Block Diagonalization Method for the Covering Graph

Marvin Minei [a] Howard Skogman [b]

[a] *Department of Mathematics*
University of California
Irvine, CA 92697 – 3875
mminei@math.uci.edu

[b] *Department of Mathematics*
State University of New York College at Brockport
Brockport, NY 14420 – 2914
hskogman@brockport.edu

Abstract. We present a block diagonalization method for the adjacency matrices of two types of covering graphs. A graph Y is a covering graph of a base graph X if there exists an onto graph map $\pi : Y \longrightarrow X$ such that for each $x \in X$ and for each $y \in \{y \mid \pi(y) = x\}$, the collection of vertices adjacent to y maps onto the collection of vertices adjacent to $x \in X$. The block diagonalization method requires the irreducible representations of the Galois group of Y over X. The first type of covering graph is the Cayley graph over the finite ring $\mathbb{Z}/p^n\mathbb{Z}$. The second type of covering graph resembles large lattices with vertices $\mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$ for large n. For one lattice, the block diagonalization method allows us to obtain explicit formulas for the eigenvalues of its adjacency matrix. We use these formulas to analyze the distribution of its eigenvalues. For another lattice, the block diagonalization method allows us to find non-trivial bounds on its eigenvalues.

Key words. Block diagonalization, affine group over a finite ring, Cayley graph, eigenvalues, adjacency matrix, irreducible representations, covering graph, Ramanujan graph, lattice, Ihara zeta function.

1 Introduction

In this paper, we present a theorem that block diagonalizes the adjacency matrix of a covering graph. We begin with a definition of the covering graph. Let X and Y be finite, connected, undirected graphs with no self-loops and no multiple edges. Then, Y is a covering graph of base graph X if there exists an onto graph map $\pi : Y \longrightarrow X$ such that for each $x \in X$ and for each $y \in \{y \mid \pi(y) = x\}$, the collection of vertices adjacent to y maps bijectively onto the collection of vertices adjacent to $x \in X$. Figure 1 presents an example of a covering graph Y and its base graph X. In this case, our mapping $\pi : Y \longrightarrow X$ is defined as

$$\pi(a_1) = a,\ \pi(a_2) = a,$$

$$\pi(b_1) = b,\ \pi(b_2) = b,$$

$$\pi(c_1) = c,\ \pi(c_2) = c,$$

$$\pi(d_1) = d,\ \pi(d_2) = d.$$

Intuitively, Y is constructed by first taking two copies of X then carefully editing the edges of these copies to satisfy the existence of the mapping π.

Our theorem block diagonalizes the adjacency matrix of Y with respect to X. In particular, the theorem requires the irreducible representations of the Galois group of Y over X, Gal(Y/X). This group is the set of graph automorphisms of Y such that if $\sigma \in \text{Gal}(Y/X)$ then $\pi(\sigma(y)) = \pi(y)$ for all $y \in Y$. For Figure 1, it is not difficult to show that Gal(Y/X) is isomorphic to $\mathbb{Z}/2\mathbb{Z}$. We also look at graphs X with multiple edges. This changes the definition of π slightly. In Section 2, we present these definitions formally.

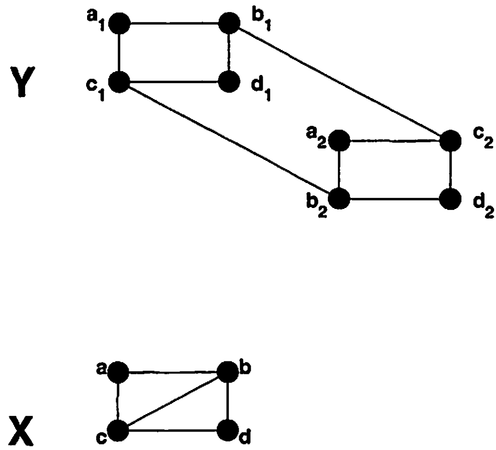

Fig. 1. The covering graph Y with base graph X.

The block diagonalization of a covering graph will reduce the computation time required to calculate the eigenvalues of the adjacency matrix of the covering graph. In one of the graphs, the block diagonalization form will lead us all the way to explicit eigenvalue formulas for our graph. In another graph, the blocks will give us upper and lower bounds on the spectrum of the graph.

In Section 2, we present the definitions and block diagonalization theorem for the covering graph. In Bell and Minei [2], this theorem was presented for a covering graph Y with base graph X where both X and Y are finite, connected, undirected, graphs with no self-loops and no multiple edges. In this paper, we allow X to have multiple edges.

In Section 3, we apply the block diagonalization theorem to two Cayley graphs of the affine group over the finite ring $\mathbb{Z}/p^n\mathbb{Z}$ where p is prime and n is a positive integer. A Cayley graph is defined as follows. If S is a subset of a finite group G, the Cayley graph $X(G,S)$ has as its vertex set G. Edges connect vertices $g \in G$ and $gs \in G$ for all $s \in S$. Usually, we assume $S = S^{-1}$ and S generates the group G so that $X(G,S)$ is an undirected, connected graph. The set S is referred to as the generating set of the graph. For this section, we set G to be the affine group over the ring $\mathbb{Z}/p^n\mathbb{Z}$, Aff $(\mathbb{Z}/p^n\mathbb{Z})$, where p is a prime number and n is an integer greater than or equal to 1. We refer to $X(G,S)$ as an affine graph. The eigenvalues of the adjacency matrix of the graph are referred to as the eigenvalues of the graph.

In this section, we block diagonalize $Y\left(\text{Aff}\left(\mathbb{Z}/p^{n+1}\mathbb{Z}\right), S\right)$ as a covering graph of $X\left(\text{Aff}\left(\mathbb{Z}/p^n\mathbb{Z}\right), S\right)$ for general S. This leads to a decomposition of the adjacency matrix of Y into p^2 blocks of dimension $p^n\left(p^n - p^{n-1}\right)$. We then apply this theorem to the following two affine graphs. For the first graph, define

$$T =$$
$$\{(g^m \;\; m) \mid m = 1, 2, \ldots, (p-2)\} \cup \{(1 \;\; 1), (g \;\; 0)\} \pmod{p}.$$

Define

$$S_p = T \cup T^{-1} \pmod{p}$$

and

$$S_{p^2} = T \cup T^{-1} \left(\text{mod } p^2\right)$$

where $T^{-1} \pmod{p}$ is the set of inverses of the elements of T computed modulo p, $T^{-1} \pmod{p^2}$ is the set of inverses of the elements of T computed modulo p^2, and g is a primitive root of p. Then, we block diagonalize $Y = Y\left(\text{Aff}\left(\mathbb{Z}/p^2\mathbb{Z}\right), S_{p^2}\right)$ as a covering graph of $X = X\left(\text{Aff}\left(\mathbb{Z}/p\mathbb{Z}\right), S_p\right)$. The adjacency matrix of Y decomposes into p^2 blocks of dimension $p(p-1)$. An important connection between X and Y used in the decomposition of Y is the irreducible representations of the Galois group Y over X, Gal(Y/X). For our X and Y, Gal$(Y/X) \approx \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$. In this case, we easily obtain the representations of Gal(Y/X).

For the second affine graph, we look at $Y\left(\text{Aff}\left(\mathbb{Z}/p^{n+2}\mathbb{Z}\right), S\right)$ as a covering graph of $X\left(\text{Aff}\left(\mathbb{Z}/p^n\mathbb{Z}\right), S\right)$ where

$$S = \left\{(g \;\; 0),\; (g^{-1} \;\; 0),\; (1 \;\; 1),\; (1 \;\; -1)\right\}.$$

With this simple change to Y, $\mathrm{Gal}(Y/X)$ rises significantly in complexity and makes the theory for block diagonalizing Y much more difficult. We demonstrate this by looking at $Y = Y(\mathrm{Aff}(\mathbb{Z}/p^3\mathbb{Z}), S)$ over $X = X(\mathrm{Aff}(\mathbb{Z}/p\mathbb{Z}), S)$.

An interesting computational observation resulting from this section involves the eigenvalues of $X(\mathrm{Aff}(\mathbb{Z}/p\mathbb{Z}), S)$ where

$$T =$$

$$\{(g^m \;\; m) \mid m = 1, 2, \ldots, (p-2)\} \cup \{(1 \;\; 1), (g \;\; 0)\} \pmod{p}$$

and $S = T \cup T^{-1}$. Calculations for various p show that X satisfies the Ramanujan bound. More specifically, for λ the second largest eigenvalue (in absolute values) of X, we have $\lambda \leq 2\sqrt{|S| - 1}$. Graphs that satisfy this bound are called Ramanujan graphs. Calculations for $X(\mathrm{Aff}(\mathbb{Z}/p^n\mathbb{Z}), S)$ suggest that the graphs are not Ramanujan graphs for $n \geq 2$. From past experience, we believe that a search for a larger family of Ramanujan graphs lie in $X(\mathrm{Aff}(\mathbb{F}_{p^n}), S)$ where $\mathbb{F}_{p^n}$ is the finite field of order p^n. We say more about this in the final section of this paper.

There exists a more effective method for block diagonalizing a Cayley graph $X(G, S)$ by using the irreducible representations of G. This is the case for our affine graphs where the representations of the affine group are worked out in [2]. The strength of our block diagonalization theorem comes from its application to non-Cayley graphs which is the subject of Section 4.

In Section 4, we look at two non-Cayley graphs with vertices $\mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$ where n is a large integer. For $n \longrightarrow \infty$, both graphs resemble lattices covering the first quadrant of the xy-plane. For the first graph, we determine explicit formulas for its eigenvalues. These formulas are used to analyze how the eigenvalues distribute themselves on the real line. For this graph, its base graph is a multiple-edged graph.

For the second graph of this section, we decompose its adjacency matrix into a set of 4-dimensional blocks. Using these blocks, we show that the eigenvalues of the graph are bounded by -2.303 and 3. Empirical data shows that -2.303 and 3 provide tight bounds on the eigenvalues. For this graph, its base graph is not a multiple-edged graph.

In Section 5, we state future work connected with this paper. The first topic is the Ihara zeta function of our graphs of Section 4. The second topic is related to the potential Ramanujan graph of Section 3.

2 Block Diagonalization Theorem of the Covering Graph

In this section, we present the definition of the covering graph and a theorem for block diagonalizing its adjacency matrix.

Let X and Y be finite, connected, undirected graphs with no self-loops. Suppose both X and Y have no multiple edges. We call Y a finite covering of the base graph X if there exists a covering map $\pi : Y \longrightarrow X$ which is an onto graph map (i.e., adjacent vertices map to adjacent vertices) such that for each $x \in X$ and for each $y \in \{y \mid \pi(y) = x\}$, the collection of vertices adjacent to y maps bijectively onto the collection of vertices adjacent to $x \in X$.

Suppose Y has no multiple edges but X does have multiple edges. Then, for each $x \in X$ and for each $y \in \{y \mid \pi(y) = x\}$, (1) The collection of vertices adjacent to y maps onto the collection of vertices adjacent to $x \in X$, (2) Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be n vertices in Y such that each α_i is connected to y and $\pi(\alpha_1) = \pi(\alpha_2) = \ldots = \pi(\alpha_n)$. Then, the number of edges between $\pi(\alpha_1)$ and $\pi(y)$ is also n.

Suppose the set $\{y \mid \pi(y) = x\}$ has the same number of elements for any $x \in X$. Let d be the order of $\{y \mid \pi(y) = x\}$. Then, Y is called a d-sheeted covering of X.

In this paper, we assume all our graphs are d-sheeted coverings. The next definition gives a more intuitive way of interpreting a d-sheeted cover.

Given a covering map $\pi : Y \longrightarrow X$, we can divide the vertices of Y into discrete sets or sheets $S_1, S_2, \ldots, S_d$. Each sheet S_i is a subgraph of Y such that (1) S_i is connected, (2) π maps S_i bijectively onto X, (3) If $s \in S_i$ and $r \in Y$ (r may or may not be in S_i) are connected to each other then $\pi(s)$ and $\pi(r)$ are connected to each other.

The following example clarifies the above definitions when X does have multiple edges. For convenience, define $E_Y(\alpha, \beta)$ to be an edge connecting vertices α and β in Y and $E_X(\alpha, \beta)$ to be an edge connecting vertices α and β in X. Define X to be the graph with vertices a and b. We make X a multiple-edged graph by drawing two edges between a and b. Define Y to be the graph with vertices a_1, a_2, b_1, and b_2. Draw edges between a_1 and b_1, between a_1 and b_2, between a_2 and b_1, and between a_2 and b_2. See Figure 2.

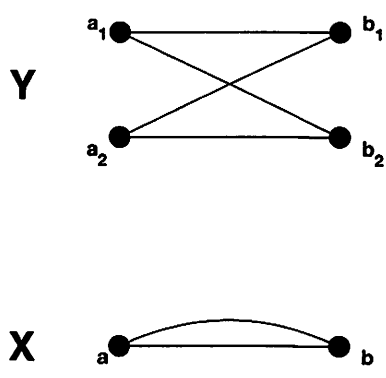

Fig. 2. The covering graph Y with multiple-edged base graph X.

To make Y a covering of X, define $\pi : Y \longrightarrow X$ as $\pi(a_1) = a$,

$\pi(a_2) = a$, $\pi(b_1) = b$, and $\pi(b_2) = b$. In addition, associate the edges of Y to the edges of X as

$$E_Y(a_1, b_1) \longmapsto E_X(\pi(a_1), \pi(b_1)),$$
$$E_Y(a_1, b_2) \longmapsto E_X(\pi(a_1), \pi(b_2)),$$
$$E_Y(a_2, b_1) \longmapsto E_X(\pi(a_2), \pi(b_1)),$$
$$E_Y(a_2, b_2) \longmapsto E_X(\pi(a_2), \pi(b_2)),$$

or

$$E_Y(a_1, b_1) \longmapsto E_X(a, b),$$
$$E_Y(a_1, b_2) \longmapsto E_X(a, b),$$
$$E_Y(a_2, b_1) \longmapsto E_X(a, b),$$
$$E_Y(a_2, b_2) \longmapsto E_X(a, b).$$

In our example, Y is composed of two sheets, $S_1 = \{a_1, b_1\}$, and $S_2 = \{a_2, b_2\}$.

The Galois group of Y over X, Gal(Y/X), is the group of graph automorphisms of Y such that if $\sigma \in$ Gal(Y/X) then $\pi(\sigma(y)) = \pi(y)$ for all $y \in Y$. A d-sheeted covering Y is a normal covering of X if there are d graph automorphisms in Gal(Y/X).

We present a method for block diagonalizing the adjacency matrix of a covering graph. Let the first sheet of Y, S_1, have m vertices. Define the $m \times m$ matrix $A(g)$, $g \in$ Gal(Y/X), by

$$(A(g))_{i,j} = \begin{cases} 1 \text{ if } i \text{ is adjacent to } g(j) \text{ in } Y \\ 0 \text{ otherwise} \end{cases}$$

where i, j are vertices of S_1.

Let $A = (a_{i,j})$ and B be matrices. Then, the tensor product of

A and B, $A \otimes B$, is

$$A \otimes B = \begin{pmatrix} a_{1,1}B & a_{1,2}B & \cdots \\ a_{2,1}B & a_{2,2}B & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}.$$

Theorem 1 *Let Y be a normal d-sheeted covering of X with Galois group* $\mathrm{Gal}(Y/X) = \{g_1, g_2, \ldots, g_d\}$. *The adjacency matrix of Y can be block-diagonalized into blocks of the form*

$$M_\pi = \sum_{g \in \mathrm{Gal}(Y/X)} d_\pi \pi(g) \otimes A(g)$$

where π runs over the irreducible representations of $\mathrm{Gal}(Y/X)$ *and d_π is the dimension of π.*

Proof. Let S_1 be the first sheet of Y and A_Y the adjacency matrix of Y. For $a, b \in S_1$ and $g_i, g_j \in \mathrm{Gal}(Y/X)$, the entries of A_Y are

$$(A_Y)_{g_i(a),g_i(b)} = \begin{cases} 1 & \text{if } a \text{ is adjacent to } g_i^{-1}g_j(b) \\ 0 & \text{otherwise.} \end{cases}$$

Also, define the $|\mathrm{Gal}(Y/X)| \times |\mathrm{Gal}(Y/X)|$ matrix $\rho(g_k)$ as

$$(\rho(g_k))_{g_i,g_j} = \begin{cases} 1 & \text{if } g_i^{-1}g_j = g_k \\ 0 & \text{otherwise} \end{cases}$$

where $g_i, g_j, g_k \in \mathrm{Gal}(Y/X)$. Thus, ρ is the right regular representation of the Galois group $\mathrm{Gal}(Y/X)$ since if δ_a is a vector with 1 in position a, we have $\rho(g)\delta_a = \delta_{ag^{-1}}$.

With these definitions,

$$A_Y = \sum_{g \in \mathrm{Gal}(Y/X)} \rho(g) \otimes A(g).$$

Let $\{\pi_1, \pi_2, \ldots, \pi_n\}$ denote the irreducible representations of $\mathrm{Gal}(Y/X)$ and let $d_1, \ldots, d_n$ denote their respective dimensions. Then, $\rho = d_1\pi_1 \bigoplus \cdots \bigoplus d_n\pi_n$, as it is the right regular representation of $\mathrm{Gal}(Y/X)$. Hence,

$$\begin{aligned} A_Y &= \sum_{g \in \mathrm{Gal}(Y/X)} (d_1\pi_1(g) \oplus \cdots \oplus d_n\pi_n(g)) \otimes A(g) \\ &= \bigoplus_{i=1}^{n} \sum_{g \in \mathrm{Gal}(Y/X)} d_i\pi_i(g) \otimes A(g) \\ &= \bigoplus_{i=1}^{n} M_{\pi_i}. \quad \blacksquare \end{aligned}$$

3 Application to the Cayley Graph over the Affine Group

We apply our block diagonalization method to the Cayley graph over the finite affine group.

Define the affine group over the ring $\mathbb{Z}/p^n\mathbb{Z}$ as

$$\mathrm{Aff}\,(\mathbb{Z}/p^n\mathbb{Z}) = \left\{ \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} \;\middle|\; x, y \in \mathbb{Z}/p^n\mathbb{Z},\ p \nmid y \right\}$$

where p is a prime number and n is an integer greater than or equal to 1. We will refer to $X\,(\mathrm{Aff}\,(\mathbb{Z}/p^n\mathbb{Z}), S)$ as an affine graph. For simplicity, we write $\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}$ as $(y \;\; x)$. The eigenvalues of the adjacency matrix of the affine graph is called the eigenvalues of the affine graph.

Theorem 2 *Let n and m be positive integers. Define $S_{n+m} \subset \mathrm{Aff}\,(\mathbb{Z}/p^{n+m}\mathbb{Z})$ and $S_n \subset \mathrm{Aff}\,(\mathbb{Z}/p^n\mathbb{Z})$ such that* (1) $|S_{n+m}| =$

$|S_n|$, (2) *For every* $t \in S_n$ *there exists* $s \in S_{n+m}$ *such that* $s \equiv t \,(\mathrm{mod}\ p^n)$. *Let* $Y = Y\left(\mathrm{Aff}\left(\mathbb{Z}/p^{n+m}\mathbb{Z}\right), S_{n+m}\right)$ *and* $X = X\left(\mathrm{Aff}\left(\mathbb{Z}/p^n\mathbb{Z}\right), S_n\right)$ *be two affine graphs. Then,* Y *is a normal finite covering of* X *and* $\mathrm{Gal}(Y/X)$ *is isomorphic to the group*

$$\Gamma = \left\{(a \quad b) \mid a, b \in \mathbb{Z}/p^{n+m}\mathbb{Z},\ p^n|(a-1),\ p^n|b\right\}.$$

Proof. The projection $\pi : Y \longrightarrow X$ is the reduction of the coordinates modulo p^{n+m} to modulo p^n. This preserves vertex adjacencies. Moreover, given $g \in X$, if we take a vertex $g' \in Y$ in $\pi^{-1}g$, we see that the vertices in Y adjacent to g' have the form $g's$, for $s \in S_{n+m}$. The vertices adjacent to g in X are of the same form except computed modulo p^n. Also, π maps these adjacent vertices in Y bijectively onto those in X. Thus, π is a covering map that makes Y a finite covering of X.

If $(a \quad b) \in \Gamma$, we define the Galois group element

$$\gamma_{(a,b)}\left((x \quad y)\right)\left(\mathrm{mod}\ p^{n+m}\right) = (a \quad b)(x \quad y)\left(\mathrm{mod}\ p^{n+m}\right).$$

It follows that $\pi \circ \gamma = \pi$ since $(a, b) \equiv (1 \quad 0)\,(\mathrm{mod}\ p^n)$ and π reduces everything mod p^n. By the definition of π, these γ's are the only automorphisms of Y such that $\pi \circ \gamma = \pi$ and hence $\mathrm{Gal}(Y/X) \approx \Gamma$. ■

Theorem 3 *For* X *and* Y *defined as above but with* $m = 1$,

$$\mathrm{Gal}(Y/X) \approx \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}.$$

Proof. See [2]. ■

We demonstrate a practical application of the block diagonalization method for the case $m = 1$. In this case, $Y = Y\left(\mathrm{Aff}\left(\mathbb{Z}/p^{n+1}\mathbb{Z}\right), S\right)$. Define

$$A = \{(a \quad b) \mid a, b \in \mathbb{Z}/p^n\mathbb{Z},\ p \not| a\}$$

and

$$\Gamma = \left\{(a \;\; b) \mid a, b \in \mathbb{Z}/p^{n+1}\mathbb{Z},\; p^n|(a-1),\; p^n|b\right\}.$$

Then, $|A| = p^n\left(p^n - p^{n-1}\right)$ and $|\Gamma| = p^2$. For each $g_i \in \Gamma$, the sets $g_i A$ make up the p^2 sheets of Y.

Construct the $p^n\left(p^n - p^{n-1}\right) \times p^n\left(p^n - p^{n-1}\right)$ matrix $M^{(i)}$ as follows. Associate an element of A to each row of $M^{(i)}$. Associate an element of $g_i A$ to each column of $M^{(i)}$. For each $s \in S$ and $a_j \in A$, if $a_j s = g_i a_k$ for some $a_k \in A$ then put a 1 in the (j,k) entry of $M^{(i)}$. Otherwise, the entries of $M^{(i)}$ are 0's. There are p^2 of these matrices $M^{(i)}$.

Define

$$\gamma_{\alpha,\beta,\delta,\psi} = \exp\left(\frac{2\pi i\alpha\beta}{p}\right) \cdot \exp\left(\frac{2\pi i\delta\psi}{p}\right)$$

for $\alpha, \beta, \delta, \psi$ in $\{1, \ldots, p\}$. These are the irreducible representations of the Galois group.

Then, for each fixed α and δ, the matrices

$$\sum_{\beta=1}^{p}\sum_{\psi=1}^{p} \gamma_{\alpha,\beta,\delta,\psi} M^{((\beta-1)p+\psi)}$$

make up the block diagonalization form of Y.

We look at the results for the following generating set. Let g be a primitive root of $\mathbb{Z}/p\mathbb{Z}$. Define the set

$$T = \{(g^m \;\; m) \mid m = 1, 2, \ldots, (p-2)\} \cup \{(1 \;\; 1), (g \;\; 0)\} \pmod{p}.$$

Define

$$S_p = T \cup T^{-1} \pmod{p}$$

and

$$S_{p^2} = T \cup T^{-1} \left(\text{mod } p^2\right)$$

It is easy to check that $Y\left(\text{Aff}\left(\mathbb{Z}/p^2\mathbb{Z}\right), S_{p^2}\right)$ and $X\left(\text{Aff}\left(\mathbb{Z}/p\mathbb{Z}\right), S_p\right)$ are both connected graphs and that Y is a covering of X. For $p = 5$, Y decomposes into twenty-five 20-dimensional blocks and $\text{Gal}(Y/X) \approx \mathbb{Z}/5\mathbb{Z} \oplus \mathbb{Z}/5\mathbb{Z}$.

Define λ to be the second largest eigenvalue (in absolute values) of a Cayley graph $Y(G, S)$. If $\lambda \leq 2\sqrt{|S| - 1}$ then Y is called a Ramanujan graph. For our graph above with $p = 5$, $Y\left(\text{Aff}\left(\mathbb{Z}/p^2\mathbb{Z}\right), S_{p^2}\right)$ has $\lambda = 8.60$ and its Ramanujan bound is $2\sqrt{|S_{p^2}| - 1} = 6$. Thus, Y is not a Ramanujan graph.

For a computational experiment, we determine λ for $X\left(\text{Aff}\left(\mathbb{Z}/p\mathbb{Z}\right), S_p\right)$ for several p's.

p	λ	$2\sqrt{\lvert S_p - 1\rvert}$	$\frac{\lambda}{2\sqrt{\lvert S_p - 1\rvert}}$
101	16.32	28.35	0.57
131	18.46	32.31	0.57
181	22.19	38	0.58
211	23.35	41.03	0.56
541	35.71	65.75	0.54

All of these graphs satisfy its Ramanujan bounds. Our data tells us that the ratios $\frac{\lambda}{2\sqrt{|S_p - 1|}}$ are oscillating near 0.56. As mentioned in Section 1, it is unclear at this point if X is a Ramanujan graph and if X can be generalized to a larger family of Ramanujan graphs. We present a brief discussion of X in the final section of this paper.

The case $m > 1$ is more difficult to deal with because of the complicated nature of the irreducible representations of Γ. The following is such an example. Set $p = 3$, $n = 1$ and $m = 2$. Define $Y = Y\left(\text{Aff}\left(\mathbb{Z}/3^3\mathbb{Z}\right), S\right)$ and $X = X\left(\text{Aff}\left(\mathbb{Z}/3\mathbb{Z}\right), S\right)$ to be two affine graphs with

$$S = \left\{ \begin{pmatrix} 2^{\pm 1} & 0 \end{pmatrix}, \begin{pmatrix} 1 & \pm 1 \end{pmatrix} \right\}.$$

We have

$$\Gamma = \left\{ (a \;\; b) \mid a, b \in \mathbb{Z}/3^3\mathbb{Z},\ 3|(a-1),\ 3|b \right\}.$$

The classes of irreducible representations of Γ arc as follows.

1. $\pi_s^{(1)} \begin{pmatrix} y & x \end{pmatrix} = \exp\left(\frac{2\pi i s u}{18}\right)$ where $y = 2^u$ and $s = 0, 1, \ldots, 8$.

2. $\pi_s^{(2)} \begin{pmatrix} y & x \end{pmatrix} = \exp\left(\frac{2\pi i s x}{9}\right)$ where $s = 1, 2$.

3. $\pi_r^{(1)} \begin{pmatrix} y & x \end{pmatrix} \cdot \pi_s^{(2)} \begin{pmatrix} y & x \end{pmatrix}$ where $r = 1, 2, \ldots, 8,$ and $s = 1, 2$.

4. $\pi_r^{(1)} \begin{pmatrix} y & x \end{pmatrix} \cdot \pi_s^{(3)} \begin{pmatrix} y & x \end{pmatrix}$ where $r = 0, 1, 2, s = 1, 2$ where

$$\left(\pi_s^{(3)}\right)_{(1,1)} = \begin{cases} \exp\left(\frac{2\pi i s x}{27}\right) & \text{if } x = 0, 9, 18 \\ 0 \text{ otherwise} \end{cases}$$

$$\left(\pi_s^{(3)}\right)_{(1,2)} = \begin{cases} \exp\left(\frac{2\pi i s(24y+x)}{27}\right) & \text{if } 24y + x = 0, 9, 18 \\ 0 \text{ otherwise} \end{cases}$$

$$\left(\pi_s^{(3)}\right)_{(1,3)} = \begin{cases} \exp\left(\frac{2\pi i s(21y+x)}{27}\right) & \text{if } 21y + x = 0, 9, 18 \\ 0 \text{ otherwise} \end{cases}$$

$$\left(\pi_s^{(3)}\right)_{(2,1)} = \begin{cases} \exp\left(\frac{2\pi i s(x+3)}{27}\right) & \text{if } x + 3 = 0, 9, 18 \\ 0 \text{ otherwise} \end{cases}$$

$$\left(\pi_s^{(3)}\right)_{(2,2)} = \begin{cases} \exp\left(\frac{2\pi is(24y+x+3)}{27}\right) & \text{if } 24y+x+3=0,9,18 \\ 0 \text{ otherwise} \end{cases}$$

$$\left(\pi_s^{(3)}\right)_{(2,3)} = \begin{cases} \exp\left(\frac{2\pi is(21y+x+3)}{27}\right) & \text{if } 21y+x+3=0,9,18 \\ 0 \text{ otherwise} \end{cases}$$

$$\left(\pi_s^{(3)}\right)_{(3,1)} = \begin{cases} \exp\left(\frac{2\pi is(x+6)}{27}\right) & \text{if } x+6=0,9,18 \\ 0 \text{ otherwise} \end{cases}$$

$$\left(\pi_s^{(3)}\right)_{(3,2)} = \begin{cases} \exp\left(\frac{2\pi is(24y+x+6)}{27}\right) & \text{if } 24y+x+6=0,9,18 \\ 0 \text{ otherwise} \end{cases}$$

$$\left(\pi_s^{(3)}\right)_{(3,3)} = \begin{cases} \exp\left(\frac{2\pi is(21y+x+6)}{27}\right) & \text{if } 21y+x+6=0,9,18 \\ 0 \text{ otherwise} \end{cases}$$

Let M be the adjacency matrix of Y. Consider the representations $\pi_s^{(1)}$, $s = 0, 1, \ldots, 8$. Since X has order six and each $\pi_s^{(1)}$ is 1-dimensional, these representations produce nine 6-dimensional blocks for M.

The second representations $\pi_s^{(2)}$, $s = 1, 2$, are 1-dimensional. Thus, these representations contribute two 6-dimensional blocks for M.

The third representations $\pi_s^{(1)} \cdot \pi_t^{(2)}$, $s = 1, 2, \ldots, 8$, $t = 1, 2$, are 1-dimensional. These representations contribute sixteen 6-dimensional blocks for M.

The fourth representations $\pi_r^{(1)} \cdot \pi_s^{(3)}$, $r = 0, 1, 2$, $s = 1, 2$, are 3-dimensional. Thus, their blocks are 18-dimensional. The multiplicity of each block is 3. Thus, there are eighteen 18-dimensional blocks.

There exists a more efficient but more complicated method to block diagonalizing a Cayley graph $X(G,S)$. This method involves the irreducible representations of G and is demonstrated in [2] where G is the affine group over the finite rings. The strength of our block diagonalization method for covering graphs comes when the graphs are not Cayley graphs. In the following section, we look at two such examples.

4 Block Diagonalization of Two Lattice Coverings

We analyze the eigenvalues of the following covering graph. Fix N to be a positive integer. We construct the graph $Y = Y(4N^2)$ with $4N^2$ vertices. In the xy-plane, place vertices at each (x,y) where $x = 0,1,\ldots,(2N-1)$ and $y = 0,1,\ldots,(2N-1)$. For each (x,y), draw an edge connecting it with $(x+1,y)(\text{mod } 2N)$ and $(x,y+1)(\text{mod } 2N)$. Figure 3 is the graph of $Y(36)$. For clarity, we omit the edges connecting the perimeter vertices of Y. As $N \longrightarrow \infty$, the resulting graph becomes a lattice of the first quadrant of the xy-plane. For N large, we obtain the eigenvalues of the adjacency matrix of Y.

Theorem 4 *The eigenvalues of the graph* $Y = Y(4N^2)$ *are*

$$2\left(\cos\left(\tfrac{m\pi}{N}\right)+\cos\left(\tfrac{n\pi}{N}\right)\right),\ 2\left(\cos\left(\tfrac{m\pi}{N}\right)-\cos\left(\tfrac{n\pi}{N}\right)\right),$$
$$2\left(-\cos\left(\tfrac{m\pi}{N}\right)+\cos\left(\tfrac{n\pi}{N}\right)\right),\ 2\left(-\cos\left(\tfrac{m\pi}{N}\right)-\cos\left(\tfrac{n\pi}{N}\right)\right).$$

where $m = 0,1,\ldots,(N-1)$ *and* $n = 0,1,\ldots,(N-1)$.

Proof. We define a base graph X of Y as follows. Place vertices at $(0,0)$, $(1,0)$, $(0,1)$, and $(1,1)$. Insert two edges between $(0,0)$ and $(1,0)$, between $(1,0)$ and $(1,1)$, between $(1,1)$ and $(0,1)$, and between $(0,1)$ and $(0,0)$. Define the covering map $\pi : Y \longrightarrow$

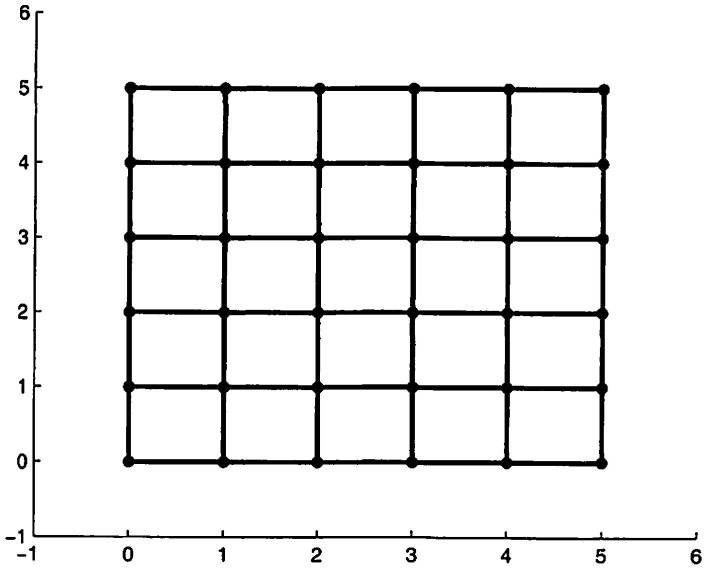

Fig. 3. The covering graph $Y(36)$. For clarity, we omit the edges connecting the perimeter vertices.

X as

$$\pi(m,n) = (0,0)$$
$$\pi(m+1,n) = (1,0)$$
$$\pi(m,n+1) = (0,1)$$
$$\pi(m+1,n+1) = (1,1)$$

where m and n are two even integers between 0 and $2N-2$. This makes Y a covering of X.

Applying our block diagonalization method to Y, we have $\mathrm{Gal}(Y/X) \approx \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}$. For $m = 0, 1, \ldots, (N-1)$ and $n = 0, 1, \ldots, (N-1)$, the adjacency matrix of Y decomposes into the blocks

$M_{Nm+n} =$

$$\begin{pmatrix} 0 & 1+e^{-2\pi in/N} & 1+e^{-2\pi im/N} & 0 \\ 1+e^{2\pi in/N} & 0 & 0 & 1+e^{-2\pi im/N} \\ 1+e^{2\pi im/N} & 0 & 0 & 1+e^{-2\pi in/N} \\ 0 & 1+e^{2\pi im/N} & 1+e^{2\pi in/N} & 0 \end{pmatrix}.$$

By direct computations, M_{Nm+n} has eigenvalues of the form

$$2\left(\cos\left(\tfrac{m\pi}{N}\right)+\cos\left(\tfrac{n\pi}{N}\right)\right),\ 2\left(\cos\left(\tfrac{m\pi}{N}\right)-\cos\left(\tfrac{n\pi}{N}\right)\right),$$
$$2\left(-\cos\left(\tfrac{m\pi}{N}\right)+\cos\left(\tfrac{n\pi}{N}\right)\right),\ 2\left(-\cos\left(\tfrac{m\pi}{N}\right)-\cos\left(\tfrac{n\pi}{N}\right)\right).\blacksquare$$

We present the eigenvalues of Y in the following manner.

Let $\left\{e_1^{(i)}, e_2^{(i)}, e_3^{(i)}, e_4^{(i)}\right\}$ be the four eigenvalues of the block M_i. For each $i = 0, 1, \ldots, N^2 - 1$, plot the points $\left\{\left(\frac{i}{N^2}, e_1^{(i)}\right), \left(\frac{i}{N^2}, e_2^{(i)}\right), \left(\frac{i}{N^2}, e_3^{(i)}\right), \left(\frac{i}{N^2}, e_4^{(i)}\right)\right\}$. We refer to this plot as the eigenvalue diagram of Y.

Figure 4 is the eigenvalue diagram for $Y(10000)$.

The following two theorems analyze the eigenvalue diagram of Y.

Theorem 5 *For $0 \le x \le 1$, define $f(x) = 2(|\cos(\pi x)| + 1)$. Then, the eigenvalue diagram of $G(4N^2)$ lies between the graphs of $f(x)$ and $-f(x)$.*

Proof. This theorem follows from the indexing of the blocks of the adjacency matrix. The eigenvalues graphed at $x = \frac{Nm+n}{N^2}$ given above are bounded by $2(|\cos(\pi m/N)|+1)$ as $\cos(n\pi/N) \le$

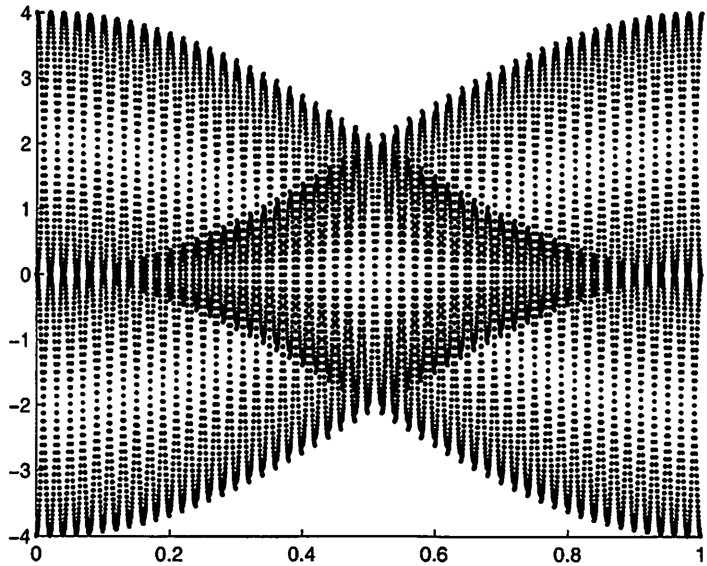

Fig. 4. The eigenvalue diagram of $Y(10000)$.

1. The outer shell of points is given by $2(|\cos(\pi m/N)| + 1)$. Thus, we have an upper bound of $2(|\cos(\pi x)| + 1)$ for this shell. Similarly, the lower bound for the shell is $-2(|\cos(\pi x)| + 1)$. ■

We analyze the eigenvalues of our graph that result from the first eigenvalue equation

$$2\cos\left(\frac{\pi m}{N}\right) + 2\cos\left(\frac{\pi n}{N}\right)$$

for $m = 0, 1, \cdots, N-1$ and $n = 0, 1, \cdots, N-1$. In particular, we determine a formula for the proportion of these eigenvalues less than or equal to $e \in (0, 4)$.

Fix N large. Let $0 < e < 4$ and define

$$L = \left[\frac{N}{\pi}\arccos\left(\frac{e}{2} - 1\right)\right]$$

where $[x]$ is the largest integer less than or equal to x.

Lemma 6 *Fix* $n \in \{L+1, L+2, \cdots, N-1\}$. *Then,*

$$2\cos\left(\frac{\pi m}{N}\right) + 2\cos\left(\frac{\pi n}{N}\right) \leq e$$

for all $m = 0, 1, \ldots, N-1$.

Proof. We begin with

$$2\cos\left(\frac{\pi m}{N}\right) \leq e - 2\cos\left(\frac{\pi n}{N}\right)$$

and determine the smallest n can be such that

$$2 \leq e - 2\cos\left(\frac{\pi n}{N}\right).$$

Solving for n, we have

$$n \geq \frac{N}{\pi} \arccos\left(\frac{e}{2} - 1\right).$$

Since n is an integer, let

$$n \geq \left[\frac{N}{\pi} \arccos\left(\frac{e}{2} - 1\right)\right] + 1.$$

Then,

$$2\cos\left(\frac{\pi m}{N}\right) + 2\cos\left(\frac{\pi n}{N}\right) \leq e$$

for all $m = 0, 1, \cdots, N-1$. ■

Lemma 7 *Fix* $n = 0, 1, \cdots, L$. *Then, the proportion of* m*'s such that*

$$2\cos\left(\frac{\pi m}{N}\right) + 2\cos\left(\frac{\pi n}{N}\right) \leq e$$

is

$$1 - \frac{1}{N}\left[\frac{N}{\pi} \arccos\left(\frac{e}{2} - \cos\left(\frac{\pi n}{N}\right)\right)\right].$$

Proof. Fix $n = 0, 1, \cdots, L$. Then,

$$2\cos\left(\frac{\pi m}{N}\right) + 2\cos\left(\frac{\pi n}{N}\right) \le e$$

provided

$$\frac{1}{\pi}\arccos\left(\frac{e}{2} - \cos\left(\frac{\pi n}{N}\right)\right) \le \frac{m}{N}.$$

Thus,

$$\frac{1}{N}\left[\frac{N}{\pi}\arccos\left(\frac{e}{2} - \cos\left(\frac{\pi n}{N}\right)\right)\right] \le \frac{m}{N}$$

since m is an integer. Since $\frac{m}{N} < 1$, the proportion of m's that make

$$2\cos\left(\frac{\pi m}{N}\right) + 2\cos\left(\frac{\pi n}{N}\right) \le e$$

true is

$$1 - \frac{1}{N}\left[\frac{N}{\pi}\arccos\left(\frac{e}{2} - \cos\left(\frac{\pi n}{N}\right)\right)\right]. \blacksquare$$

Theorem 8 *For $m = 0, 1, \cdots, N-1$, $n = 0, 1, \cdots, N-1$, the proportion of (m, n) pairs satisfying*

$$2\cos\left(\frac{\pi m}{N}\right) + 2\cos\left(\frac{\pi n}{N}\right) \le e$$

is

$$1 - \frac{1}{N^2}\sum_{n=0}^{L}\left[\frac{N}{\pi}\arccos\left(\frac{e}{2} - \cos\left(\frac{\pi n}{N}\right)\right)\right].$$

Proof. We know that for all $n = L+1, \cdots, N-1$ and $m = 0, \cdots, N-1$,

$$2\cos\left(\frac{\pi m}{N}\right) + 2\cos\left(\frac{\pi n}{N}\right) \le e.$$

Thus, there are $\alpha = N(N-1-L)$ (m, n) pairs satisfying this inequality.

For each fixed $n = 0, 1, \cdots, L$, we know that the proportion of m's that satisfy the inequality is

$$1 - \frac{1}{N}\left[\frac{N}{\pi}\arccos\left(\frac{e}{2} - \cos\left(\frac{\pi n}{N}\right)\right)\right].$$

Set

$$\beta = \sum_{n=0}^{L} 1 - \frac{1}{N}\left[\frac{N}{\pi}\arccos\left(\frac{e}{2} - \cos\left(\frac{\pi n}{N}\right)\right)\right].$$

Thus, for $m = 0, 1, \cdots, N-1$, $n = 0, 1, \cdots, N-1$, the proportion of (m, n) pairs satisfying

$$2\cos\left(\frac{\pi m}{N}\right) + 2\cos\left(\frac{\pi n}{N}\right) \leq e$$

is

$$\frac{\alpha}{N^2} + \frac{\beta}{N} = 1 - \frac{1}{N^2}\sum_{n=0}^{L}\left[\frac{N}{\pi}\arccos\left(\frac{e}{2} - \cos\left(\frac{\pi n}{N}\right)\right)\right]. \blacksquare$$

Our theorem has a closed form if we pick N carefully enough to eliminate [ ]. Choose N to be divisible by 3 and $e = 1$. Then, $L = 2N/3$. Thus, the proportion of (m, n) pairs such that

$$2\cos\left(\frac{\pi m}{N}\right) + 2\cos\left(\frac{\pi n}{N}\right) \leq 1$$

is

$$1 - \frac{1}{18N\pi}\left(3\sqrt{3} + 6\pi + 4\pi N + 9\cot\left(\frac{\pi}{2N}\right)\right) + O(1).$$

The following is data to compare our theorem with empirical data. Define "Empirical Data" to be the proportion of (m, n)'s such that

$$2\cos\left(\frac{\pi m}{N}\right) + 2\cos\left(\frac{\pi n}{N}\right) \leq 1.$$

Define "Theory" to be the value

$$1 - \frac{1}{18N\pi}\left(3\sqrt{3} + 6\pi + 4\pi N + 9\cot\left(\frac{\pi}{2N}\right)\right).$$

N	Empirical Data	Theory
33	0.6730	0.6636
66	0.6818	0.6700
99	0.6851	0.6721
132	0.6868	0.6732
165	0.6877	0.6738

We look at an asymptotic approximation to the proportion of eigenvalues in a given interval $[\alpha, \beta] \subseteq [0, 4]$. Symmetry allows us to give an asymptotic approximation for any interval contained in $[-4, 4]$. A first approximation is to consider the area given in the band relative to the total area in the region bounded by $0 \leq y \leq 2(|cos(\pi x)| + 1)$. This may seem reasonable but experimental errors were inconsistent and did not account for the fact that the eigenvalues are distributed along cosine curves. We obtain a more reasonable estimate by considering the arc lengths of the cosine curves as the asymptotic distribution of the eigenvalues. This leads to elliptic integrals that can only be evaluated numerically. However, the asymptotic values appear to agree with the above exact calculations. To approximate the asymptotic proportion of eigenvalues in a given band, we compute the arclengths of the curves $2(\cos(\pi x_1) + \cos(\pi x_2))$ that lie in the band $[\alpha, \beta]$. If we fix x_2, the arclength of the curve is given by

$$2 \int_{a_1}^{b_1} \sqrt{1 + 4\pi^2 \sin^2(\pi x_1)}\, dx_1$$

where

$$a_1 = \frac{1}{\pi} \arccos\left(\frac{\beta}{2} - \cos(\pi x_2)\right), \; b_1 = \frac{1}{\pi} \arccos\left(\frac{\alpha}{2} - \cos(\pi x_2)\right)$$

and the two accounts for the symmetry in the eigenvalues. This is an elliptic integral of the second kind. We then integrate with respect to x_2 to get the asymptotic for the eigenvalues in $[\alpha, \beta]$

as

$$\int_{a_2}^{b_2} \int_{a_1}^{b_1} \sqrt{1 + 4\pi^2 \sin^2(\pi x_1)}\, dx_1 dx_2$$

where

$$a_2 = \frac{1}{\pi} \arccos\left(\frac{\beta}{2} - 1\right), \quad b_2 = \frac{1}{\pi} \arccos\left(\frac{\alpha}{2} - 1\right).$$

The corresponding asymptotic approximation for the percentage of eigenvalues in the region $[0, 1]$ corresponding to the above data is approximately 0.65. This value is obtained by using Simpson's rule on the integral with $n = 1000$ steps.

We construct our next covering graph. Fix N to be a positive integer. We construct the graph $Z\,(4N^2)$ with $4N^2$ vertices. Place vertices at (x, y) where $x = 0, 1, \ldots, (2N - 1)$ and $y = 0, 1, \ldots, (2N - 1)$. Let m and n be two even integers between 0 and $2N-2$. Consider the four vertices (m, n), $(m+1, n)$, $(m, n+1)$, and $(m+1, n+1)$. Draw an edge between $(m+1, n)$ and (m, n), between $(m + 1, n)$ and $(m, n + 1)$, and between $(m + 1, n)$ and $(m + 1, n + 1)$. Next, draw an edge between between $(m+1, n+1)$ and $(m+2, n+1)(\text{mod } 2N)$. Finally, draw an edge between (m, n) and $(m, n - 1)(\text{mod } 2N)$.

Figure 5 is the graph Z for $N = 4$. We omit the edges connecting the perimeter vertices. As N increases to ∞, we have a tiling of the first quadrant of the xy-plane.

Theorem 9 *The eigenvalues of Z are contained in the interval* $[-2.303, 3]$.

Proof. We begin with a definition for a base graph X of Z. Place vertices at $(0, 0)$, $(1, 0)$, $(0, 1)$, and $(1, 1)$. Insert an edge between $(0, 0)$ and $(1, 0)$, between $(1, 0)$ and $(1, 1)$, between $(1, 1)$ and $(0, 1)$, between $(0, 1)$ and $(0, 0)$, and between $(1, 0)$ and $(0, 1)$.

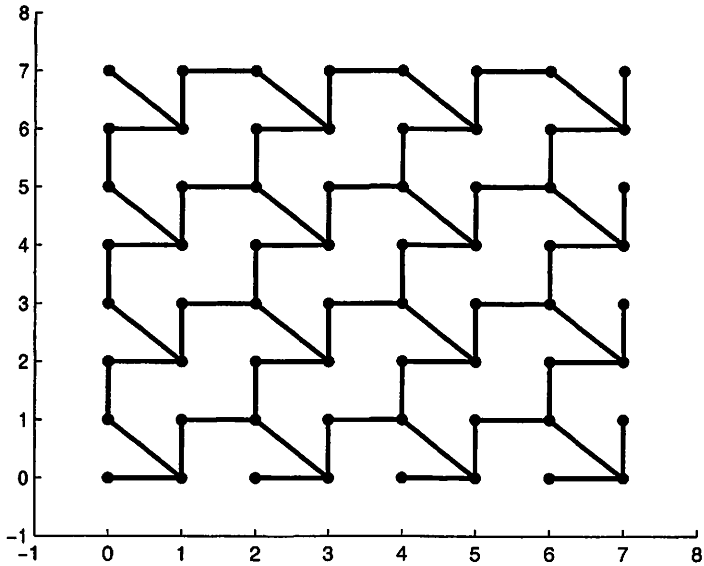

Fig. 5. The covering graph $Z(64)$. For clarity, we omit the edges connecting the perimeter vertices.

Define the covering map $\pi : Z \longrightarrow X$ as

$$\pi(m,n) = (0,0)$$
$$\pi(m+1,n) = (1,0)$$
$$\pi(m,n+1) = (0,1)$$
$$\pi(m+1,n+1) = (1,1)$$

where m and n are two even integers between 0 and $2N-2$. This makes Z a covering of X.

Applying our block diagonalization method, the adjacency matrix of Z decomposes into the blocks

$$M_{Nm+n} = \begin{pmatrix} 0 & e^{-2\pi im/N} & e^{-2\pi in/N} & 1 \\ e^{2\pi im/N} & 0 & 1 & 1 \\ e^{2\pi in/N} & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

where $m = 0, 1, \ldots, (N-1)$ and $n = 0, 1, \ldots, (N-1)$.

We obtain our eigenvalue bounds in the following manner. Consider the matrix

$$L = \begin{pmatrix} 0 & e^{-2\pi ix} & e^{-2\pi iy} & 1 \\ e^{2\pi ix} & 0 & 1 & 1 \\ e^{2\pi iy} & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

where $0 < x < 1$ and $0 < y < 1$. We show that the eigenvalues of L are contained in $[-2.303, 3]$. Using Gershgorin's circle theorem [3] on L, we know that the eigenvalues of L are in the interval $[-3, 3]$. We show that -2.303 is a tighter lower bound on the eigenvalues of L. Define $M = (2.303)I + L$ where I is the identity matrix. Our lower bound claim is equivalent to showing that the Hermitian matrix M is positive semidefinite. Thus, we show that (1) M has non-negative diagonal entries, (2) M has non-negative determinant, and (3) The determinants of the four 3×3 principal minors of M are all non-negative. Obviously, (1) is true. For (2), the determinant of M is the formula

$$f(x,y) =$$

$$\alpha + 2.606\cos(2\pi(x-y)) + 2.606\cos(2\pi x) + 2.606\cos(2\pi y)$$

where $\alpha = 2.303\,(2.303^3 - 4.909) - 3\,(2.303^2 - 1)$. Using calculus, we find that $f(x,y)$ has a lower bound of 0.01. Thus, the determinant of M is non-negative. For (3), the principal minors of M are

$$\begin{pmatrix} 2.303 & 1 & 1 \\ 1 & 2.303 & 1 \\ 1 & 1 & 2.303 \end{pmatrix}, \begin{pmatrix} 2.303 & e^{-2\pi iy} & 1 \\ e^{2\pi iy} & 2.303 & 1 \\ 1 & 1 & 2.303 \end{pmatrix},$$

$$\begin{pmatrix} 2.303 & e^{-2\pi ix} & 1 \\ e^{-2\pi ix} & 2.303 & 1 \\ 1 & 1 & 2.303 \end{pmatrix}, \begin{pmatrix} 2.303 & e^{-2\pi ix} & e^{-2\pi iy} \\ e^{2\pi ix} & 2.303 & 1 \\ e^{2\pi iy} & 1 & 2.303 \end{pmatrix}.$$

The determinants of these matrices are $2.303^3 - 3(2.303) + 2$, $2.303\,(2.303^2 - 3) + 2\cos(2\pi y)$, $2.303\,(2.303^2 - 3) + 2\cos(2\pi x)$, and $2.303\,(2.303^2 - 3) + 2\cos(2\pi(x - y))$, respectively. It is clear that they are all non-negative for any choice of x and y. Thus, M is positive semidefinite which means that the smallest eigenvalue of L is greater than or equal to -2.303. ∎

Figure 6 is the eigenvalue diagram for $Z(10000)$. Both 3 and -2.303 are tight bounds on the eigenvalues of $Z(10000)$.

5 Future Work

Our work in this paper has given us two ideas for future projects.

As one application of our block diagonalization method, we consider the Ihara zeta functions of our graphs. The Ihara zeta function associated with a finite connected graph χ is defined by Ihara [4] as

$$Z_\chi(u) = \prod_{[C]} (1 - u^{\nu(C)})^{-1}$$

where $\nu(C)$ is the length of the cycle C and the product is taken over equivalence classes of primitive, closed, backtrackless, tailless cycles. The variable u is taken sufficiently small for convergence.

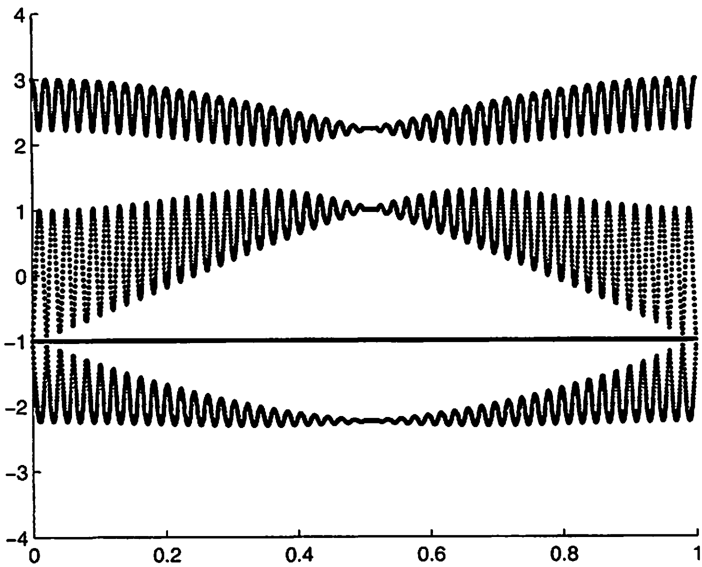

Fig. 6. The eigenvalue diagram for $Z(10000)$.

In order to calculate this zeta function efficiently we use a theorem of [4]. Let χ be a connected $(q+1)$-regular graph with v vertices, adjacency matrix A, and fundamental group rank r, $r-1=\frac{v(q-1)}{2}$. Then, the Ihara zeta function defined above is the reciprocal of a polynomial. More precisely, we have

$$Z_{\chi}^{-1}(u)=(1-u^2)^{r-1}det(I-Au+qu^2I).$$

Since our method allows us to block diagonalize A, the determinant portion of the zeta function will factor into the corresponding determinants for each block. (See Stark and Terras [7] and Stark and Terras [8] for much work done along these lines.) Following the notation used in Section 2 of this paper, assume Y is a $q+1$ regular cover of X with n distinct representations of $Gal(Y/X)$, we have

$$\begin{aligned} Z_Y^{-1}(u) &= (1-u^2)^{r-1} \prod_{i=1}^{n} det(I - M_i u + qu^2 I) \\ &= (1-u^2)^{r-1} \prod_{\lambda \in spec(Y)} (1 - \lambda u + qu^2) \end{aligned}$$

where $spec(Y)$ is used to denote the eigenvalues of Y. In the case of $Y(4N^2)$ defined in Section 4, where $q = 3, v = 4N^2$, we have

$$\begin{aligned} & Z_{Y(4N^2)}^{-1}(u) \\ &= (1-u^2)^{4N^2} \prod \left(1 - 2\left(\pm \cos\left(\tfrac{m\pi}{N}\right) \pm \cos\left(\tfrac{n\pi}{N}\right)\right) u + 3u^2\right) \end{aligned}$$

where the product is taken over m and n from 0 to $N-1$ and $\pm$ means to take all possible sign combinations. We would like to know if there are any u's such that $Z_{Y(4N^2)}^{-1}(u)$ converges for $N \longrightarrow \infty$. In addition, we are interested in the behavior of the zeros of this function for N increasing to infinity.

Recall from Section 3, the graph $X\left(\mathrm{Aff}\left(\mathbb{Z}/p\mathbb{Z}\right), S_p\right)$ where g is a primitive root of p,

$$\begin{aligned} & T = \\ & \{(g^m \;\; m) \mid m = 1, 2, \ldots, (p-2)\} \cup \{(1 \;\; 1), (g \;\; 0)\} \pmod{p}, \end{aligned}$$

and

$$S_p = T \cup T^{-1} \pmod{p}.$$

Computationally, we believe this graph to be a Ramanujan graph ([5]). If this is true, we would like to extend our definition for X as far as possible such that this extension still satisfies the Ramanujan bound. We believe this search should extend into the affine group over the finite field $\mathbb{F}_{p^n}$ instead of the affine group over $\mathbb{Z}/p^n\mathbb{Z}$. In other words, we believe that the graph

$$Y = Y\left(\mathrm{Aff}\left(\mathbb{F}_{p^n}\right), S\right)$$

defined such that Y reduces to X above when $n = 1$ will yield a larger class of Ramanujan graphs. There are several graph examples that support our belief. One example was studied by Terras

[9]. From that paper, an affine graph over $\mathbb{F}_{p^n}$ was eventually shown to be a Ramanujan graph. In particular, it was found that $\frac{\lambda}{2\sqrt{|S|-1}} \longrightarrow 0.9$ for any fixed n as $p \longrightarrow \infty$. (A later paper by Angel, Shook, Terras, and Trimble [1] proved that for the same affine graph over $\mathbb{Z}/p^n\mathbb{Z}$, $\frac{\lambda}{2\sqrt{|S|-1}} \longrightarrow \infty$.) We note that for both the graph $Y(\text{Aff}(\mathbb{F}_{p^n}), S)$ above and the graph of [9], (1) The vertex sets are matrix groups over the finite field, (2) The order of the generating sets increase in value with the order of the vertex sets.

In comparison to the above graphs, consider the graphs of Lubotzky, Phillips, and Sarnak [6]. The graphs are defined as

$$X = X(\text{PGL}(\mathbb{Z}/p\mathbb{Z}), S)$$

where the order of S is fixed and independent of the order of $\text{PGL}(\mathbb{Z}/p\mathbb{Z})$. These graphs were shown to be Ramanujan graphs. In [2], we considered the following experiment. We looked at the graph

$$X = X(\text{PGL}(\mathbb{Z}/5\mathbb{Z}), S)$$

where $|S| = 13$. As expected, this graph was a Ramanujan graph. On the other hand, the graph

$$Y = Y\left(\text{PGL}\left(\mathbb{Z}/5^2\mathbb{Z}\right), S\right)$$

where $|S| = 13$ was also a Ramanujan graph. Figure 7 is the eigenvalue histogram for Y. We note that the Ramanujan bounds are $\pm 2\sqrt{13-1} \approx \pm 6.928$. We note that for these graphs, (1) The vertex sets are matrix groups over the finite ring, (2) The order of the generating sets is independent of the order of the vertex sets.

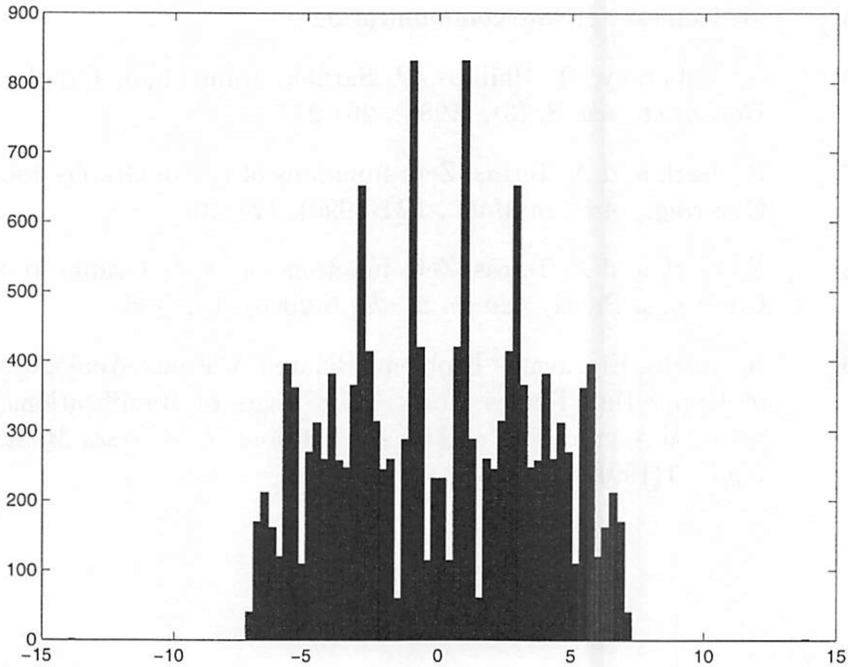

Fig. 7. The eigenvalue histogram for Y.

References

[1] J. Angel, B. Shook, A. Terras, C. Trimble, Graph Spectra for Finite Upper Half Planes over Rings, *Linear Algebra Appl.*, 226/228 (1995), 423–457.

[2] J. Bell and M. Minei, Spectral Analysis of the Affine Graph over the Finite Ring, *Linear Algebra Appl.*, 414 (2006), no. 1, 244–265.

[3] S. Grossman, Elementary Linear Algebra, Wadsworth Publishing Company (1980).

[4] Y. Ihara, On Discrete Subgroups of the Two by Two Projective Linear Group over p-adic Fields, *J. Math. Soc. Japan*, **18**(1966), 219-235.

[5] M. Komesu, private communications.

[6] A. Lubotzky, R. Phillips, P. Sarnak, Ramanujan Graphs, *Combinatorica*, **8**, (3), (1988), 261-277.

[7] H. Stark and A. Terras, Zeta functions of Finite Graphs and Coverings, *Adv. in Math.*, **121**(1996), 124–165.

[8] H. Stark and A. Terras, Zeta functions of Finite Graphs and Coverings, Pt. II, *Adv. in Math.*, **5**(2000), 132–195.

[9] A. Terras, Eigenvalue Problems Related to Finite Analogues of Upper Half Planes, Forty More Years of Ramifications: Spectral Asymptotics and its Applications, *Discourses Math. Appl.*, **1**(1991), 237–263.