j+1),(i+3,j+1)\}$ or $\{(i+1,j+2),(i+3,j+2)\}$. Without loss of generality, assume that $N_D^+((i,j)) = \{(i+1,j+1),(i+3,j+1)\}$. But then $N_D^-((i,j)) = \{(i+1,j+2),(i+3,j+2)\}$. Consequently, $d_D((i,j),(i,j+1)) > 3$, a contradiction.

Claim 3. For any vertex (i,j), the two out-neighbours of (i,j) are in different K_3-layers of D.

If there exists a vertex (i,j) such that $N_D^+((i,j))$ is contained in a single K_3-layer, then $N_D^+((i,j))$ is $\{(i+1,j+1),(i+1,j+2)\}$ or $\{(i+3,j+1),(i+3,j+2)\}$. Without loss of generality, assume that $N_D^+((i,j)) = \{(i+1,j+1),(i+1,j+2)\}$. Consequently, $N_D^-((i,j)) = \{(i+3,j+1),(i+3,j+2)\}$.

As the vertices (i,j), $(i,j+1)$ and $(i,j+2)$ are in the same partite set of the bipartite graph $C_4 \times K_3$, $d_D((i,j),\{(i,j+1),(i,j+2)\}) \leq 2$, and therefore $(i+1,j+2) \to (i,j+1)$ and $(i+1,j+1) \to (i,j+2)$. By Claims 1 and 2, we have $(i+1,j+1) \to (i+2,j)$, $(i+1,j+1) \leftarrow (i+2,j+2)$, $(i+1,j+2) \to (i+2,j)$ and $(i+1,j+2) \leftarrow (i+2,j+1)$. Now $d_D((i+1,j+1),(i+1,j+2)) > 3$, a contradiction. This contradiction proves Claim 3.

If any one of the edges of $C_4 \times K_3$ is oriented, then any strong orientation arising out of it satisfying Claims 1, 2 and 3 is isomorphic to $D'_{4,3}$ or its converse digraph. Note that $D'_{4,3}$ and its converse digraph are isomorphic (the required isomorphism f is $f((i,j)) = (i+1,j))$. But $d(D'_{4,3}) = 4$. This yields the required contradiction. Hence $\vec{d}(C_4 \times K_3) \geq 4$ and this proves $\vec{d}(C_4 \times K_3) = 4$. ■

Lemma 4.4. $\rho(C_6 \times K_3) = 2$.

Proof. By Corollary 4.1, $\rho(C_6 \times K_3) \leq 2$.

Next we show that $\rho(C_6 \times K_3) \geq 2$ i.e., $\vec{d}(C_6 \times K_3) \geq 5$. If possible assume that there is an orientation D of $C_6 \times K_3$ with $d(D) \leq 4$.

Claim 1. $d_D^+((i,j)) = 2 = d_D^-((i,j))$ for all $(i,j) \in V(C_6 \times K_3)$.

If there exists a vertex (i,j) such that (i,j) has exactly one out-neighbour, say, $(i+1, j+1)$, then $d_D((i,j),(i+5,j+2)) > 4$, a contradiction. Hence for any (i,j), $d_D^+((i,j)) \neq 1$. Similarly, $d_D^-((i,j)) \neq 1$ (can be obtained by considering the converse digraph of D) and therefore $d_D^+((i,j)) = 2 = d_D^-((i,j))$.

Claim 2. For any vertex (i,j), the two out-neighbours of (i,j) are in different C_6-layers of D.

If there exists a vertex (i,j) such that $N_D^+((i,j))$ is contained in a single C_6-layer, then $N_D^+((i,j))$ is $\{(i+1,j+1),(i+5,j+1)\}$ or $\{(i+1,j+2),(i+5,j+2)\}$. Without loss of generality, assume that $N_D^+((i,j)) = \{(i+1,j+1),(i+5,j+1)\}$. But then $N_D^-((i,j)) = \{(i+1,j+2),(i+5,j+2)\}$. Note that $C_6 \times K_3$ is bipartite with bipartition $\{(k,l) : k \in \{0,2,4\}, l \in \mathbb{Z}_3\}$, $\{(k,l) : k \in \{1,3,5\}, l \in \mathbb{Z}_3\}$ and hence if two vertices of D are in different partite sets, then they are at distance at most 3. $d_D((i,j),\{(i+1,j+2),(i+5,j+2)\}) \leq 3$ implies that $(i+1,j+1) \to (i+2,j) \to (i+1,j+2)$ and $(i+5,j+1) \to (i+4,j) \to (i+5,j+2)$. By Claim 1, $(i,j+2) \to (i+1,j+1) \to (i+2,j+2)$ or $(i,j+2) \leftarrow (i+1,j+1) \leftarrow (i+2,j+2)$.

Case 1. $(i,j+2) \to (i+1,j+1) \to (i+2,j+2)$.

We now have $d_D((i+1,j+1),(i,j+2)) \leq 3$ implies that $(i+2,j+2) \to (i+1,j) \to (i,j+2)$. $d_D((i,j),(i+3,j+2)) \leq 3$ implies that $(i+2,j) \to (i+3,j+2)$ or $(i+4,j) \to (i+3,j+2)$.

Case 1(a). $(i+2,j) \to (i+3,j+2)$.

We now have $(i+2,j) \leftarrow (i+3,j+1)$, by Claim 1. $d_D((i+2,j),(i+3,j+1)) \leq 3$ implies that $(i+3,j+2) \to (i+4,j) \to (i+3,j+1)$. $d_D((i+5,j+1),(i+2,j+1)) \leq 3$ implies that $(i+5,j+1) \to (i+4,j+2) \to (i+3,j) \to (i+2,j+1)$. By Claim 1, $(i+5,j+1) \leftarrow (i,j+2)$; again by Claim 1, $(i,j+2) \leftarrow (i+5,j)$. $d_D((i,j),(i+5,j)) \leq 3$ implies that $(i+4,j+2) \to (i+5,j)$. Recursively, by Claim 1, $(i+4,j+2) \leftarrow (i+3,j+1)$, $(i+3,j+1) \leftarrow (i+2,j+2)$, $(i+2,j+2) \leftarrow (i+3,j)$ and $(i+3,j) \leftarrow (i+4,j+1)$. $d_D((i+1,j+1),(i+4,j+1)) \leq 3$ implies that $(i+3,j+2) \to (i+4,j+1)$. By Claim 1, $(i+3,j+2) \leftarrow (i+2,j+1)$. Then $d_D((i+1,j+1),(i+2,j+1)) > 3$, a contradiction.

Case 1(b). $(i+4,j) \to (i+3,j+2)$.

We now have $(i+4,j) \leftarrow (i+3,j+1)$ by Claim 1. $d_D((i+4,j),(i+3,j+1)) \leq 3$ implies that $(i+3,j+2) \to (i+2,j) \to (i+3,j+1)$. $d_D((i+1,j+1),(i+4,j+1)) \leq 3$ implies that $(i+2,j+2) \to (i+3,j) \to (i+4,j+1)$. By Claim 1, $(i+2,j+2) \leftarrow (i+3,j+1)$; again by Claim 1,

$(i+3,j+1) \leftarrow (i+4,j+2)$. $d_D((i+2,j+2),(i+3,j+1)) \leq 3$ implies that $(i+3,j) \rightarrow (i+4,j+2)$. By Claim 1, $(i+3,j) \leftarrow (i+2,j+1)$. $d_D((i+3,j),(i,j)) \leq 3$ implies that $(i+4,j+1) \rightarrow (i+5,j+2)$. By Claim 1, $(i+5,j+2) \rightarrow (i,j+1)$. Define a mapping f on $V(C_6 \times K_3)$ by $f((k,j)) = (k+2,j)$, $f((k,j+1)) = (k+2,j+2)$, $f((k,j+2)) = (k+2,j+1)$, $k \in \mathbb{Z}_6$. Applying f to the vertices of $C_6 \times K_3$ results in the digraph of Case 1a.

Case 2. $(i,j+2) \leftarrow (i+1,j+1) \leftarrow (i+2,j+2)$.

We now have $d_D((i,j),(i+1,j)) \leq 3$ implies that $(i,j+2) \rightarrow (i+1,j)$ (since $(i,j) \rightarrow (i+1,j+1) \rightarrow (i,j+2) \rightarrow (i+1,j)$ or $(i,j) \rightarrow (i+5,j+1) \rightarrow (i,j+2) \rightarrow (i+1,j)$). $d_D((i,j),(i+3,j)) \leq 3$ implies that $(i+5,j+1) \rightarrow (i+4,j+2) \rightarrow (i+3,j)$. By Claim 1, $(i+5,j+1) \leftarrow (i,j+2)$; again by Claim 1, $(i,j+2) \leftarrow (i+5,j)$. $d_D((i+1,j+1),(i+4,j+1)) \leq 3$ implies that $(i+2,j) \rightarrow (i+3,j+2) \rightarrow (i+4,j+1)$. By Claim 1, $(i+2,j) \leftarrow (i+3,j+1)$. $d_D((i+2,j),(i+3,j+1)) \leq 3$ implies that $(i+3,j+2) \rightarrow (i+4,j) \rightarrow (i+3,j+1)$. By Claim 1, $(i+3,j+2) \leftarrow (i+2,j+1)$. Define a mapping g on $V(C_6 \times K_3)$ by $g((k,j)) = (k+4,j)$, $g((k,j+1)) = (k+4,j+2)$, $g((k,j+2)) = (k+4,j+1)$, $k \in \mathbb{Z}_6$. Applying g to the vertices of $C_6 \times K_3$ results in the digraph of Case 1.

Claim 3. For any vertex (i,j), the two out-neighbours of (i,j) are in different K_3-layers of D.

If there exists a vertex (i,j) such that $N_D^+((i,j))$ is contained in a K_3-layer, then $N_D^+((i,j))$ is $\{(i+1,j+1),(i+1,j+2)\}$ or $\{(i+5,j+1),(i+5,j+2)\}$. Without loss of generality, assume that $N_D^+((i,j)) = \{(i+1,j+1),(i+1,j+2)\}$. Consequently, $N_D^-((i,j)) = \{(i+5,j+1),(i+5,j+2)\}$.

$d_D((i,j),\{(i+5,j+1),(i+5,j+2)\}) \leq 3$ implies that $(i+1,j+1) \rightarrow (i,j+2) \rightarrow (i+5,j+1)$ and $(i+1,j+2) \rightarrow (i,j+1) \rightarrow (i+5,j+2)$. By Claims 1 and 2, we have $(i+2,j+2) \rightarrow (i+1,j+1) \rightarrow (i+2,j)$ and $(i+2,j+1) \rightarrow (i+1,j+2) \rightarrow (i+2,j)$. Now $d_D((i,j),(i+3,j)) > 4$, a contradiction. This contradiction proves Claim 3.

If any one of the edges of $C_6 \times K_3$ is oriented, then any strong orientation arising out of it satisfying Claims 1, 2 and 3 is isomorphic to $D'_{6,3}$ or its converse digraph. Note that $D'_{6,3}$ and its converse digraph are isomorphic (the required mapping f is $f((i,j)) = (i+1,j))$. But $d(D'_{6,3}) = 5$. This yields the required contradiction. Hence $\vec{d}(C_6 \times K_3) \geq 5$ and this proves $\vec{d}(C_6 \times K_3) = 5$. ∎

Lemma 4.5. *If $r \geq 4$ is even and $s \geq 7$, then $\rho(C_r \times K_s) = 0$. Furthermore, $\rho(C_4 \times K_6) = 0$.*

Proof. Orient $C_r \times K_s$ so that for any $i \in \mathbb{Z}_r$ and $j \in \mathbb{Z}_s$, $(i,j) \rightarrow \{(i+1,j+1),(i+1,j+2),\ldots,(i+1,j+\lfloor\frac{s}{2}\rfloor)$, $(i-1,j+2)$, $(i-1,j+3)$, $\ldots,(i-1,j+\lfloor\frac{s}{2}\rfloor)$, $(i-1,j-1)\}$, if i is even and $(i,j) \rightarrow \{(i+1,j+2)$,

$(i+1,j+3), \ldots, (i+1,j+\lfloor\frac{s-1}{2}\rfloor), (i+1,j-1), (i-1,j+1), (i-1,j+2), \ldots, (i-1,j+\lfloor\frac{s-1}{2}\rfloor)\}$, if i is odd. Let D be the resulting digraph.

We claim that $d(D) = \frac{r}{2}$ if $r \geq 6$ and $d(D) = 3$ if $r = 4$. To show this, by the nature of the orientation, it is enough to show that the eccentricities of the vertices $(0,0)$ and $(1,0)$ are both equal to $\frac{r}{2}$ if $r \geq 6$ and 3 if $r = 4$. We leave the verification to the reader.

For $(r,s) = (4,6)$, the above orientation of $C_r \times K_s$ results in a digraph D for which $d(D) = 3$, and hence $\rho(C_4 \times K_6) = 0$. ∎

Lemma 4.6. *If* $k \geq 2$, *then* $\vec{d}(C_{2k+1} \times K_3) \leq k+2$.

Proof. Orient $C_{2k+1} \times K_3$ so that for any $j \in \mathbb{Z}_3$, $(p,j) \to (q,j+1)$ whenever $(p,q) \in \{(0,1),(1,0),(2,3),(3,2),\ldots,(2k-2,2k-1),(2k-1,2k-2),(2k,0),(0,2k)\}$ and $(p,j) \to (q,j+2)$ whenever $(p,q) \in \{(1,2),(2,1),(3,4),(4,3),\ldots,(2k-1,2k),(2k,2k-1)\}$. It can be verified that the resulting digraph D is of diameter $\leq k+2$. ∎

Clearly, $d(C_5 \times K_3) = 3$ and for $k \geq 3$, $d(C_{2k+1} \times K_3) = k$; hence we have

Corollary 4.2. $\rho(C_5 \times K_3) \leq 1$ *and for* $k \geq 3$, $\rho(C_{2k+1} \times K_3) \leq 2$. ∎

Lemma 4.7. *If* $k \geq 8$, *then* $\vec{d}(C_{2k+1} \times K_4) = k$; *if* $4 \leq k \leq 7$, *then* $\vec{d}(C_{2k+1} \times K_4) \leq k+1$; *and if* $k \in \{2,3\}$, *then* $\vec{d}(C_{2k+1} \times K_4) \leq k+2$.

Proof. Orient $C_{2k+1} \times K_4$ so that for any $j \in \mathbb{Z}_4$, $(0,j) \to \{(1,j+1),(1,j+2),(2k,j+2),(2k,j+3)\}$ and $(2k,j) \to \{(2k-1,j+2),(2k-1,j+3),(0,j+3)\}$, and for all other vertices (i,j), $(i,j) \to \{(i+1,j+1),(i+1,j+2),(i-1,j+2),(i-1,j+3)\}$ if i is even and $(i,j) \to \{(i+1,j+3),(i-1,j+1)\}$ if i is odd. It can be verified that the resulting digraph D is of diameter k for $k \geq 8$, $k+1$ for $4 \leq k \leq 7$, and $k+2$ for $k \in \{2,3\}$. ∎

Clearly, $d(C_5 \times K_4) = 3$ and for $k \geq 3$, $d(C_{2k+1} \times K_4) = k$; hence we have

Corollary 4.3. *If* $k \geq 8$, *then* $\rho(C_{2k+1} \times K_4) = 0$; *if* $k \in \{2,4,5,6,7\}$, *then* $\rho(C_{2k+1} \times K_4) \leq 1$ *and* $\rho(C_7 \times K_4) \leq 2$. ∎

Lemma 4.8. *For* $k \geq 7$, $\rho(C_{2k} \times K_4) = 0$ *Furthermore,* $\rho(C_{12} \times K_4) \leq 1$, $\rho(C_{10} \times K_4) \leq 2$, $\rho(C_8 \times K_4) \leq 1$ *and* $\rho(C_6 \times K_4) \leq 2$.

Proof. For $k \geq 3$, orient $C_{2k} \times K_4$ so that for any $i \in \mathbb{Z}_{2k}$ and $j \in \mathbb{Z}_4$, $(i,j) \to \{(i+1,j+1),(i+1,j+2),(i-1,j+2),(i-1,j+3)\}$ whenever i is even and $(i,j) \to \{(i+1,j+3),(i-1,j+1)\}$ whenever i is odd. The resulting digraph D satisfies the requirement. ∎

Lemma 4.9. $\rho(C_4 \times K_4) = 0$.

Proof. The following orientation of $C_4 \times K_4$ defines a digraph D for which $d(D) = 3$; define, for any $j \in \mathbb{Z}_4$, $(0,j) \to \{(1,j+1), (1,j+2), (3,j+2)\}$, $(1,j) \to \{(0,j+1), (2,j+1), (2,j+3)\}$, $(2,j) \to \{(1,j+2), (3,j+2), (3,j+3)\}$ and $(3,j) \to \{(0,j+1), (0,j+3), (2,j+3)\}$. It can be verified that $d(D) = 3$. ■

Lemma 4.10. $\rho(C_4 \times K_5) = 0$.

Proof. The following orientation of $C_4 \times K_5$ defines a digraph D for which $d(D) = 3$; define, for any $j \in \mathbb{Z}_5$, $(0,j) \to \{(1,j+1), (1,j+2), (3,j+1), (3,j+3)\}$, $(1,j) \to \{(0,j+1), (0,j+2), (2,j+1), (2,j+3)\}$, $(2,j) \to \{(1,j+1), (1,j+3), (3,j+3), (3,j+4)\}$ and $(3,j) \to \{(2,j+3), (2,j+4), (0,j+1), (0,j+3)\}$. It is not difficult to check that $d(D) = 3$. ■

Lemma 4.11. *If* $s \geq 5$, *then* $\rho(C_5 \times K_s) = 0$.

Proof. Orient $C_5 \times K_s$ so that for any $j \in \mathbb{Z}_s$, $(p,j) \to \{(q,j+1), (q,j+2), \ldots, (q,j+\lfloor\frac{s-1}{2}\rfloor)\}$ whenever $(p,q) \in \{(0,1),(1,0),(3,4),(4,3)\}$, $(p,j) \to \{(q,j+1), (q,j+2), \ldots, (q,j+\lfloor\frac{s-3}{2}\rfloor), (q,j+\lceil\frac{s+1}{2}\rceil)\}$ whenever $(p,q) \in \{(1,2),(2,1)\}$, $(p,j) \to \{(q,j+\lceil\frac{s+1}{2}\rceil), (q,j+\lceil\frac{s+3}{2}\rceil), \ldots, (q,j-1)\}$ whenever $(p,q) \in \{(2,3),(3,2)\}$ and $(p,j) \to \{(q,j+2), (q,j+3), \ldots, (q,j+\lfloor\frac{s-1}{2}\rfloor), (q,j-1)\}$ whenever $(p,q) \in \{(0,4),(4,0)\}$. In addition, if s is even, orient $(0,j) \to (1,j+\frac{s}{2})$, $(2,j) \to \{(1,j+\frac{s}{2}), (3,j+\frac{s}{2})\}$ and $(4,j) \to \{(3,j+\frac{s}{2}), (0,j+\frac{s}{2})\}$. It can be verified that the resulting digraph D is of diameter 3. ■

Lemma 4.12. *If* $s \geq 9$, *then* $\rho(C_7 \times K_s) = 0$.

Proof. Orient $C_7 \times K_s$ so that for any $j \in \mathbb{Z}_s$, $(p,j) \to \{(q,j+2), (q,j+3), \ldots, (q,j+\lfloor\frac{s-1}{2}\rfloor), (q,j-1)\}$ whenever $(p,q) \in \{(0,1),(1,0),(2,3),(3,2),(4,5),(5,4)\}$, $(p,j) \to \{(q,j+1), (q,j+2), \ldots, (q,j+\lfloor\frac{s-3}{2}\rfloor), (q,j+\lceil\frac{s+1}{2}\rceil)\}$ whenever $(p,q) \in \{(1,2),(2,1),(3,4),(4,3),(5,6),(6,5)\}$ and $(p,j) \to \{(q,j+2), (q,j+3), \ldots, (q,j+\lfloor\frac{s-3}{2}\rfloor), (q,j+\lceil\frac{s+1}{2}\rceil), (q,j-1)\}$ whenever $(p,q) \in \{(0,6),(6,0)\}$. Let D be the resulting digraph.

To prove that $d(D) = 3$, by the nature of the orientation, it is enough to check that the eccentricities of the vertices $(i,0)$, $i \in \mathbb{Z}_7$, are all equal to 3. We leave the verification to the reader. ■

Lemma 4.13. *1. If* $r \geq 8$, *then* $\rho(C_r \times K_6) = 0$.
2. If $(r,s) \in \{(8,5),(9,5),(9,8)\}$, *then* $\rho(C_r \times K_s) = 0$.
3. If $(r,s) \in \{(6,5),(7,5),(6,6),(7,6),(7,7),(7,8)\}$, *then* $\rho(C_r \times K_s) \leq 1$.

Proof. Let

$$\begin{aligned}
A_{i,j} &= \{(i+1,j+1),(i+1,j+2),\ldots,(i+1,j+\lfloor\tfrac{s}{2}\rfloor),\\
&\qquad (i-1,j+2),(i-1,j+3),\ldots,(i-1,j+\lfloor\tfrac{s}{2}\rfloor),(i-1,j-1)\},\\
B_{i,j} &= \{(i+1,j+2),(i+1,j+3),\ldots,(i+1,j+\lfloor\tfrac{s-1}{2}\rfloor),(i+1,j-1),\\
&\qquad (i-1,j+1),(i-1,j+2),\ldots,(i-1,j+\lfloor\tfrac{s-1}{2}\rfloor)\},\\
C_{0,j} &= \{(1,j+1),(1,j+2),\ldots,(1,j+\lfloor\tfrac{s}{2}\rfloor),\\
&\qquad (r-1,j+1),(r-1,j+2),\ldots,(r-1,j+\lfloor\tfrac{s}{2}\rfloor)\} \text{ and}\\
D_{r-1,j} &= \{(r-2,j+2),(r-2,j+3),\ldots,(r-2,j+\lfloor\tfrac{s}{2}\rfloor),(r-2,j-1),\\
&\qquad (0,j+1),(0,j+2),\ldots,(0,j+\lfloor\tfrac{s-1}{2}\rfloor)\}.
\end{aligned}$$

If r is even, orient $C_r \times K_s$ so that for any $i \in \mathbb{Z}_r$ and $j \in \mathbb{Z}_s$, $(i,j) \to A_{i,j}$ whenever i is even and $(i,j) \to B_{i,j}$ whenever i is odd.

If r is odd, orient $C_r \times K_s$ so that for any $j \in \mathbb{Z}_s$, $(0,j) \to C_{0,j}$ and $(r-1,j) \to D_{r-1,j}$. For all other vertices (i,j), $(i,j) \to A_{i,j}$ whenever i is even and $(i,j) \to B_{i,j}$ whenever i is odd.

The resulting digraphs satisfy the requirement. ■

Combining all the results of this section, we have the proof of Theorem B.

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