Chromatic Uniqueness Of Certain Complete t-partite Graphs

G.C. Lau[a,b,c] and Y.H. Peng[b,c,1]

[a] *Faculty of I. T. and Quantitative Science*
Universiti Teknologi MARA (Segamat Campus)
85010 Johor, Malaysia

[b] *Department of Mathematics, and*
[c] *Institute for Mathematical Research*
Universiti Putra Malaysia
43400 UPM Serdang, Malaysia

ABSTRACT

Let $P(G,\lambda)$ be the chromatic polynomial of a graph G. A graph G is chromatically unique if for any graph H, $P(H,\lambda) = P(G,\lambda)$ implies H is isomorphic to G. In his Ph.D. thesis, Zhao [Theorems 5.4.2 and 5.4.3] proved that for any positive integer $t \geq 3$, the complete t-partite graphs $K(p-k,p,p,\ldots,p)$ with $p \geq k+2 \geq 4$ and $K(p-k,p-1,p,\ldots,p)$ with $p \geq 2k \geq 4$ are chromatically-unique. In this paper, by expanding the technique employed by Zhao, we prove that the complete t-partite graph $K(p-k,\underbrace{p-1,\ldots,p-1}_{d+1},\underbrace{p,\ldots,p}_{t-d-2})$ is chromatically-unique for integers $p \geq k+2 \geq 4$ and $t \geq d+3 \geq 3$.

MSC: 05C15;05C60

Keywords:
Complete t-partite graph;Chromatic polynomial; Chromatic Uniqueness

1. Introduction

All graphs considered in this paper are finite, undirected, simple and loopless. For a graph G, we denote by $P(G;\lambda)$ (or $P(G)$), the chromatic polynomial of G. Two graphs G and H are said to be *chromatically equivalent*, or *χ-equivalent*, denoted $G \sim H$ if $P(G) = P(H)$. It is clear that the relation " $\sim$ " is an equivalence relation on the family of graphs. We denote by $[G]$ the equivalence class determined by G under " $\sim$ ". A graph G is said

[1] Corresponding author. E-mail: yhpeng@fsas.upm.edu.my

to be *chromatically unique*, or χ*-unique*, if $[G] = \{G\}$, i.e., $H \sim G$ implies that $H \cong G$. Many families of χ-unique graphs are known (see [7, 8]).

For each integer $t \geq 2$, a complete t-partite graph, denoted by $K(p_1, p_2, \ldots, p_t)$, is a graph whose vertex set $V(G)$ can be partitioned into t disjoint non-empty subsets V_i with $|V_i| = p_i$ for $i = 1, 2, \ldots, t$ such that every vertex in V_i is adjacent to every vertex in V_j for $i \neq j$.

In [4], Chia et al. showed that the complete tripartite graphs $K(p, p, p+k)$ for $p \geq 2$, $1 \leq k \leq 3$, and $K(p-k, p, p)$ for $p \geq k+2$, $0 \leq k \leq 3$ are χ-unique. Zou [12] also gives partial solutions to certain complete t-partite graphs for $t \geq 3$. In [4, 7], Chia et al., and Koh and Teo conjectured that for any integers p and k with $p \geq k+2 \geq 4$, $K(p-k, p, p)$ is χ-unique. Zhao [11] proved that for any positive integer $t \geq 3$, the complete t-partite graphs $K(p-k, p, p, \ldots, p)$ with $p \geq k+2 \geq 4$ and $K(p-k, p-1, p, \ldots, p)$ with $p \geq 2k \geq 4$ are χ-unique (also see Chia and Ho [5], and Liu et al. [9] for $t = 3$). In this paper, by expanding the technique employed in [11], we prove that the complete t-partite graph $G = K(p-k, \underbrace{p-1, \ldots, p-1}_{d+1}, \underbrace{p, \ldots, p}_{t-d-2})$ is χ-unique for integers $p \geq k+2 \geq 4$ and $t \geq d+3 \geq 3$.

2. Preliminary results and notations

Let $\chi(G), V(G)$, $E(G)$ and $t(G)$ be the chromatic number, the vertex set, the edge set and the number of triangles of G, respectively. Denote by $K(G)$ (respectively $Q(G)$) the number of subgraphs K_4 (respectively induced subgraphs C_4) in a graph G. By $\overline{G}$, we denote the complement of G. Then we let $O_p = \overline{K_p}$, where K_p denotes the complete graph with p vertices. Let S be a set of edges of G with $|S| = s$, and denote by $G - S$ the graph obtained by deleting all edges in S from G.

For a graph G and a positive integer k, a partition $\{A_1, A_2, \ldots, A_k\}$ of $V(G)$ is called a *k-independent partition* in G if each A_i is a non-empty independent set of G. Let $\alpha(G, k)$ denote the number of k-independent partitions in G. If G is of order p, then $P(G, \lambda) = \sum_{k=1}^{p} \alpha(G, k)(\lambda)_k$ where $(\lambda)_k = \lambda(\lambda-1)\cdots(\lambda-k+1)$ (see [10]). Therefore, $\alpha(G, k) = \alpha(H, k)$ for each $k = 1, 2, \ldots$, if $G \sim H$.

Let G be a graph with p vertices. Then the polynomial $\sigma(G, x) = \sum_{k=1}^{p} \alpha(G, k)x^k$ is called the *σ-polynomial* of G (see [2]). Clearly, $P(G, \lambda) = P(H, \lambda)$ if and only if $\sigma(G, x) = \sigma(H, x)$.

For disjoint graphs G and H, $G \cup H$ denotes the disjoint union of G and H; $G + H$ denotes the graph whose vertex-set is $V(G) \cup V(H)$ and whose edge-set is $\{xy|x \in V(G) \text{ and } y \in V(H)\} \cup E(G) \cup E(H)$. Throughout this paper, all the t-partite graphs G under consideration are 2-connected with $\chi(G) = t$. For terms used but not defined here we refer to [1].

For convenience, simply denote $\sigma(G, x)$ by $\sigma(G)$, and $G \cong H$ by $G = H$.

Lemma 1. (Koh and Teo [7]) *If $H \sim G$, then both G and H have the same number of vertices, edges and triangles with $\chi(G) = \chi(H)$. Moreover, $\alpha(G, k) = \alpha(H, k)$ for each $k = 1, 2, \ldots,$ and*

$$-Q(G) + 2K(G) = -Q(H) + 2K(H)$$

Note that if $\chi(G) = 3 = \chi(H)$, then $G \sim H$ implies that $Q(G) = Q(H)$.

Lemma 2. (Brenti [2]) *Let G and H be two disjoint graphs. Then*

$$\sigma(G + H, x) = \sigma(G, x)\sigma(H, x).$$

In particular,

$$\sigma(K(n_1, n_2, \ldots, n_t), x) = \prod_{i=1}^{t} \sigma(O_{n_i}, x).$$

Lemma 3. (Koh and Teo [7]) *The graph $K(m, n)$ is χ-unique if $n \geq m \geq 2$ and $K(m, n) - e$ is χ-unique if $n \geq m \geq 3$.*

Let $H = K(x_1, x_2, x_3, \ldots, , x_t)$ and $H' = K(x_1, x_2, \ldots, x_i + 1, \ldots, x_j - 1, \ldots, x_t)$. If $i < j$ and $x_j - x_i \geq 2$, then H' is called an *improvement* of H.

Lemma 4. *Suppose $H' = K(x_1, x_2, \ldots, x_i + 1, \ldots, x_j - 1, \ldots, x_t)$ is an improvement of $H = K(x_1, x_2, x_3, \ldots, , x_t)$, then $|E(H')| > |E(H)|$.*

Proof. Note that $|E(H)| = \sum_{1 \leq i < j \leq t} x_i x_j$ and $|E(H')| = \left(\sum_{1 \leq m < n \leq t} x_m x_n \right) + x_j - x_i - 1$. Since $x_j - x_i \geq 2$, we have immediately $|E(H')| = |E(H)| + x_j - x_i - 1 > |E(H)|$. □

Lemma 5. (Zhao [11, p.114]) *Let $G = K(p_1, p_2, \ldots, p_t)$ with $2 \leq p_1 \leq p_2 \leq \cdots \leq p_t$. If $H \sim G$, then*

(i) $H \in [G] \subset \{K(x_1, x_2, \ldots, x_t) - S | 1 \leq x_1 \leq x_2 \leq \cdots \leq x_t \leq p_t, \sum_{i=1}^{t} x_i = \sum_{i=1}^{t} p_i, S \subset E(K(x_1, x_2, \ldots, x_t))\}$;

(ii) *there exists an integer* $b \geq 2$ *such that* $x_1 \leq x_2 \leq \cdots \leq x_b \leq p_b - 1$ *and* K_{p_i} *is a component of* $\overline{H}$ *for any* $i \geq b+1$;

(iii) *if* $x_i = p_i$ *for any* $i \geq 3$, *then* $H = G$.

From Lemmas 4 and 5, we have the following corollary.

Corollary 1. (Chao and Novacky Jr. [3]) *The graph* $G = K(p_1, p_2, \ldots, p_t)$ *is* χ*-unique if* $|p_i - p_j| \leq 1$.

Lemma 6. (Dong [6]) *Suppose that* $G = (A, B; E)$ *is a bipartite graph with* $|A| \geq 2$ *and* $|B| \geq 2$. *Then*

$$Q(G) \leq |E|(|E| - |A| - |B| + 1)/4.$$

If $d_G(v) \geq 2$ *for every* $v \in V(G)$, *then the equality holds if and only if* G *is a complete bipartite graph.*

Denote by $\binom{n}{r}$ the number of r combinations chosen from a set of n distinct objects, and $\binom{n}{r} = 0$ if $n < r$.

Definition. *Let* $t \geq d + 3 \geq 3$ *be integers. If* $X = (a_1, a_2, \ldots, a_t)$ *is a sequence of integers* $a_1 \geq a_2 \geq \cdots \geq a_t \geq 0$, *then let* $g(X) = g_1(X) + g_2(X) + g_3(X)$ *where*

$$\begin{aligned}
g_1(X) &= (a_1 + a_2 - d)\left[\sum_{3 \leq i < j \leq d+2} a_i a_j - (d-1)\sum_{i=3}^{d+2} a_i + \binom{d}{2}\right] + \\
&\quad 2\sum_{3 \leq i < j < l \leq d+2} a_i a_j a_l - (d-2)\sum_{3 \leq i < j \leq d+2} a_i a_j + \binom{d}{3}; \\
g_2(X) &= \left[(a_1 + a_2 - d)\left(\sum_{i=3}^{d+2} a_i - d\right) + 2\sum_{3 \leq i < j \leq d+2} a_i a_j - \right. \\
&\quad \left. (d-1)\sum_{i=3}^{d+2} a_i\right]\sum_{i=d+3}^{t} a_i \quad \text{and} \\
g_3(X) &= \left[a_1 + a_2 + 2\sum_{i=3}^{d+2} a_i - 2d\right]\sum_{d+3 \leq i < j \leq t} a_i a_j + 2\sum_{d+3 \leq i < j < l \leq t} a_i a_j a_l.
\end{aligned}$$

The following lemma characterizes non-negativity of $g(X)$ when $a_{d+2} \geq 1$.

Lemma 7. *Let* d, t, a_i $(1 \leq i \leq t)$, *and* $g(X)$ *be as given in the definition above. If* $a_{d+2} \geq 1$, *then* $g(X) \geq 0$, *and equality holds if and only if*

(a) $a_4 = a_5 = \cdots = a_t = 0$ *for* $d = 0$;

(b) $a_4 = a_5 = \cdots = a_t = 0$, *or* $a_3 = a_4 = 1$ *and* $a_5 = a_6 = \cdots = a_t = 0$ *for* $d = 1$;

(c) $a_4 = a_5 = \cdots = a_{d+2} = 1$ *and* $a_{d+3} = \cdots = a_t = 0$, *or* $a_3 = a_4 = \cdots = a_{d+2} = 1$, $0 \le a_{d+3} \le 1$, *and* $a_{d+4} = \cdots = a_t = 0$ *for* $d \ge 2$.

Proof. **(a).** When $d = 0$, we have

$$g(X) = (a_1 + a_2) \sum_{3 \le i < j \le t} a_i a_j + 2 \sum_{3 \le i < j < l \le t} a_i a_j a_l \ge 0$$

and equality holds if and only if $a_4 = \cdots = a_t = 0$.

(b). When $d = 1$, we have $a_3 \ge 1$ and

$$g(X) = (a_1 + a_2 - 1)(a_3 - 1) \sum_{i=4}^{t} a_i + (a_1 + a_2 + 2(a_3 - 1)) \sum_{4 \le i < j \le t} a_i a_j +$$

$$2 \sum_{4 \le i < j < l \le t} a_i a_j a_l.$$

Hence, $g(X) \ge 0$ and equality holds if and only if $a_4 = \cdots = a_t = 0$ or $a_3 = a_4 = 1$ and $a_5 = \cdots = a_t = 0$.

(c) When $d \ge 2$, we shall first prove the following two claims.

Claim 1. $g_1(X) \ge 0$ and equality holds if and only if $a_4 = \cdots = a_{d+2} = 1$. It can be verified by direct substitution that $g_1(X) = 0$ if $a_4 = \cdots = a_{d+2} = 1$. Otherwise, since $a_{d+2} \ge 1$, we may assume there exists a $2 \le k \le d$ such that $a_4 \ge \cdots \ge a_{k+2} \ge 2$ and $a_i = 1$ for $k+3 \le i \le d+2$. We now proceed by induction on d to show that $g_1(X) > 0$. Suppose $d = 2$, then $k = 2$ and

$$g_1(X) = (a_1 + a_2 - 2)[a_3 a_4 - (a_3 + a_4) + 1] > 0.$$

Hence, the above claim holds for $d = 2$. We now assume that $g_1(X) > 0$ for $2 \le d < h$ with $a_4 \ge \cdots \ge a_{k+2} \ge 2$ and $a_{k+3} = 1$ where $2 \le k \le d$. Let $2 \le d = h$ such that $a_4 \ge \cdots \ge a_{k+2} \ge 2$ and $a_{k+3} = 1$ for $2 \le k \le h$. Then we have

$$g_1(X) \quad = \quad (a_1 + a_2 - h)\left[\sum_{3 \le i < j \le k+2} a_i a_j + (h - k) \sum_{i=3}^{k+2} a_i + \binom{h-k}{2} - \right.$$

$$
\begin{aligned}
& (h-1)\left(\sum_{i=3}^{k+2} a_i + (h-k)\right) + \binom{h}{2}\Bigg] + 2\Bigg[\sum_{3\le i<j<l\le k+2} a_i a_j a_l + \\
& (h-k)\sum_{3\le i<j\le k+2} a_i a_j + \binom{h-k}{2}\sum_{i=3}^{k+2} a_i + \binom{h-k}{3}\Bigg] - \\
& (h-2)\Bigg[\sum_{3\le i<j\le k+2} a_i a_j + (h-k)\sum_{i=3}^{k+2} a_i + \binom{h-k}{2}\Bigg] + \binom{h}{3} \\
= \; & (a_1+a_2-h)\Bigg[\sum_{3\le i<j\le k+2} a_i a_j - (k-1)\sum_{i=3}^{k+2} a_i + \binom{k}{2}\Bigg] + \\
& 2\sum_{3\le i<j<l\le k+2} a_i a_j a_l - (k-2)\sum_{3\le i<j\le k+2} a_i a_j + \binom{k}{3} + \\
& (h-k)\Bigg[\sum_{3\le i<j\le k+2} a_i a_j - (k-1)\sum_{i=3}^{k+2} a_i + \binom{k}{2}\Bigg] \\
= \; & (a_1+a_2-k)\Bigg[\sum_{3\le i<j\le k+2} a_i a_j - (k-1)\sum_{i=3}^{k+2} a_i + \binom{k}{2}\Bigg] + \\
& 2\sum_{3\le i<j<l\le k+2} a_i a_j a_l - (k-2)\sum_{3\le i<j\le k+2} a_i a_j + \binom{k}{3},
\end{aligned}
$$

and by induction hypotheses, $g_1(X) > 0$. Hence, $g_1(X) \ge 0$ and equality holds if and only if $a_4 = \cdots = a_{d+2} = 1$.

Claim 2. $g_2(X) \ge 0$ and equality holds if and only if $a_3 = \cdots = a_{d+2} = 1$, or $a_{d+3} = \cdots = a_t = 0$.

It can be verified by direct substitution that $g_2(X) = 0$ if $a_3 = \cdots = a_{d+2} = 1$, or $a_{d+3} = \cdots = a_t = 0$. Otherwise, since $a_{d+2} \ge 1$, we may assume $a_{d+3} \ge 1$ and that there exists a $1 \le k \le d$ such that $a_3 \ge \cdots \ge a_{k+2} \ge 2$ and $a_{k+3} = 1$. We now proceed by induction on d to show that $g_2(X) > 0$. Suppose $d = 2$, then $1 \le k \le 2$ and

$$
g_2(X) = \Big[(a_1+a_2-2)(a_3+a_4-2) + 2a_3a_4 - a_3 - a_4\Big]\sum_{i=5}^{t} a_i .
$$

If $k = 1$, then $a_3 \ge 2$ and $a_4 = 1$. We then have

$$
g_2(X) = [(a_1+a_2-1)(a_3-1)]\sum_{i=5}^{t} a_i > 0.
$$

If $k = 2$, then $a_3 \geq a_4 \geq 2$. We then have

$$g_2(X) = [(a_1 + a_2 - 2)(a_3 + a_4 - 2) + a_3(a_4 - 1) + a_4(a_3 - 1)] \sum_{i=5}^{t} a_i > 0.$$

Hence, the above claim holds for $d = 2$. We now assume that $g_2(X) > 0$ for $2 \leq d < h$ with $a_3 \geq \cdots \geq a_{k+2} \geq 2$ and $a_{k+3} = 1$ where $1 \leq k \leq d$. Let $2 \leq d = h$ such that $a_3 \geq \cdots \geq a_{k+2} \geq 2$ and $a_{k+3} = 1$ for $1 \leq k \leq h$. Then we have

$$\begin{aligned} g_2(X) &= \Bigg[(a_1 + a_2 - h)\left(\sum_{i=3}^{k+2} a_i + (h-k) - h\right) + \\ &\quad 2\left(\sum_{3 \leq i < j \leq k+2} a_i a_j + (h-k)\sum_{i=3}^{k+2} a_i + \binom{h-k}{2}\right) - \\ &\quad (h-1)\left(\sum_{i=3}^{k+2} a_i + (h-k)\right)\Bigg] \sum_{i=h+3}^{t} a_i \\ &= \Bigg[(a_1 + a_2 - h)\left(\sum_{i=3}^{k+2} a_i - k\right) + 2\sum_{3 \leq i < j \leq k+2} a_i a_j - \\ &\quad (k-1)\sum_{i=3}^{k+2} a_i + (h-k)\left(\sum_{i=3}^{k+2} a_i - k\right)\Bigg] \sum_{i=h+3}^{t} a_i \\ &= \Bigg[(a_1 + a_2 - k)\left(\sum_{i=3}^{k+2} a_i - k\right) + 2\sum_{3 \leq i < j \leq k+2} a_i a_j - \\ &\quad (k-1)\sum_{i=3}^{k+2} a_i\Bigg] \sum_{i=h+3}^{t} a_i, \end{aligned}$$

and by induction hypotheses, $g_2(X) > 0$. Hence, $g_2(X) \geq 0$ and equality holds if and only if $a_3 = \cdots = a_{d+2} = 1$, or $a_{d+3} = \cdots = a_t = 0$.

Observe that $g_3(X) \geq 0$ and equality holds if and only if $a_{d+4} = \cdots = a_t = 0$. Therefore, combining this fact and Claims 1 and 2 above, we have proved (c). □

Let G be a graph having at least one induced subgraph C_4 and $V(G)$ can be partitioned into $V_1, V_2, \ldots, V_i, i \geq 4$, non-empty independent sets. We say an induced subgraph C_4 of G is of Type I (respectively Type II, and Type III) if the vertices of the induced C_4 are in exactly two (respectively three, and four) of the independent sets of G. Examples of a Type I, Type II and Type III induced C_4 are shown in Figure 1 below.

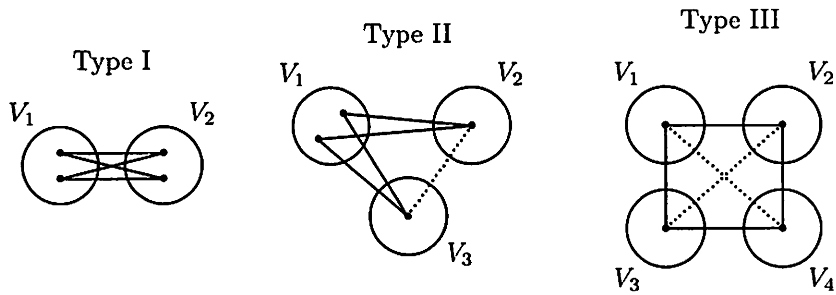

Figure 1. Type of induced C_4. Vertices joined by dotted line are not adjacent.

3. Complete t-partite graphs $K(p-k, p-1, \ldots, p-1, p, \ldots, p)$

Our main result is the following theorem.

Theorem 1. *For any integers $p \geq k+2 \geq 4$ and $t \geq d+3 \geq 3$, the complete t-partite graph $K(p-k, \underbrace{p-1, \ldots, p-1}_{d+1}, \underbrace{p, \ldots, p}_{t-d-2})$ is χ-unique.*

Proof. Let $G = K(p-k, \underbrace{p-1, \ldots, p-1}_{d+1}, \underbrace{p, \ldots, p}_{t-d-2})$ and let $H \in [G]$. Note that for $k = 0, 1$, the results can be found in [3]. From Lemma 5(i), we know that there exists a graph $F = K(x_1, x_2, \ldots, x_t)$ and $S \subset E(F)$ such that $H = F - S$ and $|S| = s$ and $1 \leq x_1 \leq x_2 \leq \ldots \leq x_t \leq p$. Clearly,

$$s = |E(F)| - |E(G)| = \sum_{1 \leq i < j \leq t} x_i x_j - (d+1)(p-k)(p-1) - (t-d-2)(p-k)p -$$

$$(d+1)(t-d-2)(p-1)p - \binom{d+1}{2}(p-1)^2 - \binom{t-d-2}{2}p^2$$

and $\sum_{i=1}^{t} x_i = pt - k - d - 1$.

By Lemma 1, $t(G) = t(H)$. Hence, we shall consider the number of triangles in G and H. Let $S = \{\epsilon_1, \epsilon_2, \ldots, \epsilon_s\} \subset E(F)$. Denote by $t(\epsilon_i)$ the number of triangles containing the edge ϵ_i in F. It is not hard to see that $t(\epsilon_i) \leq \sum_{i=3}^{t} x_i$. Then

$$t(H) \geq t(F) - s\sum_{i=3}^{t} x_i, \tag{1}$$

and the equality holds if and only if $t(\epsilon_i) = \sum_{i=3}^{t} x_i$ for all $\epsilon_i \in S$.

Let $\beta = t(F) - t(G)$. It is obvious that $t(F) = \sum_{1\leq i<j<l\leq t} x_i x_j x_l$ and

$$\begin{aligned} t(G) \;=\;& d(p-k)(p-1)^2 + (d+1)(t-d-2)(p-k)(p-1)p + \\ & \binom{d}{2}(p-k)(p-1)^2 + \binom{t-d-2}{2}(p-k)p^2 + \\ & \binom{t-d-2}{2}(d+1)(p-1)p^2 + \binom{d+1}{2}(t-d-2)(p-1)^2 p + \\ & \binom{d+1}{3}(p-1)^3 + \binom{t-d-2}{3}p^3. \end{aligned}$$

So, we have that

$$\begin{aligned} \beta \;=\;& \sum_{1\leq i<j<l\leq t} x_i x_j x_l - d(p-k)(p-1)^2 - (d+1)(t-d-2)\times \\ & (p-k)(p-1)p - \binom{d}{2}(p-k)(p-1)^2 - \binom{t-d-2}{2}(p-k)p^2 - \\ & \binom{t-d-2}{2}(d+1)(p-1)p^2 - \binom{d+1}{2}(t-d-2)(p-1)^2 p - \\ & \binom{d+1}{3}(p-1)^3 - \binom{t-d-2}{3}p^3, \end{aligned} \tag{2}$$

and

$$t(G) = t(F) - \beta. \tag{3}$$

Since $t(G) = t(H)$, from (1) and (3) it follows that

$$\beta \leq s\sum_{i=3}^{t} x_i. \tag{4}$$

Let $f(x_3, x_4, \ldots, x_t) = \beta - s\sum_{i=3}^{t} x_i$, or denoted simply by f. Since

$$\sum_{1\le i<j<l\le t} x_i x_j x_l = x_1 x_2 \sum_{i=3}^{t} x_i + (x_1+x_2)\sum_{3\le i<j\le t} x_i x_j + \sum_{3\le i<j<l\le t} x_i x_j x_l$$

and

$$\sum_{1\le i<j\le t} x_i x_j = x_1 x_2 + (x_1+x_2)\sum_{i=3}^{t} x_i + \sum_{3\le i<j\le t} x_i x_j,$$

we have by calculation, that

$$\begin{aligned} f &= (x_1+x_2)\left[\sum_{3\le i<j\le t} x_i x_j - \left(\sum_{i=3}^{t} x_i\right)^2\right] + \sum_{3\le i<j<l\le t} x_i x_j x_l - \\ &\quad \sum_{3\le i<j\le t} x_i x_j \sum_{i=3}^{t} x_i + \Bigg[(d+1)(p-k)(p-1) + \\ &\quad (t-d-2)(p-k)p + (d+1)(t-d-2)(p-1)p + \\ &\quad \binom{d+1}{2}(p-1)^2 + \binom{t-d-2}{2}p^2\Bigg]\sum_{i=3}^{t} x_i - d(p-k)(p-1)^2 - \\ &\quad (d+1)(t-d-2)(p-k)(p-1)p - \binom{d}{2}(p-k)(p-1)^2 - \\ &\quad \binom{t-d-2}{2}(p-k)p^2 - \binom{t-d-2}{2}(d+1)(p-1)p^2 - \\ &\quad \binom{d+1}{2}(t-d-2)(p-1)^2 p - \binom{d+1}{3}(p-1)^3 - \binom{t-d-2}{3}p^3 \end{aligned}$$

Substituting $x_1 + x_2 = pt - k - d - 1 - \sum_{i=3}^{t} x_i$, we have

$$\begin{aligned} f &= (pt-k-d-1-\sum_{i=3}^{t} x_i)\left[-\sum_{i=3}^{t} x_i^2 - \sum_{3\le i<j\le t} x_i x_j\right] + \\ &\quad \sum_{3\le i<j<l\le t} x_i x_j x_l - \sum_{3\le i<j\le t} x_i x_j \sum_{i=3}^{t} x_i + \Bigg[(d+1)(p-k)(p-1) + \\ &\quad (t-d-2)(p-k)p + (d+1)(t-d-2)(p-1)p + \\ &\quad \binom{d+1}{2}(p-1)^2 + \binom{t-d-2}{2}p^2\Bigg]\sum_{i=3}^{t} x_i - d(p-k)(p-1)^2 - \end{aligned}$$

$$
\begin{aligned}
& (d+1)(t-d-2)(p-k)(p-1)p - \binom{d}{2}(p-k)(p-1)^2 - \\
& \binom{t-d-2}{2}(p-k)p^2 - \binom{t-d-2}{2}(d+1)(p-1)p^2 - \\
& \binom{d+1}{2}(t-d-2)(p-1)^2 p - \binom{d+1}{3}(p-1)^3 - \binom{t-d-2}{3}p^3 \\
= \; & \sum_{i=3}^{t} x_i^3 + \sum_{3\le i<j\le t}(x_i^2 x_j + x_i x_j^2) + \sum_{3\le i<j<l\le t} x_i x_j x_l - (pt-k-d-1) \times \\
& \sum_{i=3}^{t} x_i^2 - (pt-k-d-1)\sum_{3\le i<j\le t} x_i x_j + \Big[(d+1)(p-k)(p-1) + \\
& (t-d-2)(p-k)p + (d+1)(t-d-2)(p-1)p + \\
& \binom{d+1}{2}(p-1)^2 + \binom{t-d-2}{2}p^2\Big]\sum_{i=3}^{t} x_i - d(p-k)(p-1)^2 - \\
& (d+1)(t-d-2)(p-k)(p-1)p - \binom{d}{2}(p-k)(p-1)^2 - \\
& \binom{t-d-2}{2}(p-k)p^2 - \binom{t-d-2}{2}(d+1)(p-1)p^2 - \\
& \binom{d+1}{2}(t-d-2)(p-1)^2 p - \binom{d+1}{3}(p-1)^3 - \binom{t-d-2}{3}p^3 \\
= \; & f_1 + f_2, \qquad (5)
\end{aligned}
$$

where

$$
f_1 = \sum_{i=3}^{d+2}(x_i-p+1)^2(x_i-p+k) + \sum_{i=d+3}^{t}(x_i-p)(x_i-p+1)(x_i-p+k),
$$

and

$$
\begin{aligned}
f_2 = \; & \sum_{3\le i<j\le t}(x_i^2 x_j + x_i x_j^2) + \sum_{3\le i<j<l\le t} x_i x_j x_l - (t-3)p\sum_{i=3}^{t} x_i^2 + \\
& (d-1)\sum_{i=3}^{d+2} x_i^2 + d\sum_{d+3}^{t} x_i^2 - (pt-k-d-1)\sum_{3\le i<j\le t} x_i x_j + \\
& [d(p-k)(p-1) + \binom{d+1}{2}(p-1)^2 + (t-d-2)(p-k)p + \\
& (d+1)(t-d-2)(p-1)p + \binom{t-d-2}{2}p^2]\sum_{i=3}^{t} x_i -
\end{aligned}
$$

$$\begin{aligned}&(p-1)(2p-k-1)\sum_{i=3}^{d+2} x_i - p(2p-k-1)\sum_{i=d+3}^{t} x_i - \\ &d(t-d-2)(p-k)(p-1)p - \binom{d}{2}(p-k)(p-1)^2 - \\ &\binom{t-d-2}{2}(p-k)p^2 - \binom{t-d-2}{2}(d+1)(p-1)p^2 - \\ &\binom{d+1}{2}(t-d-2)(p-1)^2 p - \binom{d+1}{3}(p-1)^3 - \binom{t-d-2}{3}p^3.\end{aligned}$$

By Lemma 5(i) and (ii), if $b < d+2$, then $x_i \le p-2$, $x_{i+1} = x_{d+2} = p-1$, $x_{d+3} = x_t = p$ for some $3 \le i \le d+1$, or else if $b = d+2$, then $x_{d+2} \le p-2$, or else if $b > d+2$, then $x_i \le p-1$, $x_{i+1} = p$ for some $d+3 \le i \le t-1$. Hence, we have $x_3 \le \cdots \le x_{d+2} \le p-1$ and $x_{d+3} \le \cdots \le x_t \le p$. Since $k \ge 2$, we claim that $x_i > p-k$ for $3 \le i \le t$. Suppose not, then $\sum_{i=1}^{t} x_i \le 3(p-k)+(d-1)(p-1)+(t-d-2)p = pt-3k-d+1 < pt-k-d-1$, a contradiction. By this claim, we have

$$f_1 \ \ge \ 0 \tag{6}$$

and equality holds if and only if $x_3 = \cdots = x_{d+2} = p-1$ and $p-1 \le x_i \le p$ for $d+3 \le i \le t$.

We may assume that $x_i = p - a_i$ for $i = 1, 2, \ldots, t$. Clearly, each a_i is a non-negative integer and $a_1 \ge a_2 \ge \cdots \ge a_t$. By substituting $x_i = p - a_i$, we can show that $f_2 = g(X)$, where $g(X)$ is as defined in Lemma 7. The readers may refer to the Appendix for details.
By Lemma 5(ii), we see that $a_i \ge 1$ for $3 \le i \le d+2$. It then follows from Lemma 7 that $f_2 \ge 0$, and equality holds if and only if

(a) $a_4 = a_5 = \cdots = a_t = 0$ for $d = 0$;

(b) $a_4 = a_5 = \cdots = a_t = 0$, or $a_3 = a_4 = 1$ and $a_5 = a_6 = \cdots = a_t = 0$ for $d = 1$;

(c) $a_4 = a_5 = \cdots = a_{d+2} = 1$ and $a_{d+3} = \cdots = a_t = 0$, or $a_3 = a_4 = \cdots = a_{d+2} = 1$, $0 \le a_{d+3} \le 1$ and $a_{d+4} = \cdots = a_t = 0$ for $d \ge 2$.

Since $f_1 \ge 0$ and $f_2 \ge 0$, we have $f \ge 0$ and equality holds if and only if $x_3 = \cdots = x_{d+2} = p-1$, $x_{d+3} = p-1$ or p, and $x_{d+4} = \cdots = x_t = p$. Therefore, inequality (4) holds if and only if $x_3 = \cdots = x_{d+2} = p-1$, $x_{d+3} = p-1$ or p, and $x_{d+4} = \cdots = x_t = p$. Hence,

$H = K(x_1, x_2, \underbrace{p-1, \ldots, p-1}_{d}, x_{d+3}, \underbrace{p, \ldots, p}_{t-d-3}) - S$ where $p - 1 \leq x_{d+3} \leq p$. Note that by Lemma 5(iii), if $x_{d+3} = p$, then $H = G$. If $x_{d+3} = p - 1$, then we consider two cases: $x_2 = p - 1$ or $x_2 < p - 1$.

Case 1: $x_2 = p-1$. In this case, $F = K(p-k+1, \underbrace{p-1, \ldots, p-1}_{d+2}, \underbrace{p, \ldots, p}_{t-d-3})$ and using Software Maple, we have

$$\begin{aligned} s &= |E(F)| - |E(G)| \\ &= (d+2)(p-k+1)(p-1) + (t-d-3)(p-k+1)p + \\ &\quad (d+2)(t-d-3)(p-1)p + \binom{d+2}{2}(p-1)^2 + \binom{t-d-3}{2}p^2 - \\ &\quad (d+1)(p-k)(p-1) - (t-d-2)(p-k)p - \\ &\quad (d+1)(t-d-2)(p-1)p - \binom{d+1}{2}(p-1)^2 - \binom{t-d-2}{2}p^2 \\ &= k-1. \end{aligned}$$

Note that $f = 0$ and equality in (4) holds. Hence, we have $\beta = s((t-d-3)p + (d+1)(p-1)) = s((t-2)p - d - 1)$. From (1) and (3), we have $t(G) = t(H) = t(F) - s((t-2)p - d - 1)$ and $t(\epsilon_i) = (t-2)p - d - 1$ for all $\epsilon_i \in S$.

Let $\{V_1, V_2, \ldots, V_t\}$ be the unique t-independent partition of F such that $|V_1| = p - k + 1$, $|V_2| = \cdots = |V_{d+3}| = p - 1$ and $|V_i| = p$ for $d + 4 \leq i \leq t$. Thus, if $k = 2$, we have $F = K(\underbrace{p-1, \ldots, p-1}_{d+3}, \underbrace{p, \ldots, p}_{t-d-3})$. Then $s = 1$ and this one deleted edge has one end-vertex in V_i and the other end-vertex in V_j for $1 \leq i < j \leq d+3$. If $k \geq 3$ (so that $x_1 \leq p-2$ and $s = k-1 \geq 2$), then each of the $k-1$ deleted edges in S must have one end-vertex in V_1 and the other end-vertex in any one of the partite sets in $\{V_2, \ldots, V_{d+3}\}$. Note that for $i \neq j$, if ϵ_i (respectively ϵ_j) is an edge in S having an end-vertex in V_1 and another end-vertex in V_i (respectively, V_j), then ϵ_i and ϵ_j do not induced a $K(1,2)$ in F. Otherwise, equality in (1) doesn't hold.

Let b_i be the number of deleted edges joining vertices in V_1 and V_{i+1} for $1 \leq i \leq d+2$. So we have $\sum_{i=1}^{d+2} b_i = s = k - 1$. The case when $k = 2$ and the only deleted edge joining V_i to V_j for $2 \leq i < j \leq d+3$ is similar to the Subcase 1.1 below. We now consider the following two subcases.

Subase 1.1: Exactly one of $b_i \neq 0$. Without loss of generality, let $b_1 \neq 0$ and $b_i = 0$ for $2 \leq i \leq d+2$. In this case, $\overline{H}$ contains $t-d-3$ copies of K_p and $d+1$ copies of K_{p-1} as its components. Set $\overline{H} = \overline{H_1} \cup \underbrace{K_{p-1} \cup \cdots \cup K_{p-1}}_{d+1} \cup \underbrace{K_p \cup \cdots \cup K_p}_{t-d-3}$. Then $H = H_1 + \underbrace{O_{p-1} + \cdots + O_{p-1}}_{d+1} + \underbrace{O_p + \cdots + O_p}_{t-d-3}$. From Lemma 2 and

$$\sigma(H) = \sigma(K(p-k, \underbrace{p-1, \ldots, p-1}_{d+1}, \underbrace{p, \ldots, p}_{t-d-2})),$$

we have

$$\sigma(H_1)[\sigma(O_{p-1})]^{d+1}[\sigma(O_p)]^{t-d-3} = \sigma(O_{p-k}+O_p)[\sigma(O_{p-1})]^{d+1}[\sigma(O_p)]^{t-d-3}.$$

So

$$\sigma(H_1) = \sigma(K(p-k, p))$$

which implies that $P(H_1) = P(K(p-k,p))$. Hence, by Lemma 3, we have $H_1 = K(p-k,p)$. So, $x_2 = p$ which is a contradiction.

Subcase 1.2: There exists an integer $c \geq 2$ such that $b_1, b_2, \ldots, b_c \neq 0$ and $b_{c+1} = \cdots = b_{d+2} = 0$. In this case, we show that $Q(G) - Q(H) + 2K(H) - 2K(G) > 0$. Note that if $t = 3$, we only need to show that $Q(G) - Q(H) > 0$. This, by Lemma 1, contradicts $G \sim H$. We first calculate $Q(G)$ and $Q(H)$. Note that G has only induced C_4 of Type I. Hence,

$$\begin{aligned} Q(G) &= \binom{p-k}{2}\left[(d+1)\binom{p-1}{2} + (t-d-2)\binom{p}{2}\right] + \binom{t-d-2}{2} \times \\ &\quad \binom{p}{2}^2 + (d+1)(t-d-2)\binom{p-1}{2}\binom{p}{2} + \binom{d+1}{2}\binom{p-1}{2}^2. \end{aligned}$$

Note that H has only induced C_4 of Type I and Type II. Recall that for $i \neq j$, if ϵ_i (respectively ϵ_j) is an edge in S having an end-vertex in V_1 and another end-vertex in V_i (respectively, V_j), then ϵ_i and ϵ_j do not induced a $K(1,2)$ in F. Thus, the number of induced C_4 of Type II in H is $\left[(d+1)\binom{p-1}{2} + (t-d-3)\binom{p}{2}\right] \sum_{i=1}^{c} b_i$. Since $b_c = k - 1 - \sum_{i=1}^{c-1} b_i$, $1 \leq b_i \leq k-c$ and $c - 1 \leq \sum_{i=1}^{c-1} b_i \leq k-2$, by Lemma 6, we have

$$\begin{aligned} Q(H) &\leq \sum_{i=1}^{c} \frac{1}{4}\Big[(p-k+1)(p-1) - b_i\Big]\Big[(p-k+1)(p-1) - b_i - \\ &\quad (p-k+1) - (p-1) + 1\Big] + (d-c+2)\binom{p-k+1}{2}\binom{p-1}{2} \end{aligned}$$

$$
\begin{aligned}
& +(t-d-3)\binom{p-k+1}{2}\binom{p}{2}+\binom{d+2}{2}\binom{p-1}{2}^2+ \\
& (d+2)(t-d-3)\binom{p}{2}\binom{p-1}{2}+\binom{t-d-3}{2}\binom{p}{2}^2+ \\
& \left[(d+1)\binom{p-1}{2}+(t-d-3)\binom{p}{2}\right]\sum_{i=1}^{c} b_i \\
= \; & \frac{1}{4}c(p-k+1)(p-1)((p-k+1)(p-1)-2p+k+1)- \\
& \frac{1}{4}\left(2(p-k+1)(p-1)-2p+k+1\right)\sum_{i=1}^{c} b_i+\frac{1}{4}\sum_{i=1}^{c} b_i^2+ \\
& (d-c+2)\binom{p-k+1}{2}\binom{p-1}{2}+(t-d-3)\binom{p-k+1}{2}\binom{p}{2} \\
& +\binom{d+2}{2}\binom{p-1}{2}^2+(d+2)(t-d-3)\binom{p}{2}\binom{p-1}{2}+ \\
& \binom{t-d-3}{2}\binom{p}{2}^2+\left[(d+1)\binom{p-1}{2}+(t-d-3)\binom{p}{2}\right]\sum_{i=1}^{c} b_i \\
= \; & \frac{1}{4}c(p-k+1)(p-1)((p-k+1)(p-1)-2p+k+1)- \\
& \frac{1}{4}(2(p-k+1)(p-1)-2p+k+1)(k-1)+ \\
& \frac{1}{2}\left[\sum_{i=1}^{c-1} b_i^2-(k-1)\sum_{i=1}^{c-1} b_i+\sum_{1\le i<j\le c-1} b_i b_j\right]+\frac{1}{4}(k-1)^2+ \\
& (d-c+2)\binom{p-k+1}{2}\binom{p-1}{2}+(t-d-3)\binom{p-k+1}{2}\binom{p}{2}+ \\
& \binom{d+2}{2}\binom{p-1}{2}^2+(d+2)(t-d-3)\binom{p}{2}\binom{p-1}{2}+ \\
& \binom{t-d-3}{2}\binom{p}{2}^2+\left[(d+1)\binom{p-1}{2}+(t-d-3)\binom{p}{2}\right](k-1) \\
= \; & \frac{1}{4}c(p-k+1)(p-1)((p-k+1)(p-1)-2p+k+1)- \\
& \frac{1}{4}(2(p-k+1)(p-1)-2p+k+1)(k-1)+ \\
& \frac{1}{2}\left[\sum_{i=1}^{c-1} b_i\left(\sum_{i=1}^{c-1} b_i-k+1\right)-\sum_{1\le i<j\le c-1} b_i b_j\right]+\frac{1}{4}(k-1)^2+
\end{aligned}
$$

$$
\begin{aligned}
& (d-c+2)\binom{p-k+1}{2}\binom{p-1}{2} + (t-d-3)\binom{p-k+1}{2}\binom{p}{2} + \\
& \binom{d+2}{2}\binom{p-1}{2}^2 + (d+2)(t-d-3)\binom{p}{2}\binom{p-1}{2} + \\
& \binom{t-d-3}{2}\binom{p}{2}^2 + \left[(d+1)\binom{p-1}{2} + (t-d-3)\binom{p}{2}\right](k-1) \\
\leq \; & \frac{1}{4}c(p-k+1)(p-1)((p-k+1)(p-1)-2p+k+1) - \\
& \frac{1}{4}(2(p-k+1)(p-1)-2p+k+1)(k-1) + \\
& \frac{1}{2}\left[(c-1)(-1) - \binom{c-1}{2}\right] + \frac{1}{4}(k-1)^2 + \\
& (d-c+2)\binom{p-k+1}{2}\binom{p-1}{2} + (t-d-3)\binom{p-k+1}{2}\binom{p}{2} + \\
& \binom{d+2}{2}\binom{p-1}{2}^2 + (d+2)(t-d-3)\binom{p}{2}\binom{p-1}{2} + \\
& \binom{t-d-3}{2}\binom{p}{2}^2 + \left[(d+1)\binom{p-1}{2} + (t-d-3)\binom{p}{2}\right](k-1) \\
= \; & \frac{1}{4}c(p-k+1)(p-1)((p-k+1)(p-1)-2p+k+1) - \\
& \frac{1}{4}(2(p-k+1)(p-1)-2p+k+1)(k-1) - \frac{1}{2}\binom{c}{2} + \\
& \frac{1}{4}(k-1)^2 + (d-c+2)\binom{p-k+1}{2}\binom{p-1}{2} + \\
& (t-d-3)\binom{p-k+1}{2}\binom{p}{2} + \binom{d+2}{2}\binom{p-1}{2}^2 + \\
& (d+2)(t-d-3)\binom{p}{2}\binom{p-1}{2} + \binom{t-d-3}{2}\binom{p}{2}^2 + \\
& \left[(d+1)\binom{p-1}{2} + (t-d-3)\binom{p}{2}\right](k-1).
\end{aligned}
$$

Thus, using Software Maple, we have

$$
\begin{aligned}
Q(G) - Q(H) \;\geq\; & \binom{p-k}{2}\left[(d+1)\binom{p-1}{2} + (t-d-2)\binom{p}{2}\right] + \\
& \binom{t-d-2}{2}\binom{p}{2}^2 + (d+1)(t-d-2)\binom{p-1}{2}\binom{p}{2} +
\end{aligned}
$$

$$\begin{aligned}
&\binom{d+1}{2}\binom{p-1}{2}^2 - \\
&\left[\frac{1}{4}c(p-k+1)(p-1)((p-k+1)(p-1)-2p+k+1) - \right.\\
&\quad \frac{1}{4}(2(p-k+1)(p-1)-2p+k+1)(k-1) - \frac{1}{2}\binom{c}{2} + \\
&\quad \frac{1}{4}(k-1)^2 + (d-c+2)\binom{p-k+1}{2}\binom{p-1}{2} + \\
&\quad (t-d-3)\binom{p-k+1}{2}\binom{p}{2} + \binom{d+2}{2}\binom{p-1}{2}^2 + \\
&\quad (d+2)(t-d-3)\binom{p}{2}\binom{p-1}{2} + \binom{t-d-3}{2}\binom{p}{2}^2 + \\
&\quad \left.\left((d+1)\binom{p-1}{2} + (t-d-3)\binom{p}{2}\right)(k-1)\right] \\
= &\ \frac{1}{2}\binom{c}{2} > 0.
\end{aligned}$$

Note that

$$\begin{aligned}
K(G) = &\binom{t-d-2}{4}p^4 + \binom{t-d-2}{3}p^3\Big((d+2)p-d-k-1\Big) + \\
&\binom{t-d-2}{2}p^2\left[\binom{d+1}{2}(p-1)^2 + (d+1)(p-k)(p-1)\right] + \\
&(t-d-2)p\left[\binom{d+1}{3}(p-1)^3 + \binom{d+1}{2}(p-1)^2(p-k)\right] + \\
&\binom{d+1}{4}(p-1)^4 + \binom{d+1}{3}(p-1)^3(p-k).
\end{aligned}$$

Since for $i \neq j$, if ϵ_i (respectively ϵ_j) is an edge in S having an end-vertex in V_1 and another end-vertex in V_i (respectively, V_j), then ϵ_i and ϵ_j do not induced a $K(1,2)$ in F, we then have

$$\begin{aligned}
K(H) = &\binom{t-d-3}{4}p^4 + \binom{t-d-3}{3}p^3((d+3)p-d-k-1) + \\
&\binom{t-d-3}{2}p^2\left[\binom{d+2}{2}(p-1)^2 + (d+2)(p-1)(p-k+1)\right] +
\end{aligned}$$

$$(t-d-3)p\left[\binom{d+2}{3}(p-1)^3+\binom{d+2}{2}(p-1)^2(p-k+1)\right]+$$
$$\binom{d+2}{4}(p-1)^4+\binom{d+2}{3}(p-1)^3(p-k+1)-$$
$$\left(\sum_{i=1}^{c}b_i\right)\left(\binom{t-d-3}{2}p^2+(t-d-3)(d+1)p(p-1)+\right.$$
$$\left.\binom{d+1}{2}(p-1)^2\right)$$
$$=\binom{t-d-3}{4}p^4+\binom{t-d-3}{3}p^3((d+3)p-d-k-1)+$$
$$\binom{t-d-3}{2}p^2\left[\binom{d+2}{2}(p-1)^2+(d+2)(p-1)(p-k+1)\right]+$$
$$(t-d-3)p\left[\binom{d+2}{3}(p-1)^3+\binom{d+2}{2}(p-1)^2(p-k+1)\right]+$$
$$\binom{d+2}{4}(p-1)^4+\binom{d+2}{3}(p-1)^3(p-k+1)-(k-1)\times$$
$$\left[\binom{t-d-3}{2}p^2+(t-d-3)(d+1)p(p-1)+\binom{d+1}{2}(p-1)^2\right].$$

Using Software Maple, we get $K(H)=K(G)$. So, we have

$$Q(G)-Q(H)+2K(H)-2K(G)\geq\frac{1}{2}\binom{c}{2}>0.$$

Therefore, by Lemma 1, this subcase is also not possible.

Case 2: $x_2<p-1$. In this case, $F=K(x_1,x_2,\underbrace{p-1,\ldots,p-1}_{d+1},\underbrace{p,\ldots,p}_{t-d-3})$.
Let $\{V_1,V_2,\ldots,V_t\}$ be the unique t-independent partition of F. Since $f=0$, from (1), (3) and (4), we have as in Case 1 that $t(G)=t(H)=t(F)-s((t-2)p-d-1)$ and $t(\epsilon_i)=(t-2)p-d-1$ for all $\epsilon_i\in S$. Thus, for each edge $\epsilon_i\in S$, one end-vertex of ϵ_i is in V_1 whereas the other end-vertex is in V_2. Hence $\overline{H}$ contains $t-d-3$ copies of K_p and $d+1$ copies of K_{p-1} as its components. Set $\overline{H}=\overline{H_1}\cup\underbrace{K_{p-1}\cup\cdots\cup K_{p-1}}_{d+1}\cup\underbrace{K_p\cup\cdots\cup K_p}_{t-d-3}$. Then

$H = H_1 + \underbrace{O_{p-1} + \ldots + O_{p-1}}_{d+1} + \underbrace{O_p + \ldots + O_p}_{t-d-3}$. From Lemma 2 and

$$\sigma(H) = \sigma(K(p-k, \underbrace{p-1, \ldots, p-1}_{d+1}, \underbrace{p, \ldots, p}_{t-d-2})),$$

we have

$$\sigma(H_1)[\sigma(O_{p-1})]^{d+1}[\sigma(O_p)]^{t-d-3} = \sigma(O_{p-k}+O_p)[\sigma(O_{p-1})]^{d+1}[\sigma(O_p)]^{t-d-3}.$$

So

$$\sigma(H_1) = \sigma(K(p-k, p))$$

which implies that $P(H_1) = P(K(p-k,p))$. Hence, by Lemma 3, we have $H_1 = K(p-k, p)$. So, $x_2 = p$ which is a contradiction. This completes the proof of this theorem. □

Note. For $d = 0$, our result improves the condition of Theorem 5.4.3 in [11] from $p \geq 2k \geq 4$ to $p \geq k+2 \geq 4$.

Acknowledgement
The authors would like to thank the referee for the valuable comments.

References

1. M. Behzad, G. Chartrand and L. Lesniak-Foster, Graphs and Digraphs (Wadsworth, Belmont, CA 1979).

2. F. Brenti, Expansions of chromatic polynomial and log-loncavity, Trans. Amer. Math. Soc. **332** (1992) 729-756.

3. C.Y. Chao and Novacky Jr., On maximally saturated graphs, *Discrete Math.* **41** (1982) 139-143.

4. G.L. Chia, B.H. Goh and K.M. Koh, The chromaticity of some families of complete tripartite graphs, SCIENTIA, Ser. A: Math. (1988) 27-37.

5. G.L. Chia and C.K. Ho, Chromatic equivalence classes of complete tripartite graphs, *Discrete Math.* (2008) doi:10.1016/j.disc. 2007.12.059

6. F.M. Dong, On chromatic uniqueness of two infinite families of graphs, J. Graph Theory **17** (1993) 387–392.

7. K.M. Koh and K.L. Teo, The search for chromatically unique graphs, *Graphs and Combinatorics* **6** (1990) 259-285.

8. K.M. Koh and K.L. Teo, The search for chromatically unique graphs - II, *Discrete Math.* **172** (1997) 59-78.

9. R.Y. Liu, H.X. Zhao and C.F. Ye, A complete solution to a conjecture on chromatic uniqueness of complete tripartite graphs, *Discrete Math.* **289** (2004)175-179.

10. R.C. Read and W.T. Tutte, Chromatic polynomials, in: L.W. Beineke, R.J. Wilson (Eds.), Selected Topics in Graph Theory, Vol. 3, Academic Press, New York, 1988, 15-42.

11. H.X. Zhao, Chromaticity and adjoint polynomials of graphs, Ph.D. *thesis* (2005), University of Twente, Netherland.

12. H.W. Zou, The chromatic uniqueness of certain complete t-partite graphs, *Discrete Math.* **275** (2004) 375-383.

Appendix

$$\begin{aligned}
f_2 = & \sum_{3\le i<j\le t}\left((p-a_i)^2(p-a_j)+(p-a_i)(p-a_j)^2\right)+ \\
& \sum_{3\le i<j<l\le t}(p-a_i)(p-a_j)(p-a_l)-(t-3)p\sum_{i=3}^{t}(p-a_i)^2+ \\
& (d-1)\sum_{i=3}^{d+2}(p-a_i)^2+d\sum_{i=d+3}^{t}(p-a_i)^2-(pt-k-d-1)\times \\
& \sum_{3\le i<j\le t}(p-a_i)(p-a_j)+\left[d(p-k)(p-1)+\binom{d+1}{2}(p-1)^2+\right. \\
& \left.(t-d-2)(p-k)p+(d+1)(t-d-2)(p-1)p+\binom{t-d-2}{2}p^2\right]\times \\
& \sum_{i=3}^{t}(p-a_i)-(p-1)(2p-k-1)\sum_{i=3}^{d+2}(p-a_i)-p(2p-k-1)\times \\
& \sum_{i=d+3}^{t}(p-a_i)-d(t-d-2)(p-k)(p-1)p-\binom{d}{2}(p-k)(p-1)^2
\end{aligned}$$

$$-\binom{t-d-2}{2}(p-k)p^2-\binom{t-d-2}{2}(d+1)(p-1)p^2-$$

$$\binom{d+1}{2}(t-d-2)(p-1)^2p-\binom{d+1}{3}(p-1)^3-\binom{t-d-2}{3}p^3$$

$$=\sum_{3\le i<j\le t}\left(2p^3-3p^2(a_i+a_j)+p(a_i^2+a_j^2)+4pa_ia_j-a_i^2a_j-a_ia_j^2\right)$$

$$+\sum_{3\le i<j<l\le t}\left(p^3-p^2(a_i+a_j+a_l)+p(a_ia_j+a_ia_l+a_ja_l)-a_ia_ja_l\right)$$

$$-(t-3)p\sum_{i=3}^{t}(p^2-2pa_i+a_i^2)+(d-1)\sum_{i=3}^{d+2}(p^2-2pa_i+a_i^2)+$$

$$d\sum_{i=d+3}^{t}(p^2-2pa_i+a_i^2)-(pt-k-d-1)\sum_{3\le i<j\le t}\Big(p^2-p(a_i+a_j)+$$

$$a_ia_j\Big)+\Big[d(p-k)(p-1)+\binom{d+1}{2}(p-1)^2+(t-d-2)(p-k)p+$$

$$(d+1)(t-d-2)(p-1)p+\binom{t-d-2}{2}p^2\Big]\sum_{i=3}^{t}(p-a_i)-$$

$$(p-1)(2p-k-1)\sum_{i=3}^{d+2}(p-a_i)-p(2p-k-1)\sum_{i=d+3}^{t}(p-a_i)-$$

$$d(t-d-2)(p-k)(p-1)p-\binom{d}{2}(p-k)(p-1)^2-$$

$$\binom{t-d-2}{2}(p-k)p^2-\binom{t-d-2}{2}(d+1)(p-1)p^2-$$

$$\binom{d+1}{2}(t-d-2)(p-1)^2p-\binom{d+1}{3}(p-1)^3-\binom{t-d-2}{3}p^3$$

$$=\;2p^3\binom{t-2}{2}-3p^2(t-3)\sum_{i=3}^{t}a_i+p(t-3)\sum_{i=3}^{t}a_i^2+4p\sum_{3\le i<j\le t}a_ia_j-$$

$$\sum_{3\le i<j\le t}(a_i^2a_j+a_ia_j^2)+p^3\binom{t-2}{3}-p^2\binom{t-3}{2}\sum_{i=3}^{t}a_i+$$

$$p(t-4)\sum_{3\le i<j\le t}a_ia_j-\sum_{3\le i<j<l\le t}a_ia_ja_l-(t-3)(t-2)p^3+$$

$$2p^2(t-3)\sum_{i=3}^{t} a_i - p(t-3)\sum_{i=3}^{t} a_i^2 + (d-1)dp^2 - 2(d-1)p\sum_{i=3}^{d+2} a_i +$$

$$(d-1)\sum_{i=3}^{d+2} a_i^2 + (t-d-2)dp^2 - 2dp\sum_{i=d+3}^{t} a_i +$$

$$d\sum_{i=d+3}^{t} a_i^2 - (pt-k-d-1)\left[p^2\binom{t-2}{2} - p(t-3)\sum_{i=3}^{t} a_i + \sum_{3\le i<j\le t} a_i a_j\right]$$

$$+\left[d(p-k)(p-1) + \binom{d+1}{2}(p-1)^2 + (t-d-2)(p-k)p + \right.$$

$$\left.(d+1)(t-d-2)(p-1)p + \binom{t-d-2}{2}p^2\right]((t-2)p - \sum_{i=3}^{t} a_i) -$$

$$(p-1)(2p-k-1)(dp - \sum_{i=3}^{d+2} a_i) - p(2p-k-1)((t-d-2)p -$$

$$\sum_{i=d+3}^{t} a_i) - d(t-d-2)(p-k)(p-1)p - \binom{d}{2}(p-k)(p-1)^2 -$$

$$\binom{t-d-2}{2}(p-k)p^2 - \binom{t-d-2}{2}(d+1)(p-1)p^2 -$$

$$\binom{d+1}{2}(t-d-2)(p-1)^2 p - \binom{d+1}{3}(p-1)^3 - \binom{t-d-2}{3}p^3$$

$$= (k+d+1)\sum_{3\le i<j\le t} a_i a_j - \left(dk + \binom{d+1}{2}\right)\sum_{i=3}^{t} a_i + (k+1)\sum_{i=3}^{d+2} a_i$$

$$+(d-1)\sum_{i=3}^{d+2} a_i^2 + d\sum_{i=d+3}^{t} a_i^2 - \sum_{3\le i<j\le t}(a_i^2 a_j + a_i a_j^2)$$

$$-\sum_{3\le i<j<l\le t} a_i a_j a_l + k\binom{d}{2} + \binom{d+1}{3}$$

$$= (k+d+1)\sum_{3\le i<j\le t} a_i a_j - d(k+1)\sum_{i=3}^{t} a_i - \binom{d}{2}\sum_{i=3}^{t} a_i +$$

$$(k+1)\sum_{i=3}^{d+2} a_i + (d-1)\sum_{i=3}^{d+2} a_i^2 + d\sum_{i=d+3}^{t} a_i^2 -$$

$$\sum_{3\le i<j\le t}(a_i^2 a_j + a_i a_j^2) - \sum_{3\le i<j<l\le t} a_i a_j a_l + (k+1)\binom{d}{2} + \binom{d}{3}$$

Since $\sum_{i=1}^t x_i = pt - k - d - 1$, we have $\sum_{i=1}^t a_i = k + d + 1$, and that

$$
\begin{aligned}
f_2 = & (a_1+a_2) \sum_{3\le i<j\le t} a_i a_j + 2 \sum_{3\le i<j<l\le t} a_i a_j a_l - \\
& (d-1)\left[\sum_{i=1}^{t} a_i - d\right] \sum_{i=3}^{d+2} a_i - d\left[\sum_{i=1}^{t} a_i - d\right] \sum_{i=d+3}^{t} a_i - \binom{d}{2} \sum_{i=3}^{t} a_i \\
& +(d-1)\sum_{i=3}^{d+2} a_i^2 + d \sum_{i=d+3}^{t} a_i^2 + \left[\sum_{i=1}^{t} a_i - d\right]\binom{d}{2} + \binom{d}{3} \\
= & \; (a_1+a_2)\left[\sum_{3\le i<j\le d+2} a_i a_j + \sum_{i=3}^{d+2} a_i \sum_{i=d+3}^{t} a_i + \sum_{d+3\le i<j\le t} a_i a_j\right] + \\
& 2 \sum_{3\le i<j<l\le t} a_i a_j a_l - (d-1)\left[a_1 + a_2 + \sum_{i=3}^{d+2} a_i + \sum_{i=d+3}^{t} a_i - d\right] \sum_{i=3}^{d+2} a_i \\
& -d\left[a_1 + a_2 + \sum_{i=3}^{d+2} a_i + \sum_{i=d+3}^{t} a_i - d\right] \sum_{i=d+3}^{t} a_i + (d-1)\sum_{i=3}^{d+2} a_i^2 + \\
& d \sum_{i=d+3}^{t} a_i^2 + (a_1 + a_2 - d)\binom{d}{2} + \binom{d}{3} \\
= & \; (a_1+a_2)\left[\sum_{3\le i<j\le d+2} a_i a_j + \sum_{i=3}^{d+2} a_i \sum_{i=d+3}^{t} a_i + \sum_{d+3\le i<j\le t} a_i a_j\right] + \\
& 2\left[\sum_{3\le i<j<l\le d+2} a_i a_j a_l + \sum_{3\le i<j\le d+2} a_i a_j \sum_{i=d+3}^{t} a_i \right. \\
& \left. + \sum_{i=3}^{d+2} a_i \sum_{d+3\le i<j\le t} a_i a_j + \sum_{d+3\le i<j<l\le t} a_i a_j a_l\right] - (d-1)(a_1+a_2)\sum_{i=3}^{d+2} a_i \\
& -2(d-1) \sum_{3\le i<j\le d+2} a_i a_j - (d-1)\sum_{i=3}^{d+2} a_i \sum_{i=d+3}^{t} a_i + (d-1)d\sum_{i=3}^{d+2} a_i \\
& -d(a_1+a_2) \sum_{i=d+3}^{t} a_i - d\sum_{i=3}^{d+2} a_i \sum_{i=d+3}^{t} a_i - 2d \sum_{d+3\le i<j\le t} a_i a_j \\
& +d^2 \sum_{i=d+3}^{t} a_i + (a_1+a_2)\binom{d}{2} - d\binom{d}{2} + \binom{d}{3} \\
= & \; (a_1+a_2)\left[\sum_{3\le i<j\le d+2} a_i a_j - (d-1)\sum_{i=3}^{d+2} a_i + \binom{d}{2}\right]
\end{aligned}
$$

$$
\begin{aligned}
&+2\sum_{d+3\le i<j<l\le t} a_ia_ja_l + \left[(a_1+a_2)\left(\sum_{i=3}^{d+2} a_i - d\right) + 2\sum_{3\le i<j\le d+2} a_ia_j \right.\\
&\quad \left. -(2d-1)\sum_{i=3}^{d+2} a_i + d^2\right]\sum_{i=d+3}^{t} a_i +\\
&\left[a_1+a_2+2\sum_{i=3}^{d+2} a_i - 2d\right]\sum_{d+3\le i<j\le t} a_ia_j + 2\sum_{3\le i<j<l\le d+2} a_ia_ja_l -\\
&\quad 2(d-1)\sum_{3\le i<j\le d+2} a_ia_j + (d-1)d\sum_{i=3}^{d+2} a_i - d\binom{d}{2} + \binom{d}{3}\\
= &\quad (a_1+a_2-2d)\left[\sum_{3\le i<j\le d+2} a_ia_j - (d-1)\sum_{i=3}^{d+2} a_i + \binom{d}{2}\right]\\
&+2\sum_{d+3\le i<j<l\le t} a_ia_ja_l + \left[(a_1+a_2)\left(\sum_{i=3}^{d+2} a_i - d\right)\right.\\
&\quad \left. +2\sum_{3\le i<j\le d+2} a_ia_j - (2d-1)\sum_{i=3}^{d+2} a_i + d^2\right]\sum_{i=d+3}^{t} a_i\\
&\quad +\left[a_1+a_2+2\sum_{i=3}^{d+2} a_i - 2d\right]\sum_{d+3\le i<j\le t} a_ia_j + 2\sum_{3\le i<j<l\le d+2} a_ia_ja_l -\\
&\quad 2\sum_{3\le i<j\le d+2} a_ia_j - (d-1)d\sum_{i=3}^{d+2} a_i + d\binom{d}{2} + \binom{d}{3}\\
= &\quad (a_1+a_2-d)\left[\sum_{3\le i<j\le d+2} a_ia_j - (d-1)\sum_{i=3}^{d+2} a_i + \binom{d}{2}\right] +\\
&\quad 2\sum_{3\le i<j<l\le d+2} a_ia_ja_l - (d-2)\sum_{3\le i<j\le d+2} a_ia_j + \binom{d}{3} +\\
&\left[(a_1+a_2-d)\left(\sum_{i=3}^{d+2} a_i - d\right) + 2\sum_{3\le i<j\le d+2} a_ia_j - (d-1)\sum_{i=3}^{d+2} a_i\right]\sum_{i=d+3}^{t} a_i\\
&\quad +\left[a_1+a_2+2\sum_{i=3}^{d+2} a_i - 2d\right]\sum_{d+3\le i<j\le t} a_ia_j + 2\sum_{d+3\le i<j<l\le t} a_ia_ja_l\\
= &\quad g(X).
\end{aligned}
$$