Tutte Uniqueness of Locally Grid Graphs

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Abstract

A graph is said to be locally grid if the structure around each of its vertices is a 3×3 grid. As a follow up of the research initiated in [8] and [9] we prove that most locally grid graphs are uniquely determined by their Tutte polynomial.

1 Introduction

Given a graph G, the Tutte polynomial of G is a two-variable polynomial $T(G;x,y)$, which contains considerable information on G [4]. A graph G is said to be *Tutte unique* if $T(G;x,y) = T(H;x,y)$ implies $G \cong H$ for every other graph H. Since chromatic polynomial is an evaluation of Tutte polynomial [4] the concept of Tutte unique graphs is a natural extension of the concept of chromatically unique graphs [9], defined as those graphs uniquely determined by their chromatic polynomial [7]. Tutte polynomial and Tutte uniqueness have been studied more generally for matroids [2, 3, 10, 11]. From now on, all graphs considered have no isolated vertices, since if we allow graphs to have isolated vertices there are no Tutte unique graphs at all.

In Section 2 we prove that locally grid graphs are Tutte unique. In [8] was studied the Tutte uniqueness of the toroidal grid. In order to study this property in the rest of locally grid graphs, classified in [8] and [12], we generalize the definition of essential cycle given in [8], being both definitions equivalent [5], and we also calculate the exact number of shortest essential cycles in locally grid graphs.

Given a fixed graph H, a connected graph G is said to be locally H if for every vertex x the subgraph induced on the set of neighbours of x is isomorphic to H. For example, if P is the Petersen graph, then there are three locally P graphs [6]. The locally grid condition is slightly different since it involves not only a vertex and its neighbours, but also four vertices at distance two.

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We first recall some definitions and results about locally grid graphs from [8].

Let $N(x)$ be the set of neighbours of a vertex x. We say that a 4-regular connected graph G is a *locally grid graph* if for each vertex x there exists an ordering x_1, x_2, x_3, x_4 of $N(x)$ and four different vertices y_1, y_2, y_3, y_4, such that, taking the indices modulo 4,

$$\begin{aligned} N(x_i) \cap N(x_{i+1}) &= \{x, y_i\} \\ N(x_i) \cap N(x_{i+2}) &= \{x\} \end{aligned}$$

and there are no more adjacencies among $\{x, x_1, \ldots, x_4, y_1, \ldots, y_4\}$ than those required by these conditions (Figure 1).

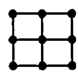


Figure 1: Locally Grid Structure

Locally grid graphs are simple, two-connected, triangle-free, and each vertex belongs to exactly four cycles of length 4.

Let $H = P_p \times P_q$ be the $p \times q$ grid, where P_l is a path with l vertices. Label the vertices of H with the elements of the abelian group $\mathbb{Z}_p \times \mathbb{Z}_q$ in the natural way. Vertices of degree four already have the locally grid property, hence we have to add edges between vertices of degree two and three in order to obtain a locally grid graph. A complete classification of locally grid graphs is given in [8] where it is also shown that all of them admit an embedding in the torus or in the Klein bottle. They fall into the following families. In all the Figures, the vertices of the graph are represented by dots and two points with the same label correspond to points that are identified in the surface.

The Torus $T^\delta_{p,q}$ with $p \geq 5$, $0 \leq \delta \leq p/2$, $\delta + q \geq 5$ if $q \geq 4$, $\delta + q \geq 6$ if $q = 2, 3$ or $4 \leq \delta < p/2$ with $\delta \neq p/3, p/4$ if $q = 1$. (Figure 2a). This graph is built as the graph $C_p \times C_q$ but moving the adjacencies in the first direction δ vertices to the right. That is,

$$\begin{aligned} E(T^\delta_{p,q}) = E(H) &\cup \{\{(i,0),(i+\delta, q-1)\}, 0 \leq i \leq p-1\} \\ &\cup \{\{(0,j),(p-1,j)\}, 0 \leq j \leq q-1\} \end{aligned}$$

For $\delta = 0$ we obtain the toroidal grid $C_p \times C_q$, in this case we will write $T_{p,q}$. We can assume that $\delta \leq p/2$. All these graphs are embeddable in the topological torus.

The Klein Bottle $K^1_{p,q}$ with $p \geq 5$, p odd, $q \geq 5$. (Figure 2b)

$$\begin{aligned} E(K^1_{p,q}) &= E(H) \cup \{\{(j,0),(p-j-1,q-1)\}, 0 \leq j \leq p-1\} \\ &\cup \{\{(0,j),(p-1,j)\}, 0 \leq j \leq q-1\} \end{aligned}$$

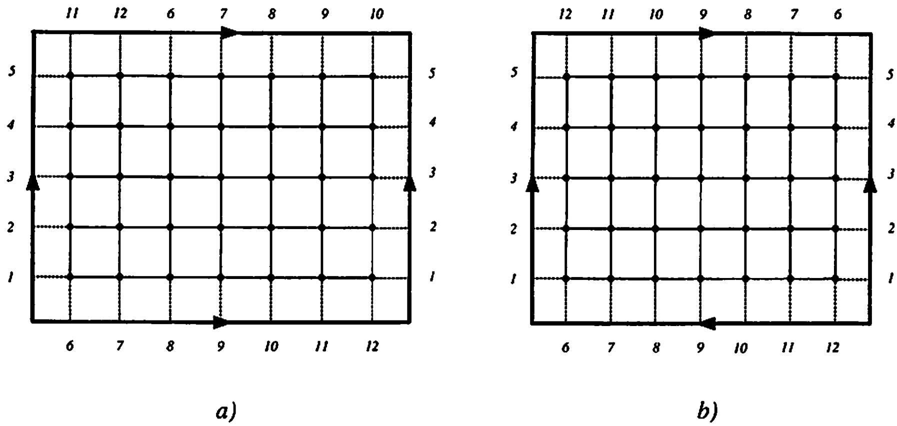


Figure 2: a) $T^2_{7,5}$ b) $K^1_{7,5}$

The Klein Bottle $K^0_{p,q}$ with $p \geq 6$, p even, $q \geq 4$ (Figure 3a).

$$\begin{aligned} E(K^0_{p,q}) &= E(H) \cup \{\{(j,0),(p-j-1,q-1)\}, 0 \leq j \leq p-1\} \\ &\cup \{\{(0,j),(p-1,j)\}, 0 \leq j \leq q-1\} \end{aligned}$$

The Klein Bottle $K^2_{p,q}$ with $p \geq 6$, p even, $q \geq 5$ (Figure 3b).

$$\begin{aligned} E(K^2_{p,q}) &= E(H) \cup \{\{(j,0),(p-j,q-1)\}, 0 \leq j \leq p-1\} \\ &\cup \{\{(0,j),(p-1,j)\}, 0 \leq j \leq q-1\} \end{aligned}$$

The graphs $K^i_{p,q}$ are built by keeping the adjacencies of the second direction untouched and reversing the ones in the first direction, thus we obtain graphs embeddable in the Klein bottle.

The graphs $S_{p,q}$ with $p \geq 3$ and $q \geq 6$. (Figure 4).

If $p \leq q$

$$\begin{aligned} E(S_{p,q}) &= E(H) \cup \{(j,0),(p-1,q-p+j)\}, 0 \leq j \leq p-1\} \\ &\cup \{\{(0,i),(i,q-1)\}, 0 \leq i \leq p-1\} \\ &\cup \{\{(0,i),(p-1,i-p)\}, p \leq i \leq q-1\} \end{aligned}$$

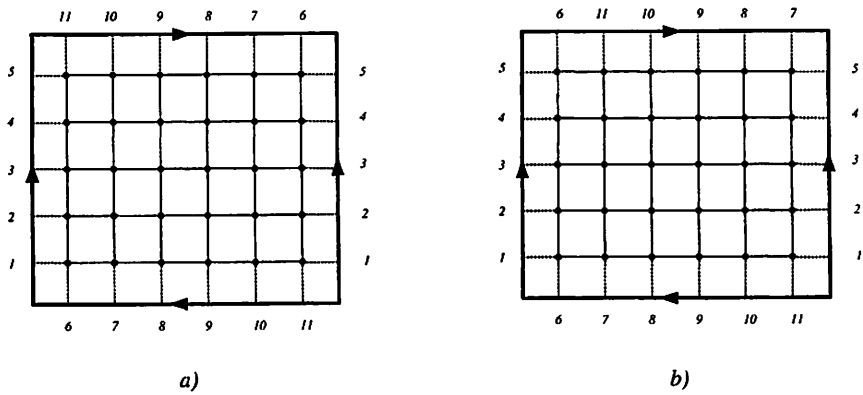


Figure 3: a) $K^0_{6,5}$ b) $K^2_{6,5}$

If $q \leq p$

$$\begin{aligned} E(S_{p,q}) &= E(H) \cup \{\{(j,0),(0,q-1-j)\}, 0 \leq j \leq q-1\} \\ &\cup \{\{(p-1-i,q-1),(p-1,i)\}, 0 \leq i \leq q-1\} \\ &\cup \{\{(i,q-1),(i+q,0)\}, 0 \leq i \leq p-q-1\} \end{aligned}$$

Figure 4 shows embeddings of the two kinds of graphs $S_{p,q}$ in the Klein bottle.

Theorem 1.1. *[8] If G is a locally grid graph with N vertices, then exactly one of the following holds:*

a) $G \cong T^{\delta}_{p,q}$ with $pq = N$, $p \geq 5$, $\delta \leq p/2$ and $\delta + q \geq 5$ if $q \geq 4$ or $\delta + q \geq 6$ if $q = 2, 3$ or $4 \leq \delta < p/2$, $\delta \neq p/3, p/4$ if $q = 1$.

b) $G \cong K^i_{p,q}$ with $pq = N$, $p \geq 5$, $i \equiv p \ (mod\ 2)$ for $i \in \{0, 1, 2\}$ and $q \geq 4 + \lceil i/2 \rceil$.

c) $G \cong S_{p,q}$ with $pq = N$, $p \geq 3$ and $q \geq 6$.

2 Tutte Uniqueness

Let $G = (V, E)$ be a graph with vertex set V and edge set E. The *rank* of a subset $A \subseteq E$ is defined by $r(A) = |A| - k(A)$, where $k(A)$ is the number of connected components of the spanning subgraph (V, A). The rank-size generating polynomial is defined as:

$$R(G; x, y) = \sum_{A \subseteq E} x^{r(A)} y^{|A|}$$

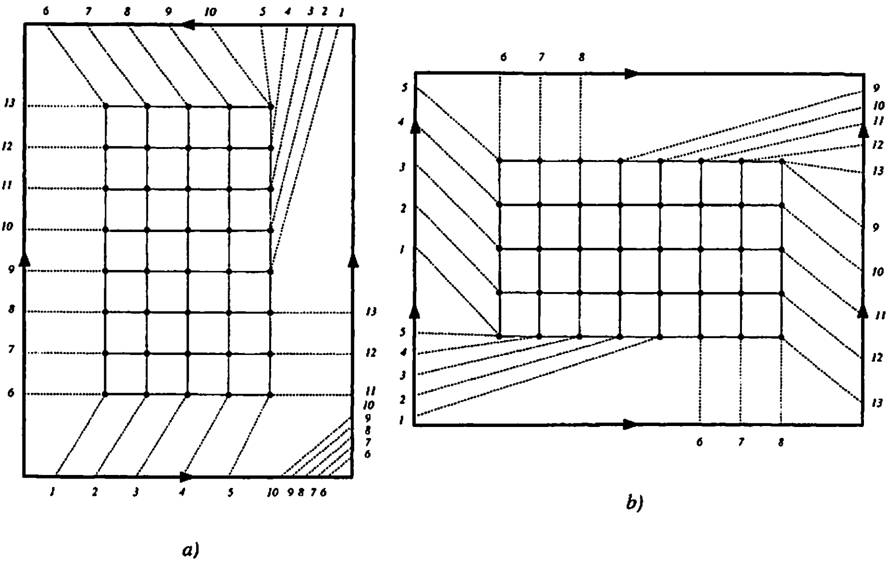


Figure 4: a) $S_{5,8}$ b) $S_{8,5}$

The coefficient of $x^i y^j$ in $R(G; x, y)$ is the number of spanning subgraphs in G with rank i and j edges. This polynomial contains exactly the same information about G as the Tutte polynomial, which is given by:

$$T(G; x, y) = \sum_{A \subseteq E} (x-1)^{r(E)-r(A)} (y-1)^{|A|-r(A)}$$

hence, the Tutte polynomial tells us for every i and j the number of edge-sets in G with rank i and size j. This fact is going to be essential in order to prove the Tutte uniqueness of locally grid graphs. Given a locally grid graph G, we show that for every locally grid graph H different from G and with $|V(G)| = |V(H)|$ there is at least one coefficient of the rank-size generating polynomial in which both graphs differ.

Given two cycles C and C' of a locally grid graph G, we say that C is *locally homotopic* to C' if there exists a cycle of length four, say H, with $C \cap H$ connected and C' is obtained from C by replacing $C \cap H$ with $H - (C \cap H)$. A *homotopy* is a sequence of local homotopies. A cycle of G is called *essential* if it is not homotopic to a cycle of length four. This definition generalizes the one given in [8] since it does not depend on the surface in which a locally grid graph is embedded [5].

Let l_G be the minimum length of an essential cycle of G. Note that l_G

is invariant under isomorphism. The number of essential cycles of length l_G contributes to the coefficient a_{l_G-1,l_G} of $R(G;x,y)$, which counts the number of edges sets with rank $l_G - 1$ and size l_G.

In order to show the Tutte uniqueness of locally grid graphs we are going to use the following results proved in [8]:

Lemma 2.1. *[8] Given two graphs G and G', if G is locally grid and $T(G;x,y) = T(G';x,y)$ then G' is locally grid.*

Lemma 2.2. *[8] Let G, G' be a pair of locally grid graphs with pq vertices then:*

- $l_G \neq l_{G'}$ *implies* $T(G;x,y) \neq T(G';x,y)$.

- *If $l_G = l_{G'}$ but G and G' do not have the same number of shortest essential cycles, then $T(G;x,y) \neq T(G';x,y)$.*

The process we are going to follow is to pairwise compare all the graphs given in the classification theorem of locally grid graphs. In those cases for which the minimum length of essential cycles or the number of cycles of this minimum length are different we have that both graphs are not Tutte equivalent, thus the relevance of the following result.

Lemma 2.3. *If G is a locally grid graph with pq vertices, then the length l_G of the shortest essential cycles and the number of these cycles are given in the following table:*

G	l_G	number of essential cycles	
$T_{p,q}$	$\min\{p,q\}$	q	*if* $p < q$
		$2p$	*if* $p = q$
		p	*if* $p > q$
$T^{\delta}_{p,q}$	$\min\{p,q+\delta\}$	q	*if* $p < q+\delta$
		$q+p\binom{q+\delta-1}{\delta}$	*if* $p = q+\delta$
		$p\binom{q+\delta-1}{\delta}$	*if* $p > q+\delta$
$K^0_{p,q}$	$\min\{p,q+1\}$	q	*if* $p < q+1$
		$5q$	*if* $p = q+1$
		$4q$	*if* $p > q+1$
$K^1_{p,q}$	$\min\{p,q\}$	q	*if* $p < q$
		$q+1$	*if* $p = q$
		1	*if* $p > q$
$K^2_{p,q}$	$\min\{p,q\}$	q	*if* $p < q$
		$q+2$	*if* $p = q$
		2	*if* $p > q$
$S_{p,q}$	$\min\{2p,q\}$	$q\binom{2p-1}{p}$	*if* $2p < q$
		$2^q+q\binom{2p-1}{p}$	*if* $2p = q$
		2^q	*if* $2p > q$

Proof. The cases $T_{p,q}$, $T^{\delta}_{p,q}$ and $K^i_{p,q}$ are proved in [8], where it is also shown that $l_{S_{p,q}} = \min(2p,q)$ and that if $q \leq p$ the number of shortest essential cycles is 2^q. Hence, we are only left with three cases in which we are given lower bounds on the number of essential cycles of length $l_{S_{p,q}}$. We are interested in calculating the exact number.

Locally grid graphs with pq vertices are constructed by adding edges to the $p \times q$ grid. These edges are called *exterior edges.* Essential cycles of shortest length can be obtained either by joining the two ends of an exterior edge by a path contained in the $p \times q$ grid or by joining the ends of two exterior edges by two paths contained in the $p \times q$ grid. The reason for which we can not use more than two exterior edges to obtain shortest essential cycles is that we have to cross twice the $p \times q$ grid (horizontally or vertically) using at least two different paths. It means that we need at least $l_G - 2$ edges to cross the grid, therefore the minimum length of any essential cycle with three or more exterior edges is $l_G + 1$.

Along the proof we are going to use that the number of shortest paths joining $(0,0)$ with (a,b) in a $p \times q$ grid is $\binom{a+b}{a}$.

In $S_{p,q}$ we distinguish the following cases.

Case 1: If $2p < q$, every exterior edge of the form $\{(0,i),(p-1,i-p)\}$ determines $\binom{2p-1}{p}$ essential cycles of length $2p$ and we have $q-p$ edges of this kind.

Using two exterior edges of the form $\{(j,0),(p-1,j+q-p)\}$ and $\{(0,i),(i,q-1)\}$ we obtain shortest essential cycles by joining $(0,i)$ with any $(j,0)$ being $0 \leq j \leq p-1$ by a path of minimum length contained in the $p \times q$ grid and analogously with $(p-1,j+q-p)$ and $(i,q-1)$. By a carefully counting one can see that for a fixed j there are $\binom{2p-1}{p}$ ways of joining the ends of these edges, therefore the number of shortest essential cycles determined by two exterior edges is $p\binom{2p-1}{p}$. Hence, if $2p < q$ the number of essential cycles of length $2p$ is $q\binom{2p-1}{p}$.

Case 2: If $p < q < 2p$, every exterior edge of the form $\{(0,i),(i,q-1)\}$ or $\{(i,0),(p-1,q-p+i)\}$ with $0 \leq i \leq p-1$ generates $\binom{q-1}{i}$ essential cycles of length q. Hence the number of shortest essential cycles in $S_{p,q}$ using one exterior edge is $2\sum_{i=0}^{p-1}\binom{q-1}{i}$.

To obtain shortest essential cycles in $S_{p,q}$ with $p < q < 2p$ using two exterior edges we can take either the edges $\{(i,0),(p-1,i+q-p)\}$ and $\{(0,j),(p-1,j-p)\}$ or $\{(0,i),(i,q-1)\}$ and $\{(0,j),(p-1,j-p)\}$ with $0 \leq i \leq p-1$ and $p \leq j \leq q-1$. In the first case, we can join $(i,0)$ being $2p-q \leq i \leq p-1$ with any $(p-1,j-p)$ being $p \leq j \leq i-p+q$ by a shortest path contained in the $p \times q$ grid. Analogously with the other two ends of the edges. In the second case, for every j we can join the vertex $(0,j)$ with any $(i,q-1)$ being $0 \leq i \leq j-p$ by a shortest path contained in the $p \times q$ grid and analogously with $(0,i)$ and $(p-1,j-p)$. In both cases, by a carefully counting and using properties of binomials numbers, we obtain $\sum_{j=p}^{q-1}\binom{q-1}{j}$ essential cycles of length q. Hence, if $p < q < 2p$ the number of shortest essential cycles is:

$$2\sum_{j=0}^{p-1}\binom{q-1}{j} + 2\sum_{j=p}^{q-1}\binom{q-1}{j} = 2^q$$

Case 3: If $q = 2p$, it is easy to prove that the number of shortest essential cycles in $S_{p,q}$ is the sum of the numbers obtained in the previous cases.

□

An edge-set is called a *normal edge-set* if it does not contain any essential cycle. In [8] it is proved that locally grid graphs are locally orientable and they use that to establish a canonical way to represent edge-sets with a set Γ of words over a given alphabet. Therefore given $\gamma \in \Gamma$, $B(\gamma)$ denotes the edge-set determined by γ and every connected normal edge-set is the result of an unique word [8]. An edge-set A is a *forbidden edge-set for a word* γ if it contains an essential cycle and a subset $B \subseteq A$ such that B is normal and has γ as its word. A word γ' contains γ if $B(\gamma)$ is a subgraph of $B(\gamma')$. Denote by $N(\gamma, G, m, r)$ the number of edge-sets in a locally grid graph G with rank r and size m, and such that its word γ' contains γ.

Lemma 2.4. *[8] Let $\gamma \in \Gamma$ be such that $B(\gamma)$ contains at least one cycle. Then the quantity $N(\gamma, G, m, r)$ is the same for all locally grid graphs G with pq vertices, no forbidden edge-set for γ of size m, and such that $l_G \geq m-2$.*

Theorem 2.5. *Let $p, q \geq 6$ verify the following conditions:*
1.- $p\binom{q+\delta-1}{\delta} \neq 2^n$ and $q+p\binom{q+\delta-1}{\delta} \neq 2^n$ for $n \in \mathbb{N}$.
2.- $pq \neq p'q'$ for all $p', q' \geq 6$ with $p = q+\delta = q'+\delta' < p'$ and $q+p\binom{p-1}{\delta} = p'\binom{p-1}{\delta'}$.
Then $T^{\delta}_{p,q}$ is Tutte unique for all $\delta \leq p/2$.

Proof. Let $p, q \geq 6$ and G be a graph with $T(G; x, y) = T(T^{\delta}_{p,q}, x, y)$. By Lemma 2.1, G is a locally grid graph, hence G has to be isomorphic to exactly one of the following graphs: $T_{p',q'}$, $T^{\delta'}_{p',q'}$, $K^{i}_{p',q'}$, $S_{p',q'}$. We prove that G is isomorphic to $T^{\delta'}_{p',q'}$ with $p = p'$, $q = q'$ and $\delta = \delta'$ assuming that G is isomorphic to each one of the previous graphs and obtaining a contradiction in all the cases except in the aforementioned case. In [8] $T_{p,q}$ was shown to be Tutte unique, thus we can consider $\delta > 0$ and G not isomorphic to $T_{p',q'}$.

Case 1. Suppose $G \cong K^{0}_{p',q'}$. By Lemma 2.2, $l_{T^{\delta}_{p,q}} = l_{K^{0}_{p',q'}}$ and the number of shortest essential cycles has to be the same in both graphs.

Case 1.1. $l_{T^{\delta}_{p,q}} = p$, $l_{K^{0}_{p',q'}} = p'$ with $p < q+\delta$ and $p' < q'+1$.

As a result of Lemma 2.2, $p = p'$ and $q = q'$. Our aim is to prove that the number of edge sets with rank q and size $q+1$ is different for each graph. This would lead to a contradiction since this number is the coefficient of $x^q y^{q+1}$ in the rank-size generating polynomial.

If $T^{\delta}_{p,q}$ has k essential cycles of length $q+1$ ($\delta > 1$) or $k+pq$ ($\delta = 1$), then $K^{0}_{p,q}$ would have $k+4q$ such cycles. Therefore if we can show that there exits a bijection between edge sets with rank q and size $q+1$ that are not essential cycles, we would have proved what we want. For every r with $0 \leq r \leq q-2$ denote by E_r the set $\{((i,r),(i,r+1))\,;\, 0 \leq i \leq p-1\}$. Let A be an edge set that is not an essential cycle with rank q and size $q+1$ in $T^{\delta}_{p,q}$. Define $s(A)$ as $\min\{r \in [0,q-2]\,;\, A \cap E_r = \emptyset\}$. If $A \subset E(T^{\delta}_{p,q})$ the minimum always exits. For every r with $0 \leq r \leq q-2$ we define the bijection, φ_r between $\{A \subseteq E(T^{\delta}_{p,q}) | r(A) = q, |A| = q+1, s(A) = r\}$ and $\{A \subseteq E(K^{0}_{p,q}) | r(A) = q, |A| = q+1, s(A) = r\}$ as follows:

If $A \subset E(T^{\delta}_{p,q})$, $\varphi_r(A) = \cup_{((i,j),(i',j'))\in A}\psi(((i,j),(i',j')))$ where $\psi(((i,j),(i',j')))$ is equal to

$$\begin{array}{lll} ((i,j),(p-1-i'+\delta,j')) & \text{if} & j'=q-1, j=0 \\ ((i,j),(i',j')) & \text{if} & j,j' \in [0,r] \\ ((p-1-i+\delta,j),(p-1-i'+\delta,j')) & \text{if} & r+1 \leq j,j' \leq q-1 \end{array}$$

Case 1.2. $l_{T^{\delta}_{p,q}} = p$, $l_{K^{0}_{p',q'}} = p' = q'+1$ with $p < q+\delta$ or $l_{T^{\delta}_{p,q}} = q+\delta < p$, $l_{K^{0}_{p',q'}} = p'$ with $p' < q'+1$.

The contradiction in these two cases is produced due to the equality of shortest essential cycles, number of these cycles and number of vertices on each graph.

Case 1.3. $l_{T^{\delta}_{p,q}} = p$, $l_{K^{0}_{p',q'}} = q'+1$ with $p < q+\delta$ and $q'+1 < p'$.

To obtain a contradiction, we are going to prove that there are more edge-sets with rank $q'+2$ and size $q'+3$ in $T^{\delta}_{p,q}$ than in $K^{0}_{p,q}$. Basically, we are going to follow the same procedure that was developed in [8]. The previous sets can be classified into three groups:

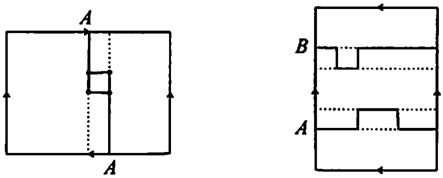
1.- Normal edge-sets.

2.- Sets containing an essential cycle of length $q'+1$ and two other edges (Figure 5a).

3.- Essential cycles of length $q'+3$ (Figures 5b and 6).

(1) By Lemma 2.4 we know that $T^{\delta}_{p,q}$ and $K^{0}_{p',q'}$ have the same number of normal edge-sets with rank $q'+2$ and size $q'+3$ that do not contain a cycle of length four. We are going to prove that the number of normal edge-sets with rank $q'+2$ and size $q'+3$ containing a cycle of length four is greater in $T^{\delta}_{p,q}$ than in $K^{0}_{p',q'}$.

Again by Lemma 2.4, the number of edge-sets with rank $q'+1$ and size $q'+2$ containing a cycle of length four is the same in both graphs, call it $s_{q'+1}$. Add one edge to each of these sets in order to obtain a set with rank $q'+2$ and size $q'+3$. This set can be one of the following types depending on which edge we are adding:



a) b)

Figure 5: a) A set of edges in $K^0_{p',q'}$ containing an essential cycle of length $q'+1$ and two other edges. b) Essential cycles of length $q'+3$ in $T^\delta_{p,q}$

Figure 6: Essential cycles of length $q'+3$ in $K^0_{p',q'}$

(A) A normal edge set with rank $q'+2$.

(B) A normal edge set containing two non essential cycles and having rank $q'+1$.

(C) An edge set containing an essential cycle of length $q'+1$ and a non essential cycle of length four.

Let $A(G)$, $B(G)$ and $C(G)$ (where G is either $T^\delta_{p,q}$ or $K^0_{p',q'}$), the number of edge-sets in G that belong to the groups A, B and C respectively. We recall the following equality from [8]:

$$s_{q'+1}(2pq-q'-2) = A(G)(q'-1) + \sum_{B\in B(G)} (q'+3-\delta(B)) + C(G)(q'-1)$$

where $\delta(B)$ is the number of edges of B which do not belong to any cycle of length four in B. Since $C(T^\delta_{p,q})=0$ and $C(K^0_{p',q'})\neq 0$ we have:

$$A(T^\delta_{p,q})(q'-1) + \textstyle\sum_{B\in B(T^\delta_{p,q})}(q'+3-\delta(B)) = A(K^0_{p',q'})(q'-1)+$$
$$+\textstyle\sum_{B\in B(K^0_{p',q'})}(q'+3-\delta(B)) + C(K^0_{p',q'})(q'-1)$$

Applying Lemma 2.4 several times we get that:

$$\sum_{B\in B(T^\delta_{p,q})} (q'+3-\delta(B)) = \sum_{B\in B(K^0_{p',q'})} (q'+3-\delta(B))$$

hence

$$A(T^{\delta}_{p,q})(q'-1) = A(K^{0}_{p',q'})(q'-1) + C(K^{0}_{p',q'})(q'-1)$$

(2) In $T^{\delta}_{p,q}$, every essential cycle of length $q'+1$ plus two edges has rank $q'+2$, but in $K^{0}_{p',q'}$ there are essential cycles for which if we add two edges we obtain sets with rank $q'+1$. By hypothesis, both graphs have the same number of shortest essential cycles therefore the number of edge-sets in this case is greater in $T^{\delta}_{p,q}$ than in $K^{0}_{p',q'}$.

(3) For every essential cycle of length $p = q'+1$ in $T^{\delta}_{p,q}$ we have $2\binom{p}{2}$ ways of adding two edges in order to obtain a new essential cycle, hence in $T^{\delta}_{p,q}$ there are $2q\binom{p}{2}$ essential cycles of length $q'+3$. In [8] it is proved that in $K^{0}_{p',q'}$ there are $4q'\binom{q'}{2} + 4\binom{q'+2}{3}$ essential cycles of length $q'+3$. Since $p = q'+1$ and $q = 4q'$, the number of essential cycles of length $q'+3$ is greater in $T^{\delta}_{p,q}$ than in $K^{0}_{p',q'}$.

Case 1.4. $l_{T^{\delta}_{p,q}} = p = q+\delta$, $l_{K^{0}_{p',q'}} = p'$ with $q'+1 > p'$.

Suppose $p = q+\delta = p'$ then $q = q'$, hence $\delta < 1$. We get a contradiction because $\delta \geq 1$.

Case 1.5. $l_{T^{\delta}_{p,q}} = p = q+\delta$, $l_{K^{0}_{p',q'}} = p' = q'+1$.

If the length of the shortest essential cycles and the number of these cycles coincide in both graphs, we would have $p = p'$, $q = q'$, $\delta = 1$ and $q + pq = 5q'$ therefore $p = 4$.

Case 1.6. $l_{T^{\delta}_{p,q}} = p = q+\delta$, $l_{K^{0}_{p',q'}} = q'+1$ with $p' > q'+1$.

$$q'+1 = p = q+\delta \quad \Rightarrow \quad 4q' = q + p\binom{q'}{\delta}$$

$$p > 4,\ q' = \binom{q'}{q'-1} < \binom{q'}{\delta} \quad \Rightarrow \quad 4q' < p\binom{q'}{\delta}$$

Case 1.7. $l_{T^{\delta}_{p,q}} = q+\delta < p$, $l_{K^{0}_{p',q'}} = p' = q'+1$.

$$q'+1 = p' = q+\delta \quad \text{and} \quad 5q' = p\binom{q+\delta-1}{\delta}$$

If $\delta = 1$ then $q = q'$, $p = p' = q+\delta < p$ hence $\delta > 1$.

$$p > 5 \text{ and } \binom{q+\delta-1}{\delta} = \binom{q'}{\delta} > q' \quad \Rightarrow \quad 5q' < p\binom{q+\delta-1}{\delta}$$

Case 1.8. $l_{T^{\delta}_{p,q}} = q + \delta < p$, $l_{K^0_{p',q'}} = q' + 1 < p'$.

Now, $q' + 1 = q + \delta$, so we can assume that $\delta > 1$ because if $\delta = 1$, then $q = q'$, $p = p'$ and the number of shortest essential cycles would not be the same in both graphs. The contradiction in this case is similar to the one obtained in the previous case because $4q' = p\begin{pmatrix} q' \\ \delta \end{pmatrix}$.

After these eight cases we can conclude that $T^{\delta}_{p,q}$ is not isomorphic to $K^0_{p,q}$.

Case 2. Suppose G isomorphic to $T^{\delta'}_{p',q'}$.

Case 2.1. $l_{T^{\delta}_{p,q}} = p$, $l_{T^{\delta'}_{p',q'}} = p'$ with $p < q + \delta$ and $p' < q' + \delta'$.

As a result of Lemma 2.2, $p = p'$ and $q = q'$. Suppose $\delta' < \delta$, as in case 1.1 our purpose is to prove that the number of edge sets with rank $q + \delta' - 1$ and size $q + \delta'$ is different in each graph. If $T^{\delta}_{p,q}$ has x essential cycles of length $q + \delta'$, $T^{\delta'}_{p',q'}$ has $x + p(\begin{matrix} q + \delta' - 1 \\ \delta' \end{matrix})$, therefore if we show that there exits a bijection between the edge sets with rank $q + \delta' - 1$ and size $q + \delta'$ that are not essential cycles, we would have proved what we want. For every edge set A with rank $q + \delta' - 1$ and size $q + \delta'$ that is not an essential cycle it is defined $s(A)$ as in case 1.1. For every r with $0 \leq r \leq q - 2$ we define the following bijection, φ_r between $\{A \subseteq E(T^{\delta'}_{p,q}) | r(A) = q + \delta' - 1, |A| = q + \delta', s(A) = r\}$ and $\{A \subseteq E(T^{\delta}_{p,q}) | r(A) = q + \delta' - 1, |A| = q + \delta', s(A) = r\}$.

If $A \subset E(T^{\delta'}_{p,q})$, $\varphi_r(A) = \cup_{((i,j),(i',j')) \in A} \psi(((i,j),(i',j')))$ where $\psi(((i,j),(i',j')))$ is equal to

$$\begin{array}{ll} ((i,j),(i' + \delta - \delta', j')) & \text{if} \quad j' = q - 1, j = 0 \\ ((i,j),(i',j')) & \text{if} \quad j, j' \in [0, r] \\ ((i + \delta - \delta', j),(i' + \delta - \delta', j')) & \text{if} \quad r + 1 \leq j, j' \leq q - 1 \end{array}$$

Case 2.2. $l_{T^{\delta}_{p,q}} = p$, $l_{T^{\delta'}_{p',q'}} = p' = q' + \delta'$ with $p < q + \delta$.

Suppose $T(T^{\delta'}_{p',q'}; x, y) = T(T^{\delta}_{p,q}; x, y)$ then $q = q' + p'\begin{pmatrix} q' + \delta' - 1 \\ \delta' \end{pmatrix}$ and $p = p'$. Since $pq = p'q'$ we obtain $q = q'$, a contradiction.

Case 2.3. $l_{T^{\delta}_{p,q}} = p$, $l_{T^{\delta'}_{p',q'}} = q' + \delta'$ with $p < q + \delta$ and $q' + \delta' < p'$.

$$q' + \delta' = p \quad \Rightarrow \quad q = p'\begin{pmatrix} p - 1 \\ \delta' \end{pmatrix}$$

$$pq = p'q' \quad \Rightarrow \quad p\begin{pmatrix} q' + \delta' - 1 \\ \delta' \end{pmatrix} = q' = p - \delta'$$

$\delta' < p-1$ then $p\binom{q'+\delta'-1}{\delta'} > p$. This contradiction was obtained by having assumed that both graphs have the same Tutte polynomial.

Because of hypothesis 2, we have that the case $l_{T^{\delta}_{p,q}} = p = q+\delta$, $l_{T^{\delta'}_{p',q'}} = q'+\delta'$ with $q'+\delta' < p'$ cannot occur.

With an analogous process to the one followed in case 1.3 we prove that the number of edge-sets with rank $q'+\delta'+1$ and size $q'+\delta'+2$ are different in $T^{\delta}_{p,q}$ and $T^{\delta'}_{p',q'}$. Therefore, $l_{T^{\delta}_{p,q}} = q+\delta$, $l_{T^{\delta'}_{p',q'}} = q'+\delta' < p'$ is not possible.

The rest of the cases are analogous to the previous ones, hence just one case can occur, namely, $l_{T^{\delta}_{p,q}} = p = q+\delta$, $l_{T^{\delta'}_{p',q'}} = p' = q'+\delta'$, which implies $p = p'$, $q = q'$ and $\delta = \delta'$.

Case 3. Suppose $G \simeq K^1_{p',q'}$, then $T(T^{\delta}_{p,q};x,y) = T(K^1_{p',q'};x,y)$. Because of Lemmas 2.2 and 2.3, we cannot have $p' > q'$.

Case 3.1. $l_{T^{\delta}_{p,q}} = p < q+\delta$, $l_{K^1_{p',q'}} = p' < q'$.

As in case 1.1 we have to obtain a bijection to prove that the number of edge-sets with rank $q-1$ and size q are different in each graph.

If $A \subset A(T^{\delta}_{p,q})$, $\varphi_r(A) = \cup_{((i,j),(i',j'))\in A}\psi(((i,j),(i',j')))$ where $\psi(((i,j),(i',j')))$ is defined as follows:

$$\begin{array}{ll} ((i,j),(p-1-i'+\delta,j')) & \text{if } \ j'=q-1, j=0 \\ ((i,j),(i',j')) & \text{if } \ j,j' \in [0,r] \\ ((p-1-i+\delta,j),(p-1-i'+\delta,j')) & \text{if } \ r+1 \le j,j' \le q-1 \end{array}$$

The rest of the cases cannot occur because the length of shortest essential cycles, the number of these cycles and the number of vertices do not coincide. We omit the proof for the sake of brevity.

The case $G \simeq K^2_{p',q'}$ is similar to the previous ones, hence we just specify the bijection in the case $p = p' < q'$ and $q = q'$:

If $A \subset A(T^{\delta}_{p,q})$, $\varphi_r(A) = \cup_{((i,j),(i',j'))\in A}\psi(((i,j),(i',j')))$ where $\psi(((i,j),(i',j')))$ is equal to:

$$\begin{array}{ll} ((i,j),(p-i'+\delta,j')) & \text{if } \ j'=q-1, j=0 \\ ((i,j),(i',j')) & \text{if } \ j,j' \in [0,r] \\ ((p-i+\delta,j),(p-i'+\delta,j')) & \text{if } \ r+1 \le j,j' \le q-1 \end{array}$$

Case 4. Finally, we are going to assume that $G \simeq S_{p',q'}$. By hypothesis 1 we cannot have $l_{T^{\delta}_{p,q}} = q+\delta \le p$, $l_{S_{p',q'}} = q'$ with $q' < 2p'$.

Case 4.1. $l_{T^{\delta}_{p,q}} = p < q+\delta$, $l_{S_{p',q'}} = q'$ with $q' \le p'$

Using the same ideas as in case 1.3 we prove that the number of edge-sets with rank $q'+1$ and size $q'+2$ is greater in $T^{\delta}_{p,q}$ than in $S_{p',q'}$. For the sake of brevity we only give a sketch of the proof. These sets are classified into three groups: normal edge-sets, sets containing an essential cycle of length q' and two other edges and essential cycles of length $q'+2$. We prove that $T^{\delta}_{p,q}$ has more edge sets of each type than $S_{p',q'}$. The ideas are similar to case 1.3, so we just mention the last type. For every essential cycle of length q' in $T^{\delta}_{p,q}$ we have $2\binom{p}{2}$ ways of adding two edges in order to get a new essential cycle. On the other hand, there are essential cycles in $S_{p',q'}$ (see Figure 7) in which the ways of adding two edges is smaller than $2\binom{q'}{2}$. Since $p = q'$ and the number of shortest essential cycles is the same in both graphs, we have more essential cycles of length $q' + 2$ in $T^{\delta}_{p,q}$ than in $S_{p',q'}$.

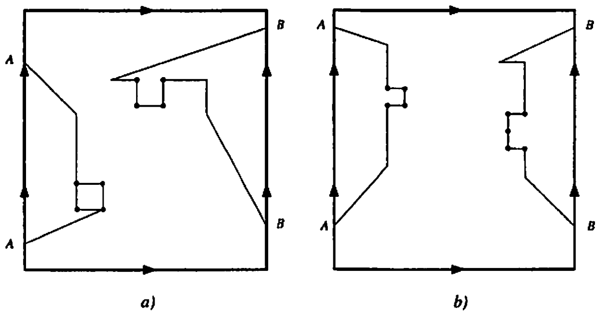


Figure 7: a) Edge sets in $S_{p',q'}$ with $p' \geq q'$ containing an essential cycle of length q' and two other edges. b) Essential cycles of length $q'+2$ in $S_{p',q'}$.

Case 4.2. $l_{T^{\delta}_{p,q}} = p = q + \delta$, $l_{S_{p',q'}} = 2p' < q'$.

Given that $2p' = p = q + \delta$, we have $q' = 2q$. We will obtain a contradiction by assuming we have equality for the number of shortest essential cycles in both graphs. In this case and in the next ones we are going to use the following property: $\binom{2p'-1}{n} < \binom{2p'-1}{m}$ if $n < m \leq [(2p'-1)/2] = p' - 1$. Since $\binom{p-1}{p/2} > q = p - \delta$ then

$$2q\binom{p-1}{p/2} > q+(2q-1)\binom{p-1}{p/2} \geq q+p\binom{p-1}{p/2} > q+p\binom{p-1}{\delta}$$

Case 4.3. $l_{T^{\delta}_{p,q}} = q + \delta < p$, $l_{S_{p',q'}} = 2p' < q'$.

Suppose that $pq = p'q'$, $2p' = q+\delta$ and $q'\binom{2p'-1}{p'} = p\binom{2p'-1}{\delta}$

then $\delta < q$ and

$$\binom{2p'-1}{p'} = \frac{p}{q'}\binom{2p'-1}{\delta} = \frac{q+\delta}{2q}\binom{2p'-1}{\delta} < \binom{2p'-1}{\delta}$$

The contradiction is due to $\binom{2p'-1}{p'} > \binom{2p'-1}{\delta}$.

Case 4.4. $l_{T^{\delta}_{p,q}} = q+\delta < p$, $l_{S_{p',q'}} = 2p' = q'$.

If $2p' = q' = q+\delta$ and $p\binom{q+\delta-1}{\delta} = q'\binom{2p'-1}{p'} + 2^{q'}$ then

$$p'\binom{2p'}{\delta} = \frac{pq}{q'}\frac{2p'}{2p'-\delta}\binom{2p'-1}{\delta} = \frac{pq}{q'}\frac{q'}{q}\binom{2p'-1}{\delta} =$$

$$= q'\binom{2p'-1}{p'} + 2^{q'} = \frac{q'}{2}\binom{2p'}{p'} + 2^{q'} = p'\binom{2p'}{p'} + 2^{q'}$$

We obtain a contradiction by assuming we have equality for the number of shortest essential cycles in both graphs.

The rest of the cases cannot occur because the length of shortest essential cycles, the number of these cycles and the number of vertices do not coincide. Therefore we obtain a contradiction, since G and $S_{p',q'}$ do not have the same Tutte polynomial. We omit the proof for the sake of brevity. □

Theorem 2.6. *$K^0_{p,q}$ is Tutte unique for all $p, q \geq 6$.*

Proof. Let $p, q \geq 6$ and G a graph with $T(G; x, y) = T(K^0_{p,q}; x, y)$. Due to Lemma 2.1 and Theorem 2.5, G has to be isomorphic to exactly one of the following graphs: $K^i_{p',q'}$, $S_{p',q'}$. We are going to prove that G is isomorphic to $K^0_{p',q'}$ with $p = p'$, $q = q'$.

Suppose G isomorphic to $K^0_{p',q'}$ then $l_{T^{\delta}_{p,q}} = l_{K^0_{p',q'}}$ and the number of shortest essential cycles has to be the same in both graphs. We just have to study the case in which $l_{K^0_{p,q}} = p < q+1$, $l_{K^0_{p',q'}} = q'+1$ with $p' > q'+1$. This is so because, if $l_{K^0_{p,q}} = q+1 < p$ and $l_{K^0_{p',q'}} = p'$ with $p' < q'+1$ the reasoning would be analogous and in these cases it is easy to verify that the number of vertices and the length of shortest essential cycles can not coincide in both graphs.

If $l_{K^0_{p,q}} = p < q+1$, $l_{K^0_{p',q'}} = q'+1$ with $p' > q'+1$ we can show that the number of edge-sets with rank $q'+2$ and size $q'+3$ is different in $K^0_{p,q}$

and $K^0_{p',q'}$. We omit the proof because it uses the same arguments as those in case 1.3.

Suppose $G \cong K^1_{p',q'}$. Since p is even and p' odd, all the cases in which the length of shortest essential cycles in $K^0_{p,q}$ is p and in $K^1_{p',q'}$ is p' are proved. By Lemma 2.3 we know that the number of shortest essential cycles in $K^0_{p,q}$ is always bigger than one, hence we obtain a contradiction in all those cases in which the number of shortest essential cycles in $K^1_{p',q'}$ is one. Therefore we just have to study two cases: $l_{K^0_{p,q}} = q + 1 < p$, $l_{K^1_{p',q'}} = p' < q'$ and $l_{K^0_{p,q}} = q + 1 < p$, $l_{K^1_{p',q'}} = p' = q'$. In the first case we obtain a contradiction by proving that the number of edge-sets with rank $p' + 1$ and size $p' + 2$ is different in each graph (following the same reasoning as in case 1.3 of Theorem 2.5). In the second case we show that if $p' = q' = q + 1$, $pq = p'q'$, p are even and p' is odd it must then be the case that q' is even. By Lemmas 2.2 and 2.3, if $p' > q'$ then $T(K^0_{p,q}; x, y) \neq T(K^2_{p',q'}; x, y)$ therefore G is not isomorphic to $K^2_{p',q'}$. Following the same reasoning as in case 1.1 of Theorem 2.5 we show that it cannot be that $l_{K^0_{p,q}} = p < q+1$ and $l_{K^2_{p',q'}} = p' < q'$. We just specify the bijection between $\{A \subseteq E(K^0_{p,q}) | r(A) = q, |A| = q + 1, s(A) = r\}$ and $\{A \subseteq E(K^2_{p',q'}) | r(A) = q, |A| = q + 1, s(A) = r\}$.

If $A \subset A(K^0_{p,q})$, $\varphi_r(A) = \cup_{((i,j),(i',j')) \in A} \psi(((i,j),(i',j')))$ where

$$\psi(((i,j),(i',j'))) = \begin{cases} ((i,j),(i'+1,j')) & \text{if} \quad j' = q-1, j = 0 \\ ((i,j),(i',j')) & \text{if} \quad j, j' \in [0,r] \\ ((i+1,j),(i'+1,j')) & \text{if} \quad r+1 \leq j, j' \leq q-1 \end{cases}$$

On the other hand, we prove (as in case 1.3 of Theorem 2.5) that if $l_{K^0_{p,q}} = q + 1 < p$ and $l_{K^2_{p',q'}} = p' < q'$ the number of edge-sets of rank $p' + 1$ and size $p' + 2$ is different for each graph.

The other four cases obtained by considering the possible combinations of the lengths of shortest essential cycles in $K^0_{p,q}$ and $K^2_{p',q'}$, are not possible since the length of shortest essential cycles, the number of these cycles and the number of vertices cannot coincide in both graphs.

Finally, suppose $G \simeq S_{p',q'}$.

Following the same reasoning as in case 1.3 of Theorem 2.5, if $l_{K^0_{p,q}} = p < q + 1$ and $l_{S_{p',q'}} = q' \leq p'$ we show that the number of edge-sets with rank $q' + 1$ and size $q' + 2$ is different in $K^0_{p,q}$ and $S_{p',q'}$, hence these graphs do not have the same Tutte polynomial.

It is easy to prove that the rest of the cases cannot occur because the length of shortest essential cycles, the number of these cycles and the number of vertices do not coincide in both graphs given that $q \geq 6$. □

Theorem 2.7. *The graph $K^1_{p,q}$ is Tutte unique for all $p, q \geq 6$.*

Proof. The argument of this proof is basically the same as those followed in Theorems 2.5 and 2.6. Due to the Tutte uniqueness of $T^\delta_{p,q}$ and $K^0_{p,q}$ we only have to prove that $T(G; x, y) \neq T(K^1_{p,q}; x, y)$ with $G \in \{K^1_{p',q'}$ (except if $p = p'$ and $q = q'$), $K^2_{p',q'}, S_{p',q'}\}$. In every case we are going to suppose that $T(G; x, y) = T(K^1_{p,q}; x, y)$ and we will obtain a contradiction.

Case 1. If $G \simeq K^1_{p',q'}$ it is easy to prove that the length of shortest essential cycles, the number of these cycles and the number of vertices only coincide if $p = p'$ and $q = q'$.

Case 2. If $G \simeq K^2_{p',q'}$, by Lemmas 2.2 and 2.3 we get a contradiction in all those cases for which the number of shortest essential cycles in $K^1_{p,q}$ is one or the number of shortest essential cycles in $K^2_{p',q'}$ is two. In the other cases a contradiction is reached because p is odd and p' is even.

Case 3. If $G \simeq S_{p',q'}$ we can consider $p \leq q$ because if $p > q$ the number of shortest essential cycles in $K^1_{p,q}$ is one and $p, q \geq 6$.

If $l_{K^1_{p,q}} = p < q$ and $l_{S_{p',q'}} = 2p' < q'$ or $l_{S_{p',q'}} = q'$ with $p' < q' < 2p'$, by Lemmas 2.2 and 2.3 it is easy to obtain a contradiction. The same is true if $l_{K^1_{p,q}} = p = q$ and $l_{S_{p',q'}} = q'$ with $q' < 2p'$ or $l_{S_{p',q'}} = 2p'$ with $2p' < q'$.

If $l_{K^1_{p,q}} = p < q$ and $l_{S_{p',q'}} = q' \leq p'$ we prove (as in previous cases) that there are different number of edge-sets with rank $q' + 1$ and size $q' + 2$ in both graphs, hence they can not have the same Tutte polynomial. □

Theorem 2.8. *The graph $K^2_{p,q}$ is Tutte unique for $p, q \geq 6$.*

Proof. Due to Theorems 2.5, 2.6 and 2.7 we just have to prove that $T(G; x, y) \neq T(K^2_{p,q}; x, y)$ with $G \in \{K^2_{p',q'}$ (except if $p = p'$ and $q = q'$), $S_{p',q'}\}$ and $pq = p'q'$.

By Lemmas 2.2 and 2.3, $T(K^2_{p',q'}; x, y) \neq T(K^2_{p,q}; x, y)$ if $p \neq p'$ and $q \neq q'$ because the length of shortest essential cycles, the number of these cycles and the number of vertices only coincide if $p = p'$ and $q = q'$.

If $G \simeq S_{p',q'}$ we can assume that $p \leq q$ otherwise if $q < p$, $K^2_{p,q}$ has two shortest essential cycles and by Lemma 2.2 we obtain a contradiction.

If $l_{K^2_{p,q}} = p < q$ and $l_{S_{p',q'}} = q' \leq p'$ we prove as in previous cases that the number of edge-sets with rank $q' + 1$ and size $q' + 2$ is different in both graphs. Hence these two graphs cannot have the same Tutte polynomial, therefore we get a contradiction and G can not be isomorphic to $S_{p',q'}$.

In the other cases we obtain a contradiction because the length of shortest essential cycles, the number of these cycles and the number of vertices cannot coincide in both graphs. □

Theorem 2.9. *The graph $S_{p,q}$ is Tutte unique for all $p, q \geq 6$ verifying that $2^q \neq p' \binom{q'+\delta-1}{\delta}$ and $q' + p' \binom{q'+\delta-1}{\delta} \neq 2^q$ for all p', q' with $p'q' = pq$ and $\delta > 0$.*

Proof. Suppose that $S_{p,q}$ is not Tutte unique. Then, by Theorems 2.5, 2.6, 2.7, 2.8 and Lemma 2.2, $S_{p,q}$ is isomorphic to $S_{p',q'}$ with $p' \neq p$, $q' \neq q$ and $pq = p'q'$.

If $l_{S_{p,q}} = 2p < q$ and $l_{S_{p',q'}} = q'$ with $q' < 2p'$, by Lemma 2.2 $q' = 2p$, $q = 2p'$ and $2^{2p} = q \binom{2p-1}{p} = q/2 \binom{2p}{p}$ with $p < q/2 < 2p$ and $p, q \geq 6$. If $q/2$ is close to $2p$ then $2^{2p} < q/2 \binom{2p}{p}$. By induction we prove that we have the same inequality when $q/2$ is close to p, that is to say $2^{2p} < (p+1) \binom{2p}{p}$ with $p \geq 6$.

The contradiction in the other cases is obtained as in the previous one from Lemma 2.2. □

We have shown that locally grid graphs are Tutte uniques for $p, q \geq 6$, but our techniques do not apply to $p = 3, 4, 5$. An interesting open problem is to prove that the number $p \binom{p-1}{\delta}$ is not a power of two for $p \geq 6$. We have studied computationally this problem and we have not found any value of p not verifying the previous property. Besides its own interest, this proof would give a more general result about the Tutte uniqueness of $T^{\delta}_{p,q}$ and $S_{p,q}$.

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References

[1] R. Ankney, J.E. Bonin, *Caracterizations of $PG(n-1,q)/PG(k-1,q)$ by numerical and polynomial invariants*, Adv. Appl. Math. **28** (2002), 287-301.

[2] J.E. Bonin, A. de Mier, *T-Uniqueness of some families of k-chordal matroids*, Adv. Appl. Math. **32** (2004), 10-30.

[3] J.E. Bonin, W.P. Miller *Caracterizing combinatorial geometries by numerical invariants*, European J. Combin. **20** (1999), 713-724.

[4] T. Brylawski, J. Oxley, *The Tutte polynomial and its applications, in: N. White (Ed.), Matroid Applications*, Cambridge University Press, Cambridge, (1992).

[5] D. Garijo, *Polinomio de Tutte de Teselaciones Regulares*, Thesis (2004).

[6] J.I. Hall, *Locally Petersen graphs*, J. Graph Theory **4** (1980), 173-187.

[7] K.M. Koh, K.L. Teo, *The search for cromatically unique graphs*, Graphs Comb. **6** (1990), 259-285.

K.M. Koh, K.L. Teo, *The search for cromatically unique graphs II*, Discrete Math. **172** (1997), 59-78.

[8] A. Márquez, A. de Mier, M. Noy, M.P. Revuelta, *Locally grid graphs: Classification and Tutte uniqueness*, Discr. Math. **266** (2003), 327-352.

[9] A. de Mier, M. Noy, *On graphs determined by their Tutte polynomials*, Graphs Combin. **20** (2004), no.1, 105-119.

[10] H. Qin, *Complete principal truncations of Dowling lattices* , Adv. Appl. Math. **32** (2004), 364-379.

[11] I. Sarmiento, *A characterisation of jointless Dowling geometries*, Discrete Math. **197/198** (1999), 713-731.

[12] C. Thomassen, *Tilings of the Torus and the Klein Bottle and vertex-transitive graphs on a fixed surface*, Trans. Amer. Math. Soc. **323**(1991), no.2, 605-635.