On $(k, g; \ell)$-dicages

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Abstract

The semigirth ℓ of a digraph D is a parameter related with the number of shortest paths in D. In particular, if G is a graph the semigirth of the associated symmetric digraph G^* is $\ell(G^*) = \lfloor (g(G)-1)/2 \rfloor$, where $g(G)$ is the girth of the graph G. In this paper some bounds for the minimum number of vertices of a k-regular digraph D having girth g and semigirth ℓ, denoted by $n(k, g; \ell)$ are obtained. Moreover we construct a family of digraphs which achieve the lower bound for some particular values of the parameters.

Key words. circulant digraphs, line digraphs, semigirth.

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1 Introduction

Throughout this paper, we only consider finite and strict digraphs (with neither loops nor two arcs with the same ends and the same orientation). Unless otherwise stated we follow the book by Bondy and Murty [8] for terminology and definitions.

Let D stand for a digraph with set of vertices $V = V(D)$ and set of (directed) arcs $A = A(D)$. The converse digraph $\overleftarrow{D}$ of a digraph D is defined by reversing the direction of the arcs of D. For any pair of vertices $u, v \in V$, a path from u to v (constituted by different vertices) is called a $u \to v$ path. The *distance from* u *to* v is denoted by $d(u,v) = d_D(u,v)$, and $diam(D) = \max\{d(u,v) : u,v \in V\}$ stands for the *diameter* of D. The girth $g = g(D)$ of the digraph D is the length of a shortest directed cycle.

For every $x \in V$, $N^+(x)$ and $N^-(x)$, denote the set of out-neighbors and in-neighbors of the vertex x, their cardinalities being $d^+(x)$ and $d^-(x)$ respectively. Also we use the closed out-neighborhood $N^+[x] = \{x\} \cup N^+(x)$ and the closed in-neighborhood $N^-[x] = \{x\} \cup N^-(x)$, respectively. A digraph D is k*-regular* if both out-degree and in-degree of every vertex x satisfies $d^+(x) = d^-(x) = k$.

Fábrega and Fiol introduced in [11] the so-called *parameter* ℓ of a simple connected digraph, which, as was pointed out in the aforementioned paper "can be thought of as a generalization of the girth of a graph". In fact, this parameter has recently received the name of semigirth see The Handbook of Graph Theory [15]):

Definition 1.1 *Let D be a (di)graph with diameter $diam(D)$. The semigirth $\ell = \ell(D)$, $1 \le \ell \le diam(D)$, is defined as the greatest integer so that, for any two vertices u, v,*

(a) if $d(u,v) < \ell$, the shortest $u \to v$ path is unique and there are no paths $u \to v$ of length $d(u,v) + 1$;

(b) if $d(u,v) = \ell$, there is only one shortest $u \to v$ path.

Given a graph G, the associated symmetric digraph G^* is obtained from G by replacing each edge $xy \in E(G)$ with the two directed edges (x,y) and (y,x) forming a "digon". The close relation between the girth $g(G)$ of a graph G and the semigirth $\ell(G^*)$ of the associated symmetric digraph G^* is clear because $\ell(G^*) = \lfloor (g(G) - 1)/2 \rfloor$.

The main aim of this paper is to obtain bounds for the minimum number of vertices of a k-regular digraph D having girth g and semigirth ℓ, denoted by $n(k,g;\ell)$. We do that by constructing a family of k-regular digraphs of girth g and semigirth ℓ. We call a $(k,g;\ell)$-digraph of order $n(k,g;\ell)$ a *$(k,g;\ell)$-dicage*. Behzand, Chartrand and Wall [4] asked for the minimum order $n(k,g)$ of any (k,g)-directed graph. A (k,g)-digraph of order $n(k,g)$ used to be called *(k,g)-dicage*. It is clear that $n(k,g) \leq n(k,g;\ell)$.

For finding bounds on $n(k,g;\ell)$, the line digraph technique will be very helpful. This technique was introduced by Harary and Norman [19], and has proved to be very useful in the design of digraphs with 'good properties'. In the *line digraph* $L(D)$ of a digraph D, each vertex represents an arc of G, that is, $V(L(D)) = \{uv : (u,v) \in A(D)\}$; and a vertex uv is adjacent to a vertex wz if and only if $v = w$ (i.e., when the arc (u,v) is adjacent to the arc (w,z) in D). Setting $L^0(D) = D$, for any integer $h \geq 1$ the *h-iterated line digraph*, $L^h(D)$, is defined recursively by $L^h(D) = L(L^{h-1}(D))$. In addition, as the vertices of $L(D)$ represent the arcs of D, the order of $L(D)$ equals the size of D, that is, $|V(L(D))| = |A(D)|$; and their respective maximum and minimum degrees coincide.

Aigner [1] showed that if D is connected and different from a cycle, then $L(D)$ is connected, the relationship between their diameters being:

$$diam(L(D)) = diam(D) + 1.$$

An important property of the semigirth is its behaviour with respect to the line digraph technique. If D is a digraph with minimum degree $\delta \geq 2$ and $L(D)$ is its line digraph, then it is proved in [11] that:

$$\ell(L(D)) = \ell(D) + 1. \tag{1}$$

It is not difficult to see that the semigirth ℓ of a digraph D and of its converse $\overleftarrow{D}$ are equal.

Balbuena et al. [2] proved that a digraph D and its iterated line digraph $L^h(D)$ have the same number of cycles of length i for $i \leq \min\{2g-1, |V(D)|\}$. This means that $g(D) = g(L^h(D)) = g$ for any $h \geq 0$, and both D and $L^h(D)$ have the same number of cycles of length g. However by (1) the semigirth of $L^h(D)$ increases according to $\ell(L^h(D)) = \ell(D) + h$.

For more details about line digraphs, see, for instance, [13, 20]. Next, in Section 2 we present our results and we provide the details of the proofs in Section 3.

2 Results

We start by proving a theorem which provides bounds for $n(k,g;\ell)$ and also shows some relationship between $n(k,g;\ell)$ and $n(k,g)$.

Theorem 2.1 *Let D be a $(k,g;\ell)$-dicage with $\ell \geq 1$, $k \geq 2$ and $g \geq 3$ on $n(k,g;\ell)$ vertices. The following assertions hold.*

(i) If $\ell \geq 2$, then $n(k,g;\ell) \geq \begin{cases} 1+k+k^2 & \text{if } g \geq 3; \\ (1+k)^2 & \text{if } g \geq 4. \end{cases}$

(ii) If $\ell \geq 3$, then $n(k,g;\ell) \geq k^2+k^3$.

(iii) If $k \geq g-2$, then $n(k,g) = n(k,g;1) \leq n(k,g;\ell)$.

(iv) $n(k,g;\ell) \leq \begin{cases} k^{\ell-1}n(k,g) & \text{if } k \geq g-2; \\ k^{\ell-1}((g-1)k+1) & \text{if } k \leq g-3. \end{cases}$

In the next theorems we improve the bounds obtained in Theorem 2.1 for the particular case $k=2$ and $g=3,4$.

Theorem 2.2 *Let D be a $(2,g;\ell)$-dicage on $n(2,g;\ell)$ vertices with semi-girth $\ell \geq 2$. Then*

(i) $n(2,3;\ell) \geq 9$. (ii) $n(2,4;\ell) \geq 10$.

Let $i_1, i_2, \ldots, i_k \in \mathbb{Z}_r$. A circulant digraph $\overrightarrow{C}_r(i_1, i_2, \ldots, i_k)$ has for vertex set the elements of $\mathbb{Z}_r$, and (a,b) is an arc if and only if $b = a + i_j$ for some $i_j \in \{i_1, i_2, \ldots, i_k\}$, where the sum is taken in $\mathbb{Z}_r$. Clearly a circulant digraph $\overrightarrow{C}_r(1,2,\ldots,k)$ where $r=(g-1)k+1$ is a (k,g)-digraph. Using this digraph, in [4] was proved that $n(k,g) \leq (g-1)k+1$ and the conjecture $n(k,g) = (g-1)k+1$ was formulated. Caccetta and Haggkvist [10] proposed a generalization of this conjecture, claiming that if each vertex of a digraph D has out-degree at least k, then the girth of G is at most $\left\lfloor |V(D)|/k \right\rfloor$. Both conjectures have been proved to be true for $k=2$ by Behzad [5], for $k=3$ first by Bermond [6] and later by Hamidoune [18], for $k=4$ and for vertex-transitive digraphs by Hamidoune [16, 17].

Based on these ideas we show a family of $(k,3;\ell)$-digraphs for $\ell \geq 2$ having $k^{\ell-2}(2k^2+1)$ vertices. To do that we use the following known operation. *Splitting a vertex v* of a digraph D consists of replacing v with

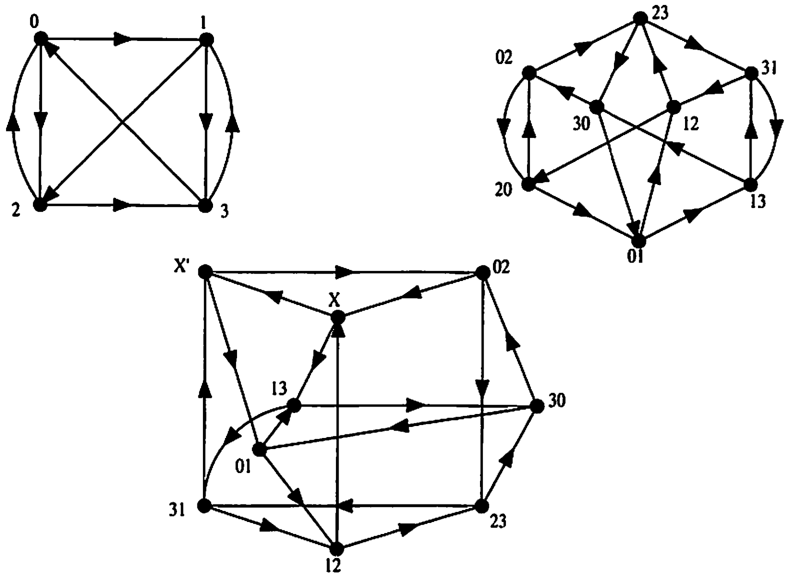

Figure 1: A $(2,3;2)$-dicage.

two new vertices v' and v'', join them by an arc (v', v'') and change every arc (w, v) and (v, z) of D to (w, v') and to (v'', z) respectively.

Theorem 2.3 *Let $\overrightarrow{C}_{2k}(1, 2, \ldots, k)$ be the circulant digraph where $k \geq 2$. Let denote by $SL(\overrightarrow{C}_{2k}(1, 2, \ldots, k))$ the digraph obtained from the line digraph $L(\overrightarrow{C}_{2k}(1, 2, \ldots, k))$ by performing the following operations:*

(a) Split vertex $(k, 0)$ into two new vertices, X and X'.

(b) Delete the $k-1$ arcs $((k+i, i),\ (i, k+i))$, $i = 1, \ldots, k-1$.

(c) Add the new arcs $(X,\ (i, k+i))$ and $((k+i, i),\ X')$, $i = 1, \ldots, k-1$.

Then $SL(\overrightarrow{C}_{2k}(1, 2, \ldots, k))$ is a $(k, 3; 2)$-digraph having $2k^2 + 1$ vertices, thus $n(k, 3; 2) \leq 2k^2 + 1$.

Figure 1 depicts the circulant digraph $\overrightarrow{C}_4(1, 2)$, its line digraph and the digraph $SL(\overrightarrow{C}_4(1, 2))$ obtained by performing in $L(\overrightarrow{C}_4(1, 2))$ the operations indicated in Theorem 2.3. Actually, this digraph is a (2, 3; 2)-dicage, such as it is derived in the following corollary.

Corollary 2.1 *Let D be a $(k,3;\ell)$-dicage with $k \geq 2$ and $\ell \geq 2$ on $n(k,3;\ell)$ vertices. Then*

(i) the digraph $SL(\overrightarrow{C}_4(1,2))$ is a $(2,3;2)$-dicage and $n(2,3;2) = 9$.

(ii) $n(k,3;\ell) \leq k^{\ell-2}(2k^2+1)$.

For any positive integers k, n, with $k \leq n$, the *dense bipartite digraph* $BD(k,n)$ introduced in [12] has set of vertices $V = \mathbb{Z}_2 \times \mathbb{Z}_n = \{(\alpha, i);\ \alpha \in \mathbb{Z}_2, i \in \mathbb{Z}_n\}$ where $\mathbb{Z}_n$ denotes the integers modulo n and each vertex (α, i) is adjacent to the vertices of $\Gamma^+(\alpha,i) = \{(1-\alpha, (-1)^\alpha k(i+\alpha)+t);\ t = 0, 1, \ldots, k-1\}$. The digraphs $BD(k, k^{r-1}+k^{r-3})$ can also be obtained as iterated line digraphs of $BD(k, k^2+1)$, which has diameter $D = 3$, $g = 4$, and parameter $\ell = 2$ [12]. Figure 2 depicts the $BD(2,5)$ bipartite digraph. As a consequence of our results we obtain that $BD(2,5)$ is a $(2,4;2)$-dicage as shown in the following corollary.

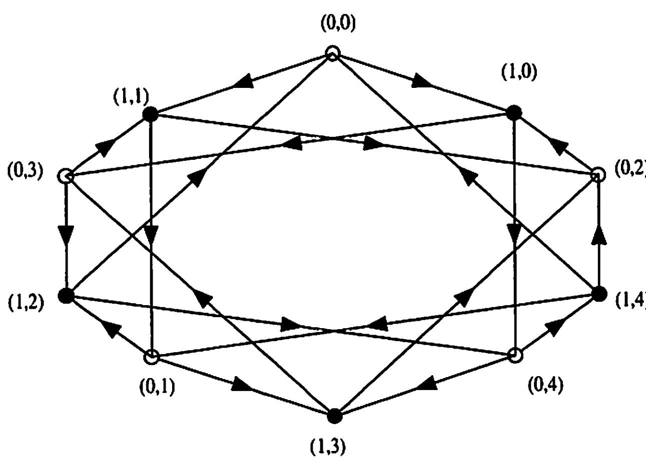

Figure 2: A (2, 4; 2)-dicage.

Corollary 2.2 *Let D be a $(k,4;\ell)$-dicage with $k \geq 2$ and $\ell \geq 2$ on $n(k,4;\ell)$ vertices. Then* *(i) the digraph $BD(2,5)$ is a $(2,4;2)$-dicage, thus $n(2,4;2) = 10$.*

(ii) $n(k,4;\ell) \leq 2k^{\ell-2}(k^2+1)$.

3 Proofs

Before proceeding with the proofs of the main results, we state the following useful lemma.

Lemma 3.1 *Let D be a digraph with $\ell \geq 2$ and $g \geq 3$. For all vertex x the following assertions hold.*

(i) The paths of length 2 are shortest paths and hence they are unique.

(ii) $N^+[x] \cap N^+(y) = \emptyset$ for all $y \in N^+[x]$, and $N^+(y) \cap N^+(z) = \emptyset$ for any two distinct vertices $y, z \in N^+(x)$.

(iii) $N^-[x] \cap N^-(y) = \emptyset$ for all $y \in N^-[x]$, and $N^-(y) \cap N^-(z) = \emptyset$ for any two distinct vertices $y, z \in N^-(x)$.

(iv) If $N^+(x) \cap N^+(y) \neq \emptyset$, then x and y are mutually at distance at least 2.

Proof. (i) Let a, b, c be a path of length 2 in D. If this path was not a shortest one, then D should contain an arc (a, c), which contradicts the definition of semigirth ℓ. Hence the paths of length 2 are shortest paths and again by definition of ℓ they must be unique.

(ii) Let $y \in N^+(x)$ be and $t \in N^+(y)$. Then clearly $t \neq x$ because $g \geq 3$. Thus, by (i) the path x, y, t is the shortest one and hence $t \notin N^+[x]$, hence $N^+[x] \cap N^+(y) = \emptyset$. By the same argument it is clear that $N^+(y) \cap N^+(z) = \emptyset$ for any two distinct vertices $y, z \in N^+(x)$.

(iii) It suffices to consider the converse digraph of D and apply item (ii).

(iv) Let $z \in N^+(x) \cap N^+(y)$. If x was adjacent to y, then from x to z we have the path x, y, z and the arc (x, z), contradicting (i). Therefore, x and y are mutually at distance at least 2. ■

Proof of Theorem 2.1: (i) Let x be any vertex of the digraph D. As $\ell \geq 2$ and $g \geq 3$ we can apply Lemma 3.1, obtaining that the number of vertices within distance 2 from x is exactly $1+k+k^2$. Besides if $g \geq 4$ then

$$N^-(x) \cap (N^+[x] \cup_{y \in N^+(x)} N^+(y)) = \emptyset. \tag{2}$$

Therefore, the number of vertices of D is at least $k+1+k+k^2 = (1+k)^2$.

(*ii*) From item (*i*) it follows that the digraph D contains the out-tree T with root in a vertex x, first level $N^+(x) = \{x_1, x_2, \ldots, x_k\}$, and second level consisting of the disjoint union $\cup_{x_i} N^+(x_i)$. As $\ell \geq 3$ the shortest paths of length at most 3 are unique, hence for every $y \in N^+(x_i)$ the set $N^+(y)$ and $V(T)$ can only share vertices of the set $N^+[x]$. Therefore there must exist at least $k^3 - k - 1$ vertices outside of tree T, implying that $|V(D)| \geq k^2 + k^3$.

(*iii*) Clearly, we have $n(k,g) \leq n(k,g;1)$. To prove the other inequality let us consider a (k,g)-dicage D and let us see that $\ell(D) = 1$. Otherwise, by applying (*i*) and using the hypotheses $k \geq 2$ and $k \geq g-2$ we get

$$|V(D)| \geq \begin{cases} 1+k+k^2 > 1+2k \geq n(k,3) & \text{if } g = 3; \\ (1+k)^2 = 1+k(k+2) > 1+k(g-1) \geq n(k,g) & \text{if } g \geq 4, \end{cases}$$

which is a contradiction because $|V(D)| = n(k,g)$. Therefore any (k,g)-dicage D has $\ell(D) = 1$ yielding $n(k,g;1) \leq n(k,g)$, then we conclude that $n(k,g) = n(k,g;1) \leq n(k,g;\ell)$.

(*iv*) Again let us consider a (k,g)-dicage D, which has $\ell(D) = 1$ if $k \geq g-2$. Then the iterated line digraph $L^{\ell-1}(G)$ is a $(k,g;\ell)$-digraph having $k^{\ell-1}n(k,g)$ vertices. Therefore $n(k,g;\ell) \leq k^{\ell-1}n(k,g)$. If $k \leq g-3$ we consider the circulant digraph $\overrightarrow{C}_r(1,2,\ldots,k)$ where $r = (g-1)k+1$ and proceed analogously by using the line digraph technique. ∎

Proof of Theorem 2.2: Let D be a $(2,g)$-digraph on $n(2,g;\ell)$ vertices for $g = 3,4$ and $\ell \geq 2$. From Theorem 2.1 it follows that the out-tree T with root x, first level $N^+(x) = \{x_1, x_2\}$, and second level consisting of $N^+(x_1) = \{x_{11}, x_{12}\}$ and $N^+(x_2) = \{x_{21}, x_{22}\}$ is included in D. We want to prove that $|V(D)| \geq 9$. We reason by contradiction assuming $|V(D)| \leq 8$, then $g = 3$, because of Theorem 2.1.

First, suppose $|V(D)| = 7$. Then we may assume that some vertex, say x_{12}, is an in-neighbor of x_2. Therefore, x_{12}, x_2, x_{2j}, $j = 1,2$ are paths of length 2, thus by (i) of Lemma 3.1, they are unique shortest paths. This means that $x_{21}, x_{22} \notin N^+(x_{12})$, and by (ii) of Lemma 3.1, we have $x_{11} \notin N^+(x_{12})$ and by (iii) of Lemma 3.1, $x \notin N^+(x_{12})$. Also $x_1 \notin N^+(x_{12})$ because $g = 3$, hence $d^+(x_{12}) = 1$ contradicting $k = 2$.

Therefore we continue the proof by assuming $|V(D)| = 8$, that is to say, $V(D) = V(T) \cup \{y\}$, where y is a new vertex. Let us study the following cases according to the in-neighbors of x_1 and x_2.

Case 1: $(x_{12}, x_2), (x_{21}, x_1) \in A(D)$. Then the other possible out-neighbor for both x_{12} and x_{21} can only be vertex y, that is, $x_{12}, x_{21} \in N^-(y)$. Hence $x_{11} \notin N^+(y)$ because otherwise we would obtain two paths of length 2, namely, x_{21}, y, x_{11} and x_{21}, x_1, x_{11}, contradicting Lemma 3.1. Similarly, $x_{22} \notin N^+(y)$ and as the girth is $g = 3$, the only possible out-neighbor for y is x, hence $d^+(y) = 1$ which gives a contradiction.

Case 2: $(y, x_1), (x_{12}, x_2) \in A(D)$. By Lemma 3.1 we have that the other possible out-neighbor of x_{12} is y, and again by using Lemma 3.1 no vertex different from x_1 can be out-neighbor of y. In other words, $d^+(y) = 1$ which gives a contradiction.

Case 3: $(y, x_1), (y, x_2) \in A(D)$. Then observe that $(y, x) \notin A(D)$ because $d^+(y) = 2$. Moreover, by Lemma 3.1, $|N^+(x_i) \cap N^-(y)| = 1$, $i = 1, 2$. Thus, without loss of generality we may assume $N^-(y) = \{x_{12}, x_{21}\}$. If $x_{12} \in N^-(y) \cap N^-(x)$, then by Lemma 3.1, the only possible out-neighbors or in-neighbors for x_{11} are x_{21}, x_{22}, which produces a digon. This is a contradiction with $g = 3$. Similarly, $x_{21} \notin N^-(y) \cap N^-(x)$, therefore $N^-(y) \cap N^-(x) = \emptyset$. This means that the only possible out-neighbor and in-neighbor of x_{12} is the vertex x_{22} which produces a digon contradicting that $g = 3$.

In any case we arrive at a contradiction and hence $|V(D)| \geq 9$ if $k = 2$, $g = 3$ and $\ell \geq 2$.

Finally, suppose $k = 2$, $g \geq 4$ and $\ell \geq 2$ hence $|V(D)| \geq 9$ by Theorem 2.1. Let us see that $|V(D)| \geq 10$. We reason for producing a contradiction assuming that $|V(D)| = 9$. Therefore the root x of the out-rooted tree T has $N^-(x) = \{y_1, y_2\}$ and from (2) it follows $y_1, y_2 \notin \{x, x_1, x_2, x_{11}, x_{12}, x_{21}, x_{22}\}$. Furthermore, we may assume that $(x_{12}, x_2), (x_{21}, x_1) \in A(D)$, because $y_1, y_2 \notin N^-(x_1) \cup N^-(x_2)$. This implies that $(x_{12}, y_1) \in A(D)$ and $(x_{21}, y_2) \in A(D)$. Moreover, notice that $(x_{11}, x_{21}) \notin A(D)$, otherwise the directed triangle x_1, x_{11}, x_{21}, x_1 would be contained in D contradicting that $g \geq 4$. Similarly, $(x_{22}, x_{12}) \notin A(D)$. Consequently $N^+(x_{11}) = \{y_2, x_{22}\}$ because $(x_{12}, y_1) \in A(D)$. Analogously, $N^+(x_{22}) = \{y_1, x_{11}\}$ producing the digon formed by the arcs (x_{11}, x_{22}) and (x_{22}, x_{11}), a contradiction. Then $|V(D)| \geq 10$. ∎

Proof of Theorem 2.3: (i) Let us consider the circulant digraph $\vec{C}_{2k}(1,2,\ldots,k)$. Notice that this digraph has exactly the following k digons:

$$(i,k+i),\ (k+i,i),\ i=0,1,\ldots,k-1.$$

Then the line digraph $L(\vec{C}_{2k}(1,2,\ldots,k))$ also contains exactly k digons which are for $i=0,1,\ldots,k-1$:

$$((i,k+i),(k+i,i)),\ ((k+i,i),(i,k+i)).$$

Let $D = SL(\vec{C}_{2k}(1,2,\ldots,k))$ be the digraph obtained from the line digraph $L(\vec{C}_{2k}(1,2,\ldots,k))$ by performing the following operations:

(a) Split vertex $(k,0)$ into two new vertices, X and X'.

(b) Delete the $k-1$ arcs $((k+i,i),(i,k+i))$, $i=1,\ldots,k-1$.

(c) Add the new arcs $(X,(i,k+i))$, $((k+i,i),X')$, $i=1,\ldots,k-1$.

Clearly D has order $2k^2+1$ vertices, is k-regular and by construction D has no digons. Moreover, we have that the out and in-neighbors of the two new vertices are

$$\begin{array}{rcl} N^-(X) & = & \{(i,k) : i=0,\ldots,k-1 \in \mathbb{Z}_{2k}\}, \\ N^+(X) & = & \{(i,k+i) : i=1,\ldots,k-1 \in \mathbb{Z}_{2k}\} \cup \{X'\}, \\ N^-(X') & = & \{(k+i,i) : i=1,\ldots,k-1 \in \mathbb{Z}_{2k}\} \cup \{X\}, \\ N^+(X') & = & \{(0,i) : i=1,\ldots,k \in \mathbb{Z}_{2k}\} \end{array} \quad (3)$$

Then D has girth $g=3$, because $(0,k),X,X',(0,k)$ is a triangle. Also note that $\ell(L(\vec{C}_{2k}(1,2,\ldots,k)))=2$, then clearly $\ell(D)\le 2$. We shall show that D has $\ell(D)=2$. In order to prove it, we say that D contains a $(1,2)$-*diamond* between x and y if there exists two internal disjoint paths between these vertices, one of them of length one (an arc) and the other of length two. The vertex x is said to be the initial vertex in the diamond, the vertex y the final vertex, an the vertex z in the path x,z,y of length 2 is the intermediate vertex. Analogously, we define a $(2,2)$-*diamond* and its initial, final and intermediate vertices (note that in this case we have two intermediate vertices).

First let us see that there are no $(1,2)$-diamonds in D. Otherwise vertex X or vertex X' must be included in the $(1,2)$-diamond, because $\ell(L(\vec{C}_{2k}(1,2,\ldots,k)))=2$.

If X is the final vertex of some $(1,2)$-diamond, then the initial and intermediate vertices are in-neighbors of X. By (3) we get that some (i,k) is adjacent to some (j,k) with $i \neq j$. Thus (i,k) and (j,k) with $i \neq j$, are adjacent in $L(\overrightarrow{C}_{2k}(1,2,\ldots,k))$, which implies that $j = k$ and hence D would contain loops which is impossible. The reasoning is similar if X' is the initial vertex of some $(1,2)$-diamond.

If X' is the final vertex of some $(1,2)$-diamond, then clearly the intermediate vertex must be X, and so the initial vertex must be adjacent to both X and X'. Then $N^-(X) \cap N^-(X') \neq \emptyset$ which is contradiction with (3). The argument is similar if X is the initial vertex and X' is the intermediate of some $(1,2)$-diamond.

In any case we conclude that D has no $(1,2)$-diamonds. Next, let us prove that there are no $(2,2)$-diamonds in D.

If X is the final vertex of some $(2,2)$-diamond, then the two intermediate vertices are in-neighbors of X, say $(i,j),(j,k)$ with $i \neq j$, which have one common in neighbor. By (3) this means that some vertex of D is adjacent to both vertices (i,k) and (j,k) which is impossible because $i \neq j$. If X is the initial vertex of some $(2,2)$-diamond, then the two intermediate vertices are out-neighbors of X having one common out-neighbor. It is easy to see that the only possibility is that the two out-neighbors of X are $(j,k+j)$ for some $j = 1,\ldots,k-1$ and vertex X', which have one common in-neighbor $(0,i)$ for some $i = 1,\ldots,k$. That is to say, some $k + j = i$ for certain $i,j \in \{1,2,\ldots,k-1\}$ which is impossible in $\mathbb{Z}_{2k}$. Analogously it is shown that X' is neither the final vertex nor the initial vertex of any $(2,2)$-diamond. Finally, if X is an intermediate vertex of some $(2,2)$-diamond, due to the adjacency rules defined by (3), the vertex $(k,k+j)$ for some $j \in \{1,\ldots,k-1\}$ has to be adjacent to the vertex $(i,k+i)$ for some $i \in \{1,\ldots,k-1\}$, which is impossible. An analogous contradiction is obtained assuming X' is an intermediate vertex.

Therefore we conclude that $\ell(D) = 2$, and the result holds. ■

Proof of Corollary 2.1: To prove (i) note that by Theorem 2.2 we know that $n(2,3;2) \geq 9$ and furthermore the digraph $SL(\overrightarrow{C}_4(1,2))$ is a $(2,3;2)$-digraph of order 9. Finally, by observing that the iterated line digraph $L^{\ell-1}(SL(\overrightarrow{C}_{2k}(1,2,\ldots,k)))$ is a $(k,3;\ell)$-digraph having $k^{\ell-2}(2k^2+1)$ vertices, then $n(k,3;\ell) \leq k^{\ell-2}(2k^2+1)$ for $\ell \geq 2$, thus item (ii) is also valid. ■

Proof of Corollary 2.2: To prove (i) note that by Theorem 2.2 we know that $n(2,4;2) \geq 10$ and the digraph $BD(2,5)$ is a $(2,4;2)$-digraph of order 10. And, finally to prove (ii) observing that the iterated line digraph $L^{\ell-2}(BD(k,k^2+1))$ is a $(k,4;\ell)$-digraph having $2k^{\ell-2}(k^2+1)$ vertices, then $n(k,3;\ell) \leq 2k^{\ell-2}(k^2+1)$ for $\ell \geq 2$ and we conclude the proof. ■

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