Broadcast domination of products of graphs

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Abstract

Broadcast domination in graphs is a variation of domination in which different integer weights are allowed on vertices and a vertex with weight k dominates its distance k-neighborhood. A distribution of weights on vertices of a graph G is called a dominating broadcast, if every vertex is dominated by some vertex with positive weight. The broadcast domination number $\gamma_b(G)$ of a graph G is the minimum weight (the sum of weights over all vertices) of a dominating broadcast of G. In this paper we prove that for a connected graph G, $\gamma_b(G) \geq \lceil 2\text{rad}(G)/3 \rceil$. This general bound and a newly introduced concept of condensed dominating broadcast are used in obtaining sharp upper bounds for broadcast domination numbers of three standard graph products in terms of broadcast domination numbers of factors. A lower bound for a broadcast domination number of the Cartesian product of graphs is also determined, and graphs that attain it are characterized. Finally, as an application of these results we determine exact broadcast domination numbers of Hamming graphs and Cartesian products of cycles.

Key words: dominating broadcasts, domination, graph products, Cartesian product, radius.

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1 Introduction

Let G be a graph. The *distance* $d_G(u,v)$ (or just $d(u,v)$ when it is understood from the context that we consider graph G) between vertices u and v of G is the length of a shortest path between u and v. *Interval* $I(u,v)$ is the set of vertices that lie on a shortest path between vertices u and v. For a vertex $v \in V(G)$ let $N[v] = \{x \in V(G) \mid d(x,v) \leq 1\}$ denote its *neighborhood*, and $N(v) = N[v] \setminus \{v\}$ its *open neighborhood*. More generally, the *k-neighborhood* of v where $k \in \mathbb{N}$, is the set $N_k[v] = \{x \in V(G) \mid d(x,v) \leq k\}$. The *eccentricity* $e_G(x)$ of the vertex x in G is maximum distance between a vertex $u \in V(G)$ and x. The minimum (resp. maximum) eccentricity of a vertex in the graph G is called the *radius* (resp. the *diameter*) of G and is denoted by $\text{rad}(G)$ (resp. $\text{diam}(G)$).

A function $f : V(G) \to \{0, 1, \ldots, \text{diam(G)}\}$ is called *a dominating broadcast* on G if for every vertex v of G there exists a vertex $x \in V(G)$ with $f(x) > 0$ such that $d(x,v) \leq f(x)$. If f is a dominating broadcast, then by D_f we denote the set of vertices x in G with $f(x) > 0$, and call it the *f-dominating set.* By definition we have

$$\bigcup_{u \in D_f} N_{f(u)}[u] = V(G).$$

The *weight* $w(f)$ of a dominating broadcast f of G is $\sum_{v \in V(G)} f(v)$. The minimum weight of a dominating broadcast in a graph G is called the *broadcast domination number* of G, and is denoted by $\gamma_b(G)$.

Broadcast domination was introduced and studied in [3], and some further results can be found in [2]. It presents a natural variation of domination where some dominating vertices may have larger influence than some other. (For a systematic treatment of various domination parameters we refer to [5].) The concept was motivated by some practical communication network problems, such as distribution of transmitters for radio stations etc.

It is easy to see that $\gamma_b(G) \leq \min\{\text{rad}(G), \gamma(G)\}$. Interestingly we could obtain also a lower bound for $\gamma_b(G)$ in terms of rad(G): we prove in the next section that $\gamma_b(G) \geq \lceil 2\text{rad(G)}/3 \rceil$ for any connected graph G. In the second part of that section we introduce the concept of *condensed* dominating broadcast. Roughly speaking, a condensed dominating broadcast is such that any broadcast, obtained from it by transferring some weights from different vertices to a single vertex, is no longer dominating. We show that every graph has a condensed dominating broadcast of weight $\gamma_b(G)$ which implies the previously known result about efficient dominating broadcasts [2, Theorem 17].

In Section 3 we study the broadcast domination in graph products. Let G and H be two graphs. For all three products of graphs G and H the vertex set of the product is $V(G) \times V(H)$. Their edge sets are defined as follows. In the *Cartesian product* $G \Box H$ two vertices (x, y) and (v, w) are adjacent if and only if either $x = v$ and $yw \in E(H)$ or $y = w$ and $xv \in E(G)$. In the *direct product* $G \times H$ two vertices (x, y) and (v, w) are adjacent if and only if $xv \in E(G)$ and $yw \in E(H)$. Finally, the edge set $E(G \boxtimes H)$ of the *strong product* $G \boxtimes H$ is the union of $E(G \Box H)$ and $E(G \times H)$. For $v \in V(H)$ let $G_v = \{(u, v) \in V(G \Box H) \,|\, u \in V(G)\}$ and for $u \in V(G)$ let $H_u = \{(u, v) \in V(G \Box H) \,|\, v \in V(H)\}$. Note that the subgraph of $G \Box H$ induced by G_v is isomorphic to G and the subgraph of $G \Box H$ induced by H_u is isomorphic to H.

We refer to [7] where domination concepts in graph products have been surveyed – the main attention is given to (upper and lower) bounds of a given domination parameter in a given graph product expressed in terms of the domination parameters of factor graphs. We prove upper bounds for the broadcast domination of all three standard graph products. For the Cartesian and the strong product we show $\gamma_b(G \Box H) \leq \frac{3}{2}(\gamma_b(G) + \gamma_b(H))$, and $\gamma_b(G \boxtimes H) \leq \frac{3}{2} \max\{\gamma_b(G), \gamma_b(H)\}$, respectively. For the direct product we obtain

$$\gamma_b(G \times H) \leq \begin{cases} 3\max\{\gamma_b(G), \gamma_b(H)\} & \text{if rad(G)} \neq \text{rad(H)} \\ 3\min\{\gamma_b(G), \gamma_b(H)\} + 1 & \text{if rad(G)} = \text{rad(H)}. \end{cases}$$

Finally, in the last section we obtain exact values for broadcast domination numbers of two classes of Cartesian products of graphs, notably *Hamming graphs* (Cartesian products of complete graphs), and Cartesian products of cycles. For instance we prove that $\gamma_b(C_m \Box C_n) = \mathrm{rad}(C_m \Box C_n) - 1$ if and only if m and n are both even, and otherwise $\gamma_b(C_m \Box C_n) = \mathrm{rad}(C_m \Box C_n)$.

2 Two remarks

2.1 Broadcast domination number vs. radius

In [3] the following lower bound was proved for the broadcast domination number of an arbitrary connected graph:

$$\gamma_b(G) \geq \left\lceil \frac{\mathrm{diam(G)} + 1}{3} \right\rceil. \tag{1}$$

The expression on the right is also known to be the lower bound for the usual domination number, hence (1) is its improvement.

Knowing $\gamma_b(G) \leq \mathrm{rad(G)}$, it seems to be interesting to find some lower bounds for the broadcast domination number expressed in terms of the radius. Since diam(G) $\geq$ rad(G), we get from (1) the following bound that holds for arbitrary connected graph G:

$$\gamma_b(G) \geq \left\lceil \frac{\text{rad(G)} + 1}{3} \right\rceil .$$

This cannot be improved directly because in some graphs diam(G) = rad(G), but we shall obtain a better bound by using a different approach. It is in many cases (but not in all) better as (1).

Lemma 2.1 *Let G be a graph, and H its spanning subgraph. Then*

$$\gamma_b(G) \leq \gamma_b(H)$$

and

$$\text{rad(G)} \leq \text{rad(H)}.$$

Proof. It is convenient to look at G as if G would be obtained from H by adding some edges. Hence any dominating broadcast in H is also a dominating broadcast in G, thus the first inequality follows. Since $d_H(u,v) \geq d_G(u,v)$ for any two vertices u and v, it is also clear that $e_H(u) \geq e_G(u)$ for any vertex u. Hence the radius of H, which is the minimum eccentricity in H, cannot be smaller than the radius of G. □

A dominating broadcast f is called an *efficient dominating broadcast* if for every vertex $x \in V(G)$ there exists exactly one vertex $u \in D_f$ such that $d(x,u) \leq f(u)$. In the following lemma we will use the previously known result [2, Theorem 17] that for every graph there exists an efficient dominating broadcast f such that $\gamma_b(G) = w(f)$.

Lemma 2.2 *Let G be a connected graph. Then there is a spanning tree T in G such that*

$$\gamma_b(G) = \gamma_b(T).$$

Proof. Let G be a connected graph, and let f be an efficient dominating broadcast of G with $w(f) = \gamma_b(G)$. Hence the neighborhoods $N_{f(u)}[u]$ of vertices from D_f are pairwise disjoint, and their union is $V(G)$. We obtain T as follows.

Consider a neighborhood $N_{f(u)}[u]$ where u is an arbitrary vertex from D_f. Let $T(u)$ be a spanning tree of the subgraph of G induced by $N_{f(u)}[u]$ (it can be obtained for instance by BFS starting in u) such that $d_{T(u)}(x,u) = d_G(x,u)$ for every vertex $x \in N_{f(u)}[u]$. Hence every vertex of $T(u)$ is f-dominated by u also in $T(u)$. Since the trees $T(u), u \in D_f$ are pairwise disjoint, their disjoint union is a disconnected subgraph of G (unless $|D_f| = 1$). Now, add edges of $E(G)$ between different trees $T(u)$ in such a way that the resulting graph is connected (this is possible, since G is connected). Clearly, in this procedure we can avoid obtaining cycles, and so the resulting graph is a spanning tree that we denote by T.

Obviously f is also a dominating broadcast of T, hence $\gamma_b(T) \leq \gamma_b(G)$. Since T is a spanning subgraph of G, we have $\gamma_b(G) \leq \gamma_b(T)$ by Lemma 2.1. □

Theorem 2.3 *Let G be a connected graph. Then*

$$\gamma_b(G) \geq \left\lceil \frac{2\text{rad(G)}}{3} \right\rceil,$$

and the bound is sharp.

Proof. Let us first prove the bound for trees. If $G = K_2$ then the bound clearly holds. Let $T \neq K_2$ be an arbitrary tree, and $v \in V(T)$ be a central vertex of T. Then there is a vertex x at distance $\text{rad}(T)$ from v. We claim that there is a vertex y, from a connected component of $T-v$ not containing x, such that $d(v,y) \geq \text{rad}(T) - 1$. Suppose there is no such y. Let u be the neighbor of v on the path between v and x. Then $d(u,x) = \text{rad}(T) - 1$, and $d(u,z) \leq \text{rad}(T) - 1$ for all other $z \in V(T)$, which is a contradiction. Hence there is a vertex y such that the path between y and x has at least $2\text{rad}(T)$ vertices. Denote this path by P, and let f be a dominating broadcast of T with $w(f) = \gamma_b(T)$. For each $x \in V(T)$ denote by x' the unique closest vertex of x in P. Let f' be the broadcast of P defined by $f'(x) = \max\{f(y) \mid y' = x\}$. Since f is a dominating broadcast of T, f' is a dominating broadcast of P, and $w(f) \geq w(f')$. Clearly $w(f') \geq \gamma_b(P)$. It was observed in [3] that $\gamma_b(P_n) = \lceil \frac{n}{3} \rceil$, and so

$$\gamma_b(T) = w(f) \geq w(f') \geq \left\lceil \frac{2\text{rad}(T)}{3} \right\rceil.$$

Let now G be an arbitrary connected graph. Then by Lemma 2.2 there exists a spanning tree T of G, such that $\gamma_b(G) = \gamma_b(T)$. Using that the bound is true for T we get

$$\gamma_b(G) = \gamma_b(T) \geq \left\lceil \frac{2\text{rad}(T)}{3} \right\rceil \geq \left\lceil \frac{2\text{rad}(G)}{3} \right\rceil.$$

The last inequality holds, because $\text{rad}(T) \geq \text{rad}(G)$ (Lemma 2.1).

The bound is achieved, for instance, in the case of even paths P_{2n}. □

2.2 Condensed dominating broadcasts

Let G be a graph and f a dominating broadcast of G. We say that f is a *condensed dominating broadcast* if for any vertex $x \in V(G)$ and any subset $D' \subseteq D_f$, such that $|D'| \geq 2$, there is a vertex $y \in D'$ such that

$$N_{f(y)}[y] \nsubseteq N_r[x],$$

where $r = \sum_{u \in D'} f(u)$. From the definition we infer the following. If there exists a vertex x and a set $D' \subseteq D_f$, such that

$$d(x,y) \leq \sum_{u \in D' \setminus \{y\}} f(u) \tag{2}$$

for all $y \in D'$, then f is not condensed. In this case one can transfer the weights of D' to the vertex x and still obtain a dominating broadcast, which is, intuitively speaking, more condensed than the original one. If f is a dominating broadcast with vertices $x, y \in D_f, x \neq y$, such that $d(x,y) \leq f(x)$, then f is clearly not condensed. (In the language of [2], a condensed dominating broadcast is also independent dominating broadcast.) In this case we can obtain a dominating broadcast f_1 by setting $f_1(x) = f(x) + f(y)$, $f_1(y) = 0$, and $f_1(z) = f(z)$ otherwise. Clearly $w(f_1) = w(f)$, and $|D_f| > |D_{f_1}|$. We can show even more.

Lemma 2.4 *If f is a condensed dominating broadcast in a graph G, then f is also an efficient dominating broadcast in G.*

Proof. Let f be a condensed dominating broadcast. We claim that the closed neighborhoods $N_{f(u)}[u]$, where $u \in D_f$ are pairwise disjoint. Suppose to the contrary, there is a vertex $z \in N_{f(u)}[u] \cap N_{f(v)}[v]$ for some $u \neq v$ in D_f. Then clearly $d(u,v) \leq f(u) + f(v)$. Now let x be a vertex on a shortest u,v-path, such that $d(x,u) = f(v)$ and set $D' = \{u,v\}$. Since $d(x,u) = f(v)$ we find that $d(x,v) \leq f(u)$ and therefore f is not condensed. □

The converse of the above implication is not true. For instance, let G be obtained from the path P_5 such that to its central vertex v a leaf is attached. Then the leaves of G form a perfect code which yields an efficient dominating broadcast (the leaves are given weight 1, and other three vertices weight 0). The broadcast is not condensed, since v is at distance at most two to every leaf.

Lemma 2.5 *Let G be a graph and f a dominating broadcast on G. Then there exists a condensed dominating broadcast g on G, such that $w(g) \leq w(f)$.*

Proof. If f is condensed dominating broadcast of G, there is nothing to prove. In the sequel assume that f is not condensed. Then there is a vertex $x \in V(G)$ and a subset $D' \subseteq D_f$, such that $|D'| \geq 2$ and

$$N_{f(y)}[y] \subseteq N_r[x]\,, \tag{3}$$

for all $y \in D'$, where $r = \sum_{u \in D'} f(u)$.

Define f_1 as follows:

$$f_1(u) = \begin{cases} f(u) & \text{if } u \in D_f \setminus D' \\ r & \text{if } u = x \\ 0 & \text{otherwise} \end{cases}$$

We claim that f_1 is a dominating broadcast on G. Let $y \in V(G)$ be an arbitrary vertex. Since f is a dominating broadcast, y is f-dominated by some vertex $v \in D_f$. If $v \notin D'$ then y is also f_1-dominated by v. Otherwise $v \in D'$. In this case it follows from (3) that $N_{f(v)}[v] \subseteq N_r[x]$, and since $y \in N_{f(v)}[v]$ we find that y is f_1-dominated by x. Clearly the weight of f_1 is equal to the weight of f and $|D_{f_1}| < |D_f|$. If f_1 is condensed then $g = f_1$ and we are done. If f_1 is not condensed, then we can define analogously as f_1, a dominating broadcast f_2 such that $|D_{f_2}| < |D_{f_1}|$. More generally, for any non-condensed dominating broadcast f_i we can define a dominating broadcast of equal weight, such that $|D_{f_i}| < |D_{f_{i-1}}|$. If there is no dominating broadcast g with $D_g > 1$ and $w(g) = w(f)$, then since $|D|$ is finite, there is a dominating broadcast g with $|D_g| = 1$ and $w(g) = w(f)$, which clearly is condensed.

□

The following result follows immediately from Lemma 2.5.

Theorem 2.6 *Let G be a graph. There exists a condensed dominating broadcast f of G such that $w(f) = \gamma_b(G)$.*

The above results implies [2, Theorem 17] about efficient dominating broadcasts since by Lemma 2.4 condensed dominating broadcasts are also efficient. Also, it might be useful, for instance, in finding graphs in which the broadcast domination number equals their radius (so called Type 2 graphs in [2]). We shall use Theorem 2.6 in the sequel.

3 Bounds for products of graphs

3.1 Cartesian product

We start with an upper bound for the Cartesian product $X = G \Box H$ of graphs G and H. It is easy to see that $d_X((u,x),(v,y)) = d_G(u,v) +$

$d_H(x, y)$, and so rad(G□H) = rad(G) + rad(H). Now,

$$\gamma_b(G\Box H) \leq \text{rad}(G\Box H) = \text{rad}(G) + \text{rad}(H) \leq \frac{3}{2}(\gamma_b(G) + \gamma_b(H)),$$

where the last inequality follows from Theorem 2.3, and $\gamma_b(G) \geq \left\lceil \frac{2\text{rad}(G)}{3} \right\rceil \geq \frac{2\text{rad}(G)}{3}$.

Proposition 3.1 *Let G and H be connected graphs. Then*

$$\gamma_b(G\Box H) \leq \frac{3}{2}(\gamma_b(G) + \gamma_b(H)),$$

and the bound is sharp.

The sharpness of the bound can be checked with $C_6 \Box C_6$.

Let us now consider a lower bound for $\gamma_b(G\Box H)$. We shall obtain the lower bound $\gamma_b(G)$ that looks natural and perhaps easy, yet the class of graphs for which it is sharp is rather large. We will also characterize graphs of this class (so called class $\mathcal{X}$), the result which will be used in the last section.

For $n \geq 2$, let X_n be the graph $K_{2,n}$, see Figure 1. A graph G is in class $\mathcal{X}$ if X_n is a spanning subgraph of G for some $n \geq 2$ and $\gamma(G) > 1$. Note that for every graph G in class $\mathcal{X}$, $\gamma(G) = 2$.

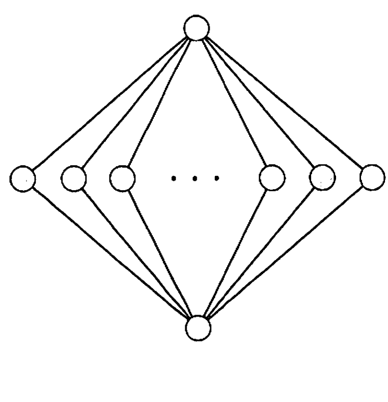

Figure 1: Graphs X_n

Theorem 3.2 *Let $X = G\Box H$ be the Cartesian product of two connected nontrivial graphs. Then $\gamma_b(X) \geq \gamma_b(G)$ and the equality holds if and only if $H = K_2$ and G is a graph from the class $\mathcal{X}$, in which case $\gamma_b(X) = \gamma(X) = \gamma_b(G) = \gamma(G) = 2$.*

Proof. Let f be a minimum dominating broadcast of $X = G\Box H$. Define the function $M : V(G) \times V(H) \to \mathbb{N}_0$ as follows

$$M(u,v) = \max\{f(u,z) - d_H(v,z) \mid z \in V(H), z \neq v\}.$$

We claim that for any $v \in V(H)$ the function

$$f_v(u,v) = \begin{cases} \max\{f(u,v), M(u,v)\} & \text{if } \max\{f(u,v), M(u,v)\} > 0 \\ 1 & \text{if } f(u,v) = M(u,v) = 0 \\ 0 & \text{otherwise.} \end{cases}$$

is a dominating broadcast of G_v. Let (u,v) be an arbitrary vertex of G_v. As f is a dominating broadcast, there is a vertex $(x,y) \in D_f$ which f-dominates (u,v), that is $d_G(x,u) + d_H(y,v) \leq f(x,y)$. If $y = v$ then it is easily seen that (x,y) f_v-dominates (u,v). If $x = u$ then clearly $M(u,v) \geq 0$ and so $f_v(u,v) \geq 1$ by the second row in the definition of f_v. Finally, let $y \neq v$ and $x \neq u$. Then $M(x,v) \geq f(x,y) - d_H(v,y) > 0$, and so $d_G(u,x) \leq M(x,v) \leq f_v(x,v)$. Thus (u,v) is f_v-dominated by (x,v), and the lower bound follows.

Now suppose that $\gamma_b(X) = \gamma_b(G)$. Let f be a minimum dominating broadcast of X, and in addition, let f be condensed (we may assume this by Theorem 2.6). We claim that $f(u,z) \leq 1$ for all $(u,z) \in V(X)$. If not, let $(u,z) \in V(X)$ be such that $f(u,z) > 1$, and let $v \in V(H)$ be a neighbor of z. Consider the function f_v. Note that $M(u,v) \geq 1$, and so for $f_v(u,v)$ we use the first row in the definition of f_v. This yields $w(f_v) < w(f)$, a contradiction. Thus indeed $f(u,z) \leq 1$ for all $(u,z) \in V(X)$.

Let $(u,v) \in V(X)$ be a vertex, such that $f(u,v) = 1$. We claim that $N[v] = V(H)$. If not, let $v' \in V(H)$ be a vertex that is not adjacent to v. Consider the dominating broadcast $f_{v'}$. As v and v' are not adjacent we find that $f_{v'}(u,v') < \sum_{z\in V(H)} f(u,z)$ and hence $w(f_{v'}) < w(f)$, a contradiction.

We now claim that H is a complete graph. If for all $x \in V(G)$, there is a vertex $y \in V(H)$, such that $f(x,y) = 1$, then $w(f) \geq |G| \geq \mathrm{rad}(G) + 1 \geq \gamma_b(G)+1$, a contradiction. Hence there is an $x \in V(G)$, such that $f(x,y) = 0$ for all $y \in V(H)$. Since f is dominating, for every $y \in V(H)$, there is a neighbor x_y of x, such that $f(x_y,y) = 1$. It follows that $N[y] = V(H)$ for all $y \in V(H)$, by the claim of the previous paragraph. Therefore H is a complete graph. Now if $|V(H)| \geq 3$, then we claim that f is not condensed. Let $x \in V(G)$ be such that $f(x,y) = 0$ for all $y \in V(H)$ and $f(x_y,y) = 1$ for all $y \in V(H)$, where x_y is a neighbor of x. Let $D' = \{(x_y,y) \mid y \in V(H)\}$ and let $v \in V(H)$ be an arbitrary fixed vertex. Then we find that

$$d_X((x,v),(x_y,y)) \leq 2 \leq \sum_{\substack{(a,b)\in D' \\ (a,b)\neq(x_y,y)}} f(u)$$

for all $(x_y, y) \in D'$. Thus it follows from (2) that f is not condensed, a contradiction. Hence $H = K_2$.

Since $H = K_2$, and the range of f is $\{0,1\}$, D_f corresponds to an ordinary dominating set, and $|D_f| = \gamma_b(X)$. Thus $\gamma_b(G\Box K_2) = \gamma(G\Box K_2) = \gamma_b(G) = \gamma(G)$, where the last equality follows from $\gamma_b(G) \leq \gamma(G) \leq \gamma(G\Box K_2)$. Let $V(H) = \{u, v\}$. Clearly $D_f \cap G_u \neq \emptyset$ and $D_f \cap G_v \neq \emptyset$ since otherwise $|D_f| = |G| \geq \text{rad(G)} + 1$. Let $D_1 = D_f \cap G_u$ and $D_2 = D_f \cap G_v$. We claim that for any vertices $(w, u) \in D_1$ and $(z, v) \in D_2$, vertices w and z are not adjacent in G. Indeed, if z and w were adjacent, f would not be condensed by (2) (for $x = (w, v)$ and $D' = \{(w, u), (z, v)\}$ we would have $d((w, u), (w, v)) = d((z, v), (w, v)) = 1)$.

Next we claim that for every $(w, u) \in D_1$ there is exactly one $(z, v) \in D_2$ such that $N(w) = N(z), w \neq z$. Let $(w, u) \in D_1$ be an arbitrary vertex. First, suppose that $\deg_G(w) = 1$. Consider the vertex (t, v), where t is the only neighbor of w in G. Since (t, v) is not dominated by (t, u) (recall that f is condensed), there exist a vertex $s \in V(G)$ such that (s, v) dominates (t, v). Hence $N(w) \subseteq N(s)$. If $N(w) \neq N(s)$, then there is a vertex $t' \in N(s)$ and a vertex s', such that (s', u) dominates (t', u). Since $x = (s, u)$ is a vertex, such that $d((s, u), (w, u)), d((s, u), (s, v)), d((s, u), (s', u)) \leq 2$, we find by setting $D' = \{(w, u), (s, v), (s', u)\}$, that f is not condensed, a contradiction. Therefore $N(w) = N(s)$ and $s = z$.

Now suppose that $\deg_G(w) > 1$ and let $(t_1, v), (t_2, v)$ be vertices such that $t_1, t_2 \in N(w)$. Since f is condensed, (t_1, v) and (t_2, v) are not f-dominated by (t_1, u) and (t_2, u), and so there exist vertices (s_1, v) and (s_2, v) from D_f, such that (s_i, v) dominates (t_i, v). We claim that $s_1 = s_2 = z$. Suppose that $s_1 \neq s_2$ and set $x = (w, v)$ and $D' = \{(s_1, v), (s_2, v), (w, u)\}$. Since $d((s_1, v), (w, v)) = d((s_2, v), (w, v)) = 2$ and $d((w, u), (w, v)) = 1$ we find that f is not condensed, a contradiction. It follows that $N(w) \subseteq N(z)$. By analogous arguments we get $N(z) \subseteq N(w)$. Therefore $N(w) = N(z)$ and w and z are not adjacent.

Let us now prove that $|D_1| = |D_2| = 1$. Let $D_1 = \{(w_1, u), \ldots, (w_n, u)\}$ and let $D_2 = \{(z_1, v), \ldots, (z_n, u)\}$ be such that $N(w_i) = N(z_i)$ for $i = 1, \ldots, n$. Consider the graph G_1 induced by $\{w_1, z_1\} \cup N(w_1)$. Clearly this graph is from the set $\mathcal{X}$ and moreover, since G is connected, there is a graph G_i induced by $\{w_i, z_i\} \cup N(w_i)$, such that some vertex from G_1 is adjacent to a vertex from G_i. We infer there is a vertex $t_1 \in N(w_1)$ adjacent to a vertex $t_i \in N(w_i)$. Setting $x = (t_1, u)$ and $D' = \{(w_1, u), (w_i, u), (z_1, v)\}$ we find that f is not condensed, a contradiction. Hence $n = 1$, $|D_f| = 2$ and G is from the class $\mathcal{X}$ as claimed.

□

We remark that $\mathcal{X}$ is the subclass of the class of graphs that appears in the following result of Hartnell and Rall [4], concerning the usual domination number of prisms.

Theorem 3.3 *[4] For a connected graph G, $\gamma(G\square K_2)=\gamma(G)$ if and only if G has a γ-set D that can be partitioned as $D_1\cup D_2$ such that $V(G)\setminus N[D_1]=D_2$ and $V(G)\setminus N[D_2]=D_1$.*

3.2 Strong product

Proposition 3.4 *Let $X=G\boxtimes H$ be the strong product of connected graphs G and H, Then*

$$\gamma_b(G\boxtimes H)\leq \frac{3}{2}\max\{\gamma_b(G),\gamma_b(H)\},$$

and the bound is sharp.

Proof. For any vertices $(u,v),(x,y)\in V(G)\times V(H)$ their distance is given by:

$$d_X((u,v),(x,y))=\max\{d_G(u,x),d_H(v,y)\}.$$

It follows that $e_X((u,v))=\max\{e_G(u),e_H(v)\}$, and therefore

$$\mathrm{rad(G\boxtimes H)}=\max\{\mathrm{rad(G)},\mathrm{rad(H)}\}.$$

We derive

$$\gamma_b(G\boxtimes H)\leq \mathrm{rad(G\boxtimes H)}=\max\{\mathrm{rad(G)},\mathrm{rad(H)}\}\leq \max\{\frac{3}{2}\gamma_b(\mathrm{G}),\frac{3}{2}\gamma_b(\mathrm{H})\},$$

where the last inequality follows from Theorem 2.3. Note that $\gamma_b(C_6)=2$ and $\gamma_b(C_6\boxtimes C_6)=3$ which shows the bound is sharp. □

We now present an upper bound of a different flavor.

Proposition 3.5 *Let D_1 and D_2 be distance k-dominating sets of G and H respectively. Then $D_1\times D_2$ is a distance k-dominating set of $G\boxtimes H$, moreover*

$$f(u,v)=\begin{cases} k & \text{if } (u,v)\in D_1\times D_2\\ 0 & \text{otherwise}\end{cases}$$

is a dominating broadcast of $G\boxtimes H$.

Proof. Let (x,y) be an arbitrary vertex of $G\boxtimes H$. Since D_1 and D_2 are distance k-dominating sets of G and H respectively, there are vertices $u\in D_1$ and $v\in D_2$ such that $d(x,u)\leq k$ and $d(y,v)\leq k$. We infer $d_{G\boxtimes H}((x,y),(u,v))=\max\{d(x,u),d(y,v)\}\leq k$. Hence $D_1\times D_2$ is a distance k-dominating set of $G\boxtimes H$ and f is a dominating broadcast. □

The following corollary is straightforward to prove.

Corollary 3.6 *For any graphs G and H, $\gamma_b(G\boxtimes H)\leq \min\{k\,\gamma_k(G)\gamma_k(H)\mid k\in \mathbb{N}\}$.*

3.3 Direct product

We will use the following lemma about the distances in direct product [1].

Lemma 3.7 *Let $X = G \times H$ and let $(x,y),(v,w)$ be vertices of X. Then $d_X((x,y),(v,w))$ is the smallest d such that there is an x,v-walk of length d in G and a y,w-walk of length d in H.*

Note that from Lemma 3.7 we infer the following. If there is an x,v-walk of the same parity as $d_H(y,w)$ but not longer than $d_H(y,w)$, then $d_X((x,y),(v,w)) = d_H(y,w)$. This is because we can always prolong the length of a walk by any even number by going back and forth along the same edge as many times as needed.

Theorem 3.8 *Let $X = G \times H$ be the direct product of connected graphs G and H. Then*

$$\gamma_b(G \times H) \leq \begin{cases} 3\max\{\gamma_b(G),\gamma_b(H)\} & \text{if } \mathrm{rad(G)} \neq \mathrm{rad(H)} \\ 3\min\{\gamma_b(G),\gamma_b(H)\}+1 & \text{if } \mathrm{rad(G)} = \mathrm{rad(H)}. \end{cases}$$

and the bound is sharp.

Proof. We may assume without loss of generality that $\mathrm{rad(H)} \leq \mathrm{rad(G)}$. Let $u \in V(G)$ be a central vertex of G, and $v \in V(H)$ a central vertex of H, and let $v' \in V(H)$ be a neighbor of v.

Suppose first that $\mathrm{rad(H)} < \mathrm{rad(G)}$. Then we set $f(u,v) = f(u,v') = \mathrm{rad(G)}$, and $f(x,y) = 0$, otherwise. Let (x,y) be an arbitrary vertex of $G \times H$. Note that $d_G(x,u) \leq \mathrm{rad(G)}$ and $d_H(y,v) < \mathrm{rad(G)}$. If $d_G(x,u)$ and $d_H(y,v)$ are of the same parity then

$$d_X((x,y),(u,v)) = \max\{d_G(x,u), d_H(y,v)\} \leq \mathrm{rad(G)},$$

hence (x,y) is f-dominated by (u,v). If they are not of the same parity, then there is a v',y-walk W of length $d_H(y,v)+1$ (which is not greater than rad(G)), and so it is of the same parity as $d_G(x,u)$. Hence $d_X((x,y),(u,v')) = \max\{d_G(x,u),|W|\} \leq \mathrm{rad(G)}$, thus (u,v') f-dominates (x,y). We infer that f is a dominating broadcast of $G \times H$. By Theorem 2.3 we have $w(f) = 2\mathrm{rad(G)} \leq 2\frac{3}{2}\gamma_b(\mathrm{G}) = 3\gamma_b(\mathrm{G})$.

Suppose now that $\mathrm{rad(H)} = \mathrm{rad(G)}$. Then we set $f(u,v) = \mathrm{rad(G)}$, $f(u,v') = \mathrm{rad(G)}+1$, and $f(x,y) = 0$, otherwise. Arguing analogously as above one can show that f is a dominating broadcast (details are left to the reader) whose weight is clearly $2\mathrm{rad(G)}+1 \leq 3\gamma_b(\mathrm{G})+1$. We also have $w(f) \leq 3\gamma_b(H)+1$.

Note that $\mathrm{rad(C_6)} = 3 \neq 2 = \mathrm{rad(P_3)}$, while $\gamma_b(C_6) = 2$ and $\gamma_b(P_3) = 1$. Since $\gamma_b(C_6 \times P_3) = 6$, this shows the bound is sharp. □

4 Broadcast domination of some classes of graphs

4.1 Hamming graphs

Let $G = K_{n_1} \Box K_{n_2} \Box \cdots \Box K_{n_p}$ be the Cartesian product of p complete graphs. If $n_i = 2$ for all i we call the graph *hypercube* or *p-cube*, and denote it by Q_p.

Consider first the case of hypercubes. Note that $\gamma_b(Q_1) = 1$, $\gamma_b(Q_2) = 2 = \gamma_b(Q_3)$, and in the latter case dominating broadcast is unique (up to automorphisms of Q_3). That is, two antipodal vertices $u, v \in V(Q_3)$ (with $d(u,v) = \mathrm{rad}(Q_3) = 3$) are given $f(u) = f(v) = 1$, and $f(x) = 0$, otherwise. Furthermore, note that similar dominating broadcasts can be obtained for arbitrary $Q_n, n \geq 3$, by setting $f(u) = k$, $f(v) = n - k - 1$ for antipodal vertices $u, v \in V(Q_n)$ $(1 \leq k \leq n-2)$, and $f(x) = 0$, otherwise. Hence $\gamma_b(Q_n) \leq n - 1$ for $n \geq 3$.

Note that $\gamma_b(Q_3) = 3-1 = 2$. By Theorem 3.2, $\gamma_b(Q_n) = \gamma_b(Q_{n-1} \Box K_2)$ is strictly greater than $\gamma_b(Q_{n-1})$ as soon as Q_{n-1} is not in class $\mathcal{X}$. Since the only hypercube in class $\mathcal{X}$ is Q_2 we infer $\gamma_b(Q_n) > \gamma_b(Q_{n-1})$ for all $n \geq 4$, and so $\gamma_b(Q_n) \geq n-1$ by induction. Combining this with an earlier observation, $\gamma_b(Q_n) = n - 1$, for $n \geq 3$.

Let $G = K_{n_1} \Box K_{n_2} \Box \cdots \Box K_{n_p}$ be a Hamming graph in which at least one n_i is greater than 2. Since Cartesian product is commutative and associative we can write

$$G = K_{n_1} \Box (\ldots(K_{n_2} \Box \cdots \Box (K_{n_{p-2}} \Box (K_{n_{p-1}} \Box K_{n_p}))\ldots)$$

where $n_p > 2$ (by reindexing of the factors, if necessary). Since for any $i \geq 2$ the graph $K_{n_i} \Box \cdots \Box K_{n_p}$ is not in $\mathcal{X}$ we get $\gamma_b(K_{n_{i-1}} \Box (K_{n_i} \Box \cdots \Box K_{n_p})) > \gamma_b(K_{n_i} \Box \cdots \Box K_{n_p})$. By induction we get $\gamma_b(G) \geq p$ and therefore, since $p = \mathrm{rad(G)}$, $\gamma_b(G) = p$. We summarize our observations in

Theorem 4.1 *Let $X = K_{n_1} \Box K_{n_2} \Box \cdots \Box K_{n_p}$ be a Hamming graph. Then*

$$\gamma_b(X) = \begin{cases} p-1 & \text{if } n_i{=}2 \text{ for all } i \text{ and } p \geq 3 \\ p & \text{otherwise} \end{cases}$$

Hence $\gamma_b(X) = \mathrm{rad}(X)$ holds for all Hamming graphs except hypercubes Q_n, where $n \geq 3$.

4.2 Cartesian products of cycles

We first consider the case when cycles are of even length. In fact we shall obtain a result about a somewhat more general class of graphs that might be of independent interest. Call a graph G *rad-antipodal* if there exist radial vertices u and v (that is, vertices with $d(u,v) = \text{rad}(G)$) such that $I(u,v) = V(G)$. That is, every vertex $x \in V(G)$ lies on a shortest path between u and v, so we have $d(u,v) = d(u,x) + d(x,v)$. (We remark that in rad-antipodal graphs $\text{diam}(G) = \text{rad}(G)$.)

Lemma 4.2 *Let G be a rad-antipodal graph with* $\text{diam}(G) \geq 3$. *Then* $\gamma_b(G) < \text{rad}(G)$.

Proof. Let u and v be two radial vertices with $I(u,v) = V(G)$. Define $f : V(G) \to \mathbb{N}_0$ by $f(u) = k$, $f(v) = \text{diam}(G) - k - 1$, where $0 < k < \text{diam}(G) - 1$, and $f(x) = 0$ otherwise. Then clearly f is a dominating broadcast of G and $w(f) = \text{rad}(G) - 1$. □

Hypercubes and even cycles are rad-antipodal graphs. More rad-antipodal graphs can be derived from the following lemma.

Lemma 4.3 *Let G and H be rad-antipodal graphs. Then $X = G \Box H$ is also rad-antipodal.*

Proof. Note that $\text{rad}(G \Box H) = \text{rad}(G) + \text{rad}(H)$. Let $u, v \in V(G)$ be radial vertices of G with $I(u,v) = V(G)$, and $x, y \in V(H)$ radial vertices of H with $I(x,y) = V(H)$. Then $d_X((u,x),(v,y)) = d_G(u,v) + d_H(x,y) = \text{rad(X)}$, hence $(u,x),(v,y)$ are antipodal vertices in X. Let (a,b) be an arbitrary vertex in X. Then $d_X((u,v),(x,y)) = d_G(u,v) + d_H(x,y) = d_G(u,a) + d_G(a,v) + d_H(x,b) + d_H(b,y) = d_X((u,x),(a,b)) + d_X((a,b),(v,y))$. Hence $I_X((u,v),(x,y)) = V(X)$. □

Since any product of even cycles X has $\text{diam}(X) > 2$ we infer

Corollary 4.4 *Let $X = C_{n_1} \Box C_{n_2} \Box \cdots \Box C_{n_k}$ be the product of even cycles (i.e. n_i even for all i). Then* $\gamma_b(X) < \text{rad(X)}$.

In the sequel we provide exact broadcast domination numbers of Cartesian products of two cycles. We denote the vertices of a cycle C_m by integers $1, 2, \ldots, m$. Note that

$$\text{rad}(C_m \Box C_n) = \begin{cases} \frac{m+n}{2} & \text{if } m \text{ and } n \text{ both even} \\ \frac{m+n-2}{2} & \text{if } m \text{ and } n \text{ are both odd} \\ \frac{m+n-1}{2} & \text{otherwise.} \end{cases}$$

In the proof of Theorem 4.6 we use a result about dominating broadcasts of grid graphs:

Theorem 4.5 *[2] For any pair m, n of integers with $m, n \geq 2$*

$$\gamma_b(P_m \Box P_n) = \mathrm{rad}(P_m \Box P_n) = \left\lfloor \frac{m}{2} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor .$$

Theorem 4.6 *Let $X = C_m \Box C_n$. If m or n is odd then $\gamma_b(X) = \mathrm{rad}(X)$, otherwise $\gamma_b(X) = \mathrm{rad}(X) - 1$.*

Proof. Let f be a minimum dominating broadcast of X and let $u \in V(X)$ be a vertex such that $f(u) \geq f(x)$ for all $x \in V(X)$. Without loss of generality assume that $m \geq n$.

Suppose that $f(u)$ is even and $f(u) + 2 < n$. Without loss of generality assume that $u = (1,1)$. Let $Y = X - N_{f(u)}[u]$ and consider the grids $G_1 \subseteq Y$ and $G_2 \subseteq Y$ defined as follows:

$$G_1 = \left\{1 + \frac{f(u)}{2}, \ldots, m - \frac{f(u)}{2}\right\} \times \left\{2 + \frac{f(u)}{2}, \ldots, n - \frac{f(u)}{2}\right\} \text{ and}$$

$$G_2 = \left\{2 + \frac{f(u)}{2}, \ldots, m - \frac{f(u)}{2}\right\} \times \left\{2 + \frac{f(u)}{2}, \ldots, n - \frac{f(u)}{2} + 1\right\} .$$

As $m \geq n > f(u) + 2$ these two grids are well defined (see Figure 2). Now observe that there are at least $f(u)$ C_m-layers and at least $f(u)$ C_n-layers of X disjoint with G_1. Similarly there are at least $f(u)$ C_m and C_n-layers disjoint with G_2. Therefore, if $x \in V(G_1)$ is a vertex such that $f(x) \leq f(u)$, then the set of f-dominated vertices by x in the subgraph G_1 is equal to the set of f-dominated vertices by x in X. Analogous statement holds if $x \in V(G_2)$. If $x \notin V(G_1)$ and x f-dominates some vertices from G_1, then consider the following projection (see also Figure 2). If $x = (x_1, x_2)$ and $x_2 < 2 + f(u)/2$, then split the weight $f(x)$ of x between two vertices $a = (x_1, 2 + f(u)/2)$ and $b = (x_1, n - f(u)/2)$ of the grid G_1 such that the vertex a receives the weight

$$f(x) - \left(\frac{f(u)}{2} + 2 - x_2\right)$$

if this value is positive. If the above value is 0, then put weight 1 on a, otherwise don't give any weight on a. We put the weight $f(x) - w$ on b, where w is the weight given to a. Call the obtained function f' and observe that the functions f and f' dominate the same set of vertices in G_1. It is easily seen that for any $x \notin V(G_1)$ (or $x \notin V(G_2)$) the weight of x can be split between two vertices of G_1 (or G_2) in such a way that the set of dominated vertices from G_1 (or G_2) stays the same. Since f dominates G_1 and G_2 and since $\gamma_b(G_1) = \left\lfloor \frac{m-f(u)}{2} \right\rfloor + \left\lfloor \frac{n-f(u)-1}{2} \right\rfloor$ and

$\gamma_b(G_2) = \left\lfloor \frac{m-f(u)-1}{2} \right\rfloor + \left\lfloor \frac{n-f(u)}{2} \right\rfloor$ we find that

$$w(f) \geq f(u) + \left\lfloor \frac{m - f(u)}{2} \right\rfloor + \left\lfloor \frac{n - f(u) - 1}{2} \right\rfloor = \left\lfloor \frac{m}{2} \right\rfloor + \left\lfloor \frac{n-1}{2} \right\rfloor$$

and at the same time

$$w(f) \geq f(u) + \left\lfloor \frac{m - f(u) - 1}{2} \right\rfloor + \left\lfloor \frac{n - f(u)}{2} \right\rfloor = \left\lfloor \frac{m-1}{2} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor .$$

Thus we find that $w(f) \geq \frac{m+n-2}{2}$ if m and n are both even or both odd and $w(f) \geq \frac{m+n-1}{2}$ otherwise. Thus $w(f) \geq \mathrm{rad}(X) - 1$ if m and n are even and $w(f) \geq \mathrm{rad}(X)$ otherwise.

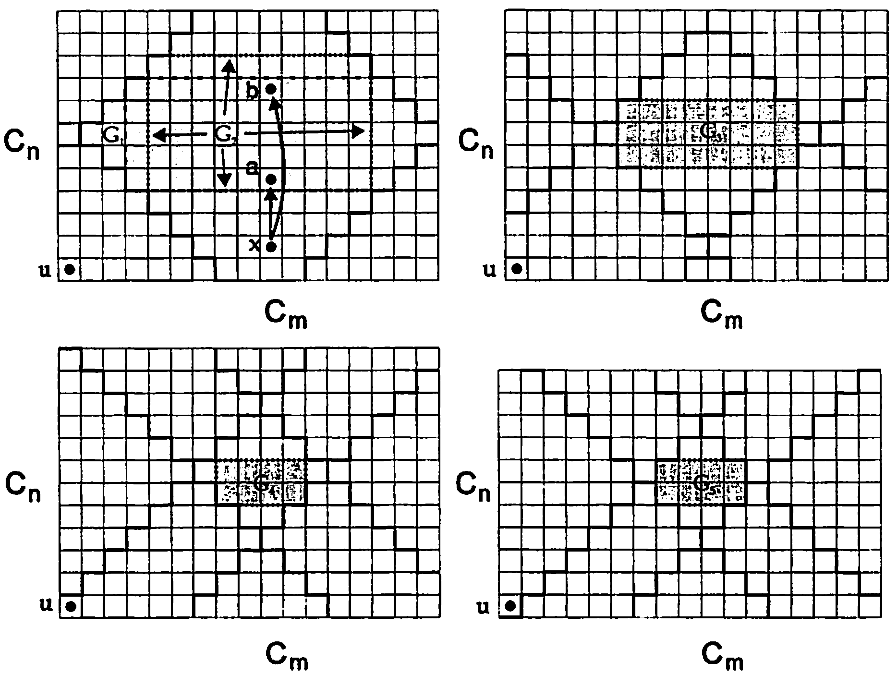

Figure 2: Grids $G_1, \ldots, G_5$. The boundary of G_2 is clearly indicated by arrows, while other grids are shaded. On the upper left picture $f(u) = 6$, on the upper right $f(u) = 9$, and $f(u) = 11$ otherwise.

Suppose that $f(u) > 1$ is odd, $f(u) + 2 < n$ and let

$$G_3 = \left\{1 + \frac{f(u)+1}{2}, \ldots, m - \frac{f(u)-1}{2}\right\} \times \left\{1 + \frac{f(u)+1}{2}, \ldots, n - \frac{f(u)-1}{2}\right\}$$

Note that there are exactly $f(u)$ C_m and C_n layers disjoint with G_3. Arguing as in the earlier two cases, we derive that to broadcast dominate G_3, as a subgraph of X, one needs as much weights as to broadcast dominate G_3, that is, the product of paths. Hence

$$w(f) \geq f(u) + \left\lfloor \frac{m - f(u)}{2} \right\rfloor + \left\lfloor \frac{n - f(u)}{2} \right\rfloor .$$

Since $f(u)$ is odd we find that $w(f) \geq \frac{m+n}{2}$ if m and n are both odd, $w(f) \geq \frac{m+n-2}{2}$ if m and n are both even and $w(f) \geq \frac{m+n-1}{2}$ otherwise. Thus $w(f) \geq \text{rad}(X)-1$ if m and n are even and $w(f) \geq \text{rad}(X)$ otherwise.

If $f(u) = 1$, then all weights are 1 and thus D_f is a dominating set. We can also assume that f is condensed and hence efficient. Let $x = (x_1, x_2) \in X - D_f$ be any vertex. Without loss of generality assume that x is f-dominated by $s = (x_1, x_2 - 1)$. Consider vertices $u = (x_1 - 1, x_2)$ and $v = (x_1 + 1, x_2)$. If u and v are dominated by one vertex $z = (x_1 + 2, x_2)$, then $w = (x_1, x_2 + 1)$ is dominated by the vertex $t = (x_1, x_2 + 2)$, and clearly $t \neq z$ (note that $m = 4$ in this case). In this case set $D' = \{s, t, z\}$. Otherwise u and v are dominated by two vertices z_1 and z_2. In this case set $D' = \{s, z_1, z_2\}$ and observe that in either case f is not condensed by (2), a contradiction.

Now assume that $n \leq f(u) + 2$ and consider the grids

$$G_4 = \left\{3 + f(u) - \frac{n}{2}, \ldots, m + \frac{n}{2} - f(u) - 1\right\} \times \left\{\frac{n}{2}, \frac{n+2}{2}\right\}$$

if n is even and

$$G_5 = \left\{2 + f(u) - \frac{n-1}{2}, \ldots, m + \frac{n-1}{2} - f(u)\right\} \times \left\{\frac{n+1}{2}, \frac{n+3}{2}\right\}$$

if n is odd. Since $\gamma_b(G_4) = \left\lfloor \frac{m+n-2f(u)-1}{2} \right\rfloor$ and $\gamma_b(G_5) = \left\lfloor \frac{m+n-2f(u)}{2} \right\rfloor$ we find that $w(f) \geq \frac{m+n-2}{2}$ if m and n are even and $w(f) \geq \frac{m+n-1}{2}$ otherwise. Hence we have proved that $\gamma_b(X) \geq \frac{m+n-2}{2}$ if m and n are even and $\gamma_b(X) \geq \frac{m+n-1}{2}$ otherwise.

Thus we have, by Corollary 4.4, that $\gamma_b(X) = \text{rad(X)} - 1$ if m and n are even and $\gamma_b(X) = \text{rad(X)}$ otherwise. □

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