On the Crossing Numbers of the k-th Power of P_n *

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Abstract Let P_n be a path with n vertices. P_n^k, the k-th power of the path P_n, is a graph on the same vertex set as P_n and the edges that join all vertices x and y if and only if the distance between them is at most k. In this paper, the crossing numbers of P_n^k are studied. Drawings of P_n^k are presented and proved to be optimal for the case $n \leq 8$ and for the case $k \leq 4$.
Keywords crossing number, the k-th power of path, drawing, non-regular graph

1 Introduction

We consider only finite undirected graphs without loops or multiple edges.

A graph $G = (V, E)$ is a set V of vertices and a subset E of the unordered pairs of vertices, called edges.

A *drawing* is a mapping of a graph into a surface. Vertices go into distinct nodes. An edge and its incident vertices map into a homeomorphic image of the closed interval [0,1] with the relevant nodes as endpoints and the interior, an arc, containing no node. A drawing is *good* if it satisfies (i) no two arcs incident with a common node have a common point; (ii) no two arcs have more than one point in common; (iii) no arc has a self-intersection; and (iv) no three arcs have a point in common. A common

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point of two arcs is a *crossing*. An *optimal drawing* in a given surface is a good drawing which exhibits the least possible number of crossings. This number is the *crossing number* of the graph for the surface. We denote the crossing number of G for the plane (or sphere) by $cr(G)$, a drawing of G in the plane (or sphere) by D, and the number of the crossings in this drawing by $\nu(D)$. It is clear that $cr(G) \leq \nu(D(G))$. We also speak of the nodes as vertices and the arcs as edges. Drawings considered in this paper are all good.

Let P_n be a path with n vertices. P_n^k, the k-th power of the path P_n, is a graph on the same vertex set as P_n and the edges that join all vertices x and y if and only if the distance between them is at most k. P_n is called the *base path* of P_n^k. The graph P_n^k is an important family of non-regular graphs, whose topological properties have been widely studied[1, 2].

The exact crossing number is known only for a few specific families of graphs. Such families include some regular graphs as generalized Petersen graphs $P(n,3)$[3], $P(2n+1,n)$ [4], Circulant graphs $C(n;\{1,3\})$[5], $C(mk;\{1,k\})$[6] and $C(2n+1;\{1,n\})$ [7, 8], Flower Snarks and related graphs[9], and the Cartesian products of cycles $C_m \square C_n$ for all $n \geq m(m+1)$ and for $m \leq 7$ [10, 11]. For non-regular graphs, the Cartesian products of some graphs with path and cycles are determined[12, 13, 14, 15, 16, 17]. We refer to recent surveys for more details[18, 19].

It seems rather interesting to determine the crossing numbers of P_n^k . Its solution can be used to explore an effective approach of the crossing numbers of non-regular graphs to provide a wider theoretical base for the application of the crossing number.

In this paper, the crossing numbers of P_n^k are studied by showing their cylinder drawings. Drawings of P_n^k are presented and proved to be optimal for the case $n \leq 8$ and for the case $k \leq 4$.

Let X be a subset of $V(G)$ or of $E(G)$ for a graph G. Then $\langle X \rangle$ denotes the subgraph of G induced by X.

Let A, B be two disjoint subsets of $E(G)$. In a drawing D, the number of the crossings crossed by an edge in A and another edge in B is denoted by $\nu_D(A,B)$. The number of the crossings that involve a pair of edges in A is denoted by $\nu_D(A)$. So $\nu(D) = \nu_D(E(G))$. If an edge is not crossed by any other edge, we say that it is *clean* in D; if it is crossed by at least one edge, we say that it is *crossed* in D. By counting the number of crossings in D, we have Lemma 1.1.

Lemma 1.1. Let A, B, C be mutually disjoint subsets of $E(G)$. Then,

$$\begin{aligned} \nu_D(C, A \cup B) &= \nu_D(C,A) + \nu_D(C,B), \\ \nu_D(A \cup B) &= \nu_D(A) + \nu_D(B) + \nu_D(A,B). \end{aligned}$$

□

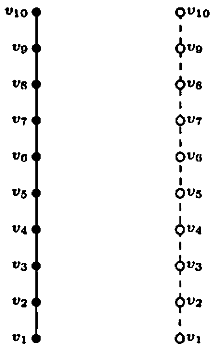


Figure 2.1. $D_{n,1}$

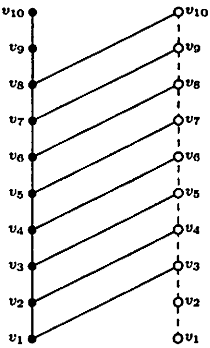


Figure 2.2. $D_{n,2}$

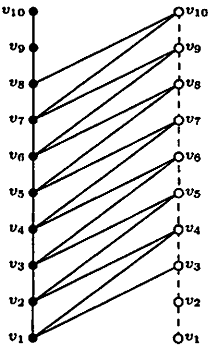


Figure 2.3. $D_{n,3}$

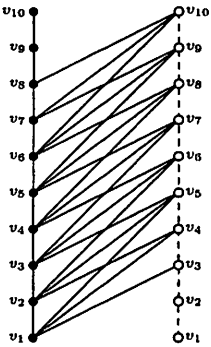


Figure 2.4. $D_{n,4}$

From [20], we have Lemma 1.2.

Lemma 1.2. Let i be the least number of edges of a graph G whose deletion from G results in a planar subgraph H of G, then $cr(G) \geq i$. □

2 $cr(P_n^k)$ for $k = 1, 2, 3, 4, 5, n-2, n-1$

Let $V(P_n^k) = \{v_1, v_2, \cdots, v_n\}$ and $E(P_n^k) = \{v_iv_j \mid n-k+1 \leq i < j \leq n\} \cup (\bigcup_{i=1}^{n-k}\{v_iv_{i+1}, \cdots, v_iv_{i+k}\})$. Then $|E(P_n^k)| = nk - \frac{k(k+1)}{2}$.

In order to state our result, we show a drawing $D_{n,k}$ of P_n^k on a cylinder (homeomorphic to a sphere), which can also be defined as the topological space Pr obtained from the region $\{(x,y) \in R^2 | (x^2 \leq 1) \wedge (y^2 \leq 1)\}$ by identifying the points (x,y) and $(-x,y)$ for every (x,y) such that $x^2 = 1$. In a drawing $D_{n,k}$, if there are some points(curves) on the boundary $|x| = 1$, these points(curves) should be only counted once though they are drawn twice in the drawing. In a drawing $D_{n,k}$, the points $(-1,y)$ and $(1,y)$ should be identified as *one* point.

We put all vertices of P_n^k and the edges in $E(P_n)$ on the boundary $x = -1$ called *left-wall*, then these vertices(edges) should also lie on $x = 1$ called *right-wall*. The edges not in $E(P_n)$ are drawn from left-wall to right-wall by straight line segments(distorted slightly if necessary to avoid concurrence). Figures 2.1-2.4 show the drawings $D_{10,k}$ for $k \leq 4$. By counting the number of crossings in $D_{n,k}$, we have Lemma 2.1.

Lemma 2.1. $cr(P_n^k) \leq \frac{(k-1)(k-2)(k-3)}{6}n - \frac{(3k+4)(k-1)(k-2)(k-3)}{24}$.

Proof. Let $B_i = \{v_iv_{i+j} | \ (2 \leq j \leq k) \wedge (v_iv_{i+j} \in E(P_n^k))\}$ for $1 \leq i \leq n$, then we have

$$|B_i| = \begin{cases} k-1, & 1 \leq i \leq n-k; \\ n-i-1, & n-k+1 \leq i \leq n-1; \\ 0, & i = n. \end{cases}$$

For $1 \le i \le n$, define function $f_B(i) = \sum_{j>i} \nu_{D_{n,k}}(B_i, B_j)$. Let x be an arbitrary crossing in $D_{n,k}$, which occurs between $v_a v_b$ and $v_c v_d$. Let $y = \min\{a, b, c, d\}$, then x is only counted in B_y, and it is counted just once. Hence, $\nu(D_{n,k}) = \sum_{1 \le i \le n} f_B(i)$ by Lemma 1.1.

For $1 \le i \le n$, the edge $v_i v_{i+1+s}$ is crossed by $(s-2)$ edges ($v_{i+1}v_{i+3}$, $\ldots$, $v_{i+1}v_{i+s-1}$, $v_{i+1}v_{i+s}$) in B_{i+1}, crossed by $(s-3)$ edges ($v_{i+2}v_{i+4}$, $\ldots$, $v_{i+2}v_{i+s-1}$, $v_{i+2}v_{i+s}$) in B_{i+2}, $\ldots$, and crossed by one edge ($v_{i+s-2}v_{i+s}$) in B_{i+s-2}. Thus, the edge $v_i v_{i+1+s}$ contributes $\sum_{y=1}^{s-2} y$ crossings to $f_B(i)$, hence we have,

$$f_B(i) = \begin{cases} \sum\limits_{s=3}^{k-1} \sum\limits_{y=1}^{s-2} y = \frac{(k-1)(k-2)(k-3)}{6}, & 1 \le i \le n-k, \\ \sum\limits_{s=3}^{n-i-1} \sum\limits_{y=1}^{s-2} y = \frac{(n-i-1)(n-i-2)(n-i-3)}{6}, & n-k+1 \le i \le n-4, \\ 0, & i \ge n-3. \end{cases}$$

$$\begin{aligned} \text{Hence, } cr(P_n^k) &\le \nu(D_{n,k}) = \textstyle\sum_{i=1}^{n} f_B(i) \\ &= (n-k)\frac{(k-1)(k-2)(k-3)}{6} + \textstyle\sum_{i=n-k+1}^{n-4} \frac{(n-i-1)(n-i-2)(n-i-3)}{6} \\ &= \frac{(k-1)(k-2)(k-3)}{6} n - \frac{(3k+4)(k-1)(k-2)(k-3)}{24}. \end{aligned}$$ □

Theorem 2.2. $cr(P_n^k) = 0$ for $k = 1, 2, 3$ and $cr(P_n^4) = n - 4$ for $n \ge 5$.
Proof. By Lemma 2.1, $cr(P_n^k) \le 0$ for $k = 1, 2, 3$ and $cr(P_n^4) \le n - 4$. We need only to prove $cr(P_n^4) \ge n - 4$.

Let i be the least number of the edges of P_n^4 whose deletion from it results in a planar subgraph Q_n^4 of P_n^4. Consider Q_n^4. Q_n^4 has n vertices, $|E(P_n^4)| = 4n-10$ edges. Let D^* be a planar drawing of Q_n^4 and r denote the number of the faces in D^*. Then, according to Euler Polyhedron Formula,

$$\begin{aligned} n - 4n + 10 + i + r &= 2, \\ r &= 3n - 8 - i. \end{aligned}$$

Since the girth of Q_n^4 is 3, by counting the number of the edges of each face in D^*, we have

$$\begin{aligned} 3 \times (3n - 8 - i) &\le 2 \times (4n - 10 - i), \\ i &\ge n - 4. \end{aligned}$$

So, $cr(P_n^4) = n - 4$ by Lemma 1.2. □

Figures 2.1-2.3 show the planar drawings of P_n^1, P_n^2 and P_n^3. Figure 2.4 shows the drawings of P_n^4 with $n - 4$ crossings.

Notice that $P_n^{n-1} \cong K_n$ and $P_n^{n-2} \cong K_n - e$, hence we have, $cr(P_n^{n-1}) = cr(K_n)$ and $cr(P_n^{n-2}) = cr(K_n - e)$. Since $cr(K_7 - e) = 6$[17], $cr(K_8 - e) = 15$[16] and $cr(K_n) = \lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor\lfloor\frac{n-2}{2}\rfloor\lfloor\frac{n-3}{2}\rfloor$ for $n \le 11$ [19], we have

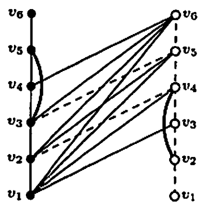


Figure 2.5. $D'_{6,5}$

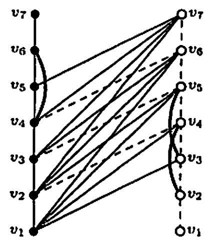


Figure 2.6. $D'_{7,5}$

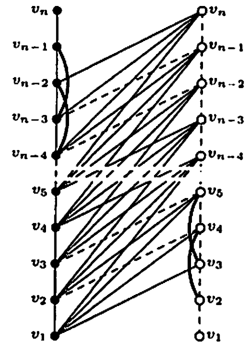


Figure 2.7. $D'_{n,5}(n \geq 8)$

Lemma 2.3. $cr(P_7^5) = 6$, $cr(P_8^6) = 15$ and $cr(P_n^{n-1}) = \frac{n(n-1)(n-2)(n-3)}{4}$ for $n \leq 12$. □

Theorem 2.4. $cr(P_n^5) \leq 4n - 23$ for $n \geq 8$.

Proof. By Lemma 2.1, $cr(P_n^5) \leq \frac{(5-1)(5-2)(5-3)}{6}n - \frac{(3\times5+4)(5-1)(5-2)(5-3)}{24} = 4n-19$. The drawing $D_{n,5}$ of P_n^5 described above can be adjusted slightly to provide a drawing $D'_{n,5}$ with fewer crossings by the following way: for $n \geq 8$, erase all the edges in $\{v_2v_4, v_3v_5, v_{n-1}v_{n-3}, v_{n-2}v_{n-4}\}$ from $D_{n,5}$, then redraw v_2v_4 and v_3v_5 from right-wall to right-wall, and redraw $v_{n-1}v_{n-3}$ and $v_{n-2}v_{n-4}$ from left-wall to left-wall(see Figure 2.7). Thus, we reduce 4 crossings. Hence, $cr(P_n^5) \leq 4n - 23$ for $n \geq 8$. □

Figure 2.5 shows a drawing of P_6^5 with 3 crossings. Figure 2.6 shows a drawing of P_7^5 with 6 crossings. Figure 2.7 shows a drawing of P_n^5 with $4n - 23$ crossings for $n \geq 8$.

Lemma 2.5. $cr(P_8^5) = 9$.

Proof. Let D be a good drawing of P_8^5. By Theorem 2.4, $cr(P_8^5) \leq 9$. It remains to prove that $\nu_D(P_8^5) \geq 9$. To the contrary, suppose that $\nu_D(P_8^5) \leq 8$.

Let $V(P_8^5) = \{v_i | \ 1 \leq i \leq 8\}$ and the base path $P_8 = v_1v_2v_3v_4v_5v_6v_7v_8$. We may divide the edges of P_8^5 into three edge sets: red edge set $E_0 = \{v_iv_j | \ 3 \leq i < j \leq 6\}$, yellow edge set $E_1 = \{v_iv_j | \ i \in \{1,2,7,8\} \wedge j \in \{3,4,5,6\}\}$ and blue edge set $E_2 = \{v_1v_2, v_2v_7, v_7v_8\}$. Note that $\langle E_0 \rangle \cong K_4$ and $\langle E_1 \rangle \cong K_{4,4}$. Let $E_1(i) = \{v_iv_j | \ j \in \{3,4,5,6\}\}(i \in \{1,2,7,8\})$, then $E_1 = \bigcup_{i\in\{1,2,7,8\}} E_1(i)$.

It is well known that there are only two drawings of K_4 in plane within isomorphism, as shown in Figures 2.8(a) and (b).

Case 1. Suppose $\nu_D(E_0) = 0$ (see Figure 2.8(a)). Since $cr(K_{4,4}) = 4$, $\nu_D(E_1) \geq 4$. Since there are only three of the four vertices v_3, v_4, v_5 and v_6 on each boundary of the drawing, we have $\nu_D(E_0, E_1) \geq 4$. From the hypothesis $\nu(D) \leq 8$, we have $\nu_D(E_1) = 4$, $\nu_D(E_0, E_1) = 4$ and $\nu_D(E_2) +$

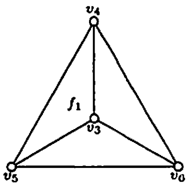


Figure 2.8(a)
$\nu_D(E_0) = 0$

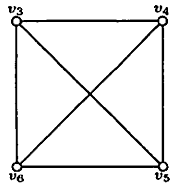


Figure 2.8(b)
$\nu_D(E_0) = 1$

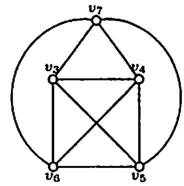


Figure 2.8(c)

$\nu_D(E_2, E_0 \cup E_1) = 0$. Thus, all the edges in E_2 have to lie in the same region, without loss of generality, we may assume that they are all in f_1. The four yellow edges incident with v_6 are all crossed by some red edge in $\{v_3v_4, v_4v_5, v_5v_3\}$, so, all the yellow edges in $\{v_iv_j| \ i \in \{1,2,7,8\} \wedge j \in \{3,4,5\}\}$ have to lie in f_1. For every pair vertices v_x, v_y of $\{v_1, v_2, v_7, v_8\}$, we have $\nu_D(\{v_xv_j|j \in \{3,4,5\}\}, \{v_yv_j| \ j \in \{3,4,5\}\}) \geq 1$, thus, $\nu_D(E_1) \geq \binom{4}{2} = 6$, a contradiction to $\nu_D(E_1) = 4$.

Case 2. Suppose $\nu_D(E_0) = 1$ (see Figure 2.8(b)). Since $cr(K_{4,4}) = 4$, $\nu_D(E_1) \geq 4$. If some vertex v_i of $\{v_1, v_2, v_7, v_8\}$ does not lie in f_1, then $\nu_D(E_1(i), E_0) \geq 2$. Then from the hypothesis $\nu(D) \leq 8$, there are at least three vertices of $\{v_1, v_2, v_7, v_8\}$ in f_1. We may assume that v_1, v_2 and v_7 are all lie in f_1.

If some yellow edge in $E_1(i)(i \in \{1,2,7\})$ crosses the red edges of E_0, then $\nu_D(E_1(i), E_0) \geq 2$. Hence there is at least one $i \in \{1,2,7\}$, say $i = 7$, such that all the four yellow edges in $E_1(7)$ do not cross the red edges (see Figure 2.8(c)). (Otherwise, we would have $\nu_D(\bigcup_{i\in\{1,2,7\}} E_1(i), E_0) \geq 6$ and $\nu(D) \geq \nu_D(E_1) + \nu_D(\bigcup_{i\in\{1,2,7\}} E_1(i), E_0) \geq 4 + 6 = 10$, a contradiction to $\nu(D) \leq 8$.)

There are at most two of the four vertices v_3, v_4, v_5 and v_6 on each boundary of the drawing in Figure 2.8(c), then for every $i \in \{1,2,8\}$ we have $\nu_D(E_1(i), E_0 \cup E_1(7)) \geq 2$. Hence we have $\nu_D(\bigcup_{i\in\{1,2,8\}} E_1(i), E_0 \cup E_1(7)) \geq 6$. Notice that the induced subgraph $\langle \bigcup_{i\in\{1,2,8\}} E_1(i) \rangle \cong K_{3,4}$ and $cr(K_{3,4}) = 2$, we would have $\nu(D) \geq \nu_D(E_0) + \nu_D(\bigcup_{i\in\{1,2,8\}} E_1(i)) + \nu_D(\bigcup_{i\in\{1,2,8\}} E_1(i), E_0 \cup E_1(7)) \geq 1+2+6 = 9$, a contradiction to $\nu(D) \leq 8$.

From Cases 1-2, the hypnosis $\nu_D(P_8^5) \leq 8$ is incorrect. So, $\nu_D(P_8^5) = 9$. □

3 Upper bounds of $cr(P_n^k)$

Lemma 3.1.

(1) $cr(P_n^k) \leq \frac{k^3-9k^2+32k-36}{6} n - \frac{3k^4-22k^3+45k^2+82k-216}{24}$ for even n and even k.

(2) $cr(P_n^k) \le \frac{k^3-9k^2+32k-36}{6}n - \frac{3k^4-22k^3+45k^2+82k-204}{24}$ for even n and odd k.

(3) $cr(P_n^k) \le \frac{k^3-9k^2+32k-36}{6}n - \frac{3k^4-22k^3+45k^2+94k-240}{24}$ for odd n.

Proof. For $k \ge 6$, the drawing $D_{n,k}$ of P_n^k can be modified to get a new drawing $ND_{n,k}$ as follows: erase the edges $v_iv_{i+2}(1 \le i \le n-2)$ from the drawing, then redraw the edges v_iv_{i+2} from left-wall to left-wall for odd i and from right-wall to right-wall for even i. Figure 3.1 shows the process modifying $D_{n,k}$ to $ND_{n,k}$ for $k = 6, 7$.

Let $B_i = \{v_iv_{i+j} | (3 \le j \le k) \wedge (v_iv_{i+j} \in E(P_n^k))\}$ and $R_i = \{v_iv_{i+2} | v_iv_{i+2} \in E(P_n^k)\}$ for $1 \le i \le n$. We have

$$|B_i| = \begin{cases} k-2, & 1 \le i \le n-k; \\ n-2-i, & n-k+1 \le i \le n-3; \\ 0, & n-2 \le i \le n. \end{cases} \tag{3.1}$$

For $1 \le i \le n$, define functions $g_B(i) = \sum_{j>i} \nu_{ND_{n,k}}(B_i, B_j)$ and $g_R(i) = \sum_{j=1}^{n} \nu_{ND_{n,k}}(R_i, B_j)$. Each crossing between an edge in R_x and another edge in B_y is only counted in $g_R(x)$. Let $x_1 < x_2$, then each crossing between an edge in B_{x_1} and another edge in B_{x_2} is only counted in $g_B(x_2)$. Hence, each crossing in $ND_{n,k}$ is counted just once, by Lemma 1.1,

$$\nu(ND_{n,k}) = \sum_{1\le i\le n}(g_B(i) + g_R(i)). \tag{3.2}$$

Compute $\sum_{1\le i\le n} g_B(i)$ firstly.

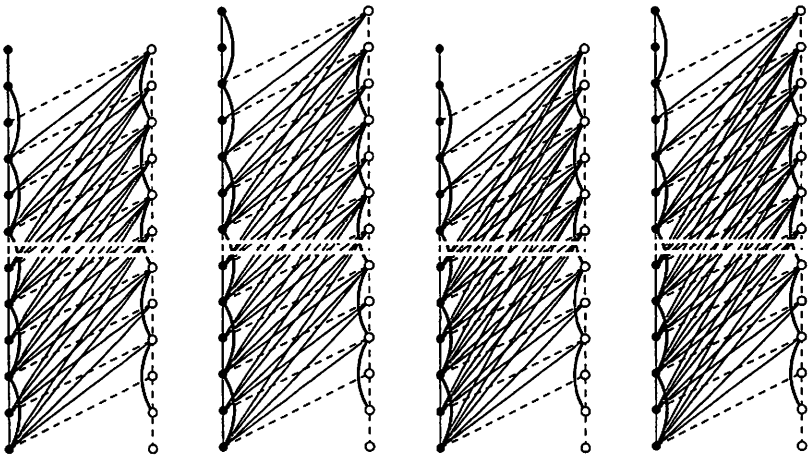
For $1 \le i \le n$, the edge v_iv_{i+2+s} is crossed by $(s-2)$ edges $(v_{i+1}v_{i+4}, \ldots, v_{i+1}v_{i+s}, v_{i+1}v_{i+s+1})$ in B_{i+1}, crossed by $(s-3)$ edges $(v_{i+2}v_{i+5}, \ldots, v_{i+2}v_{i+s}, v_{i+2}v_{i+s+1})$ in $B_{i+2}, \ldots$, and crossed by one edge $(v_{i+s-2}v_{i+s+1})$ in B_{i+s-2}. Then, the edge v_iv_{i+2+s} contributes $\sum_{y=1}^{s-2} y$ crossings to $g_B(i)$, hence we have,

$$g_B(i) = \begin{cases} \sum\limits_{s=1}^{k-2}\sum\limits_{y=1}^{s-2} y = \frac{(k-2)(k-3)(k-4)}{6}, & 1 \le i \le n-k, \\ \sum\limits_{s=1}^{n-i-2}\sum\limits_{y=1}^{s-2} y = \frac{(n-i-2)(n-i-3)(n-i-4)}{6}, & n-k+1 \le i \le n-5, \\ 0, & i \ge n-4. \end{cases}$$

Hence,

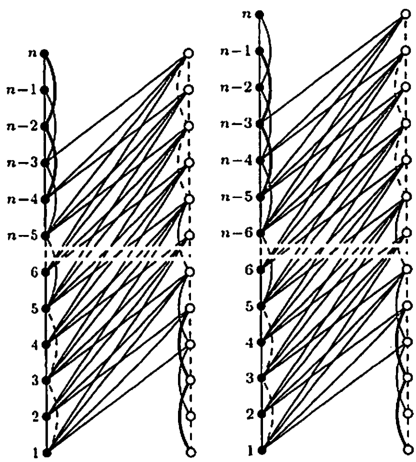
$$\begin{aligned} \sum_{i=1}^{n} g_B(i) &= (n-k)\frac{(k-2)(k-3)(k-4)}{6} + \sum_{i=n-k+1}^{n-5} \frac{(n-i-2)(n-i-3)(n-i-4)}{6} \\ &= \frac{(k-2)(k-3)(k-4)}{6}n - \frac{(3k+5)(k-2)(k-3)(k-4)}{24}. \end{aligned} \tag{3.3}$$

Then, compute $\sum_{1\le i\le n} g_R(i)$.



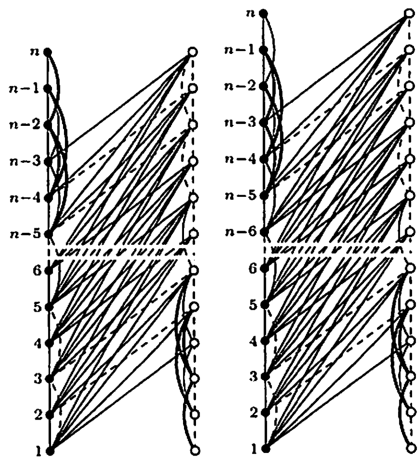
(a) P_{2t}^{6} (b) P_{2t-1}^{6} (c) P_{2t}^{7} (d) P_{2t-1}^{7}

Figure 3.1. Adjust $D_{n,k}$ to $ND_{n,k}$ for $k = 6, 7$ by adjusting dashed curves to bold ones



(a) $ND'_{2t,6}$ (b) $ND'_{2t-1,6}$

Figure 3.2 $ND'_{n,6}(n \geq 10)$



(a) $ND'_{2t,7}$ (b) $ND'_{2t-1,7}$

Figure 3.3 $ND'_{n,7}(n \geq 12)$

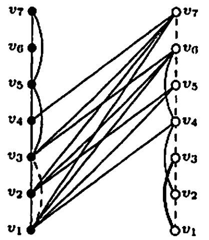


Figure 3.4(a) $ND'_{7,6}$

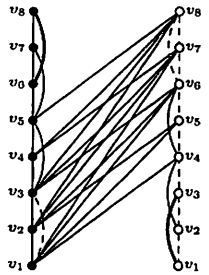


Figure 3.4(b) $ND'_{8,6}$

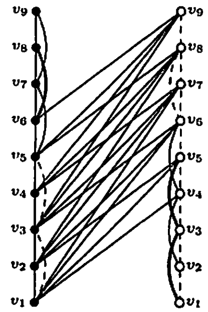


Figure 3.4(c) $ND'_{9,6}$

For odd i, the edge v_iv_{i+2} is crossed by $|B_{i+1}|$ edges incident with v_{i+1}, and for even i, v_iv_{i+2} is crossed by $|B_{n-i}|$ edges incident with v_{i+1}, i.e., $g_R(i) = |B_{i+1}|$ for odd i and $g_R(i) = |B_{n-i}|$ for even i. Hence,

$$\begin{aligned}\sum_{i=1}^{n} g_R(i) &= \sum_{i=1}^{n-2} g_R(i) = \sum_{j=1}^{\lceil \frac{n}{2}\rceil -1} g_R(2j-1) + \sum_{j=1}^{\lfloor \frac{n}{2}\rfloor -1} g_R(2j) \\ &= \sum_{j=1}^{\lceil \frac{n}{2}\rceil -1} |B_{2j}| + \sum_{j=1}^{\lfloor \frac{n}{2}\rfloor -1} |B_{n-2j}|. \qquad (3.4)\end{aligned}$$

Case 1. If n is even, then from (3.4),

$$\sum_{i=1}^{n} g_R(i) = \sum_{j=1}^{\frac{n}{2}-1} |B_{2j}| + \sum_{j=1}^{\frac{n}{2}-1} |B_{n-2j}| = 2\sum_{j=1}^{\frac{n}{2}-1} |B_{2j}|.$$

Case 1.1. If k is even, then from the equation (3.1), we have

$$\begin{aligned}\sum_{i=1}^{n} g_R(i) &= 2\sum_{j=1}^{\frac{n}{2}-1} |B_{2j}| = 2\sum_{j=1}^{\frac{n-k}{2}} |B_{2j}| + 2\sum_{j=\frac{n-k}{2}+1}^{\frac{n}{2}-1} |B_{2j}| \\ &= 2\sum_{j=1}^{\frac{n-k}{2}} (k-2) + 2\sum_{j=\frac{n-k}{2}+1}^{\frac{n}{2}-1} (n-2-2j) \\ &= (k-2)(n-k) + \frac{(k-2)(k-4)}{2} \\ &= (k-2)n - \frac{(k-2)(k+4)}{2}. \qquad (3.5)\end{aligned}$$

Case 1.2. If k is odd, then from (3.1), we have,

$$\begin{aligned}\sum_{i=1}^{n} g_R(i) &= 2\sum_{j=1}^{\frac{n}{2}-1} |B_{2j}| = 2\sum_{j=1}^{\frac{n-k-1}{2}} |B_{2j}| + 2\sum_{j=\frac{n-k-1}{2}+1}^{\frac{n}{2}-2} |B_{2j}| \\ &= 2\sum_{j=1}^{\frac{n-k-1}{2}} (k-2) + 2\sum_{j=\frac{n-k-1}{2}+1}^{\frac{n}{2}-2} (n-2-2j) \\ &= (k-2)(n-k-1) + \frac{(k-1)(k-3)}{2} \\ &= (k-2)n - \frac{(k-2)(k+4)+1}{2}. \qquad (3.6)\end{aligned}$$

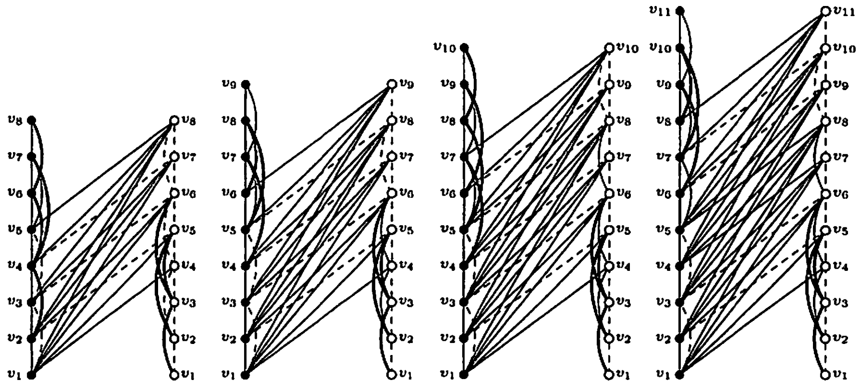


Figure 3.5(a) $ND'_{8,7}$ Figure 3.5(b) $ND'_{9,7}$ Figure 3.5(c) $ND'_{10,7}$ Figure 3.5(d) $ND'_{11,7}$

Case 2. If n is odd, then from (3.4) we have, $\sum_{i=1}^{n} g_R(i) = \sum_{j=1}^{\frac{n-1}{2}} |B_{2j}| + \sum_{j=1}^{\frac{n-3}{2}} |B_{n-2j}|$. From (3.1), we have

$$\begin{aligned}\sum_{i=1}^{n} g_R(i) &= \sum_{j=1}^{\frac{n-1}{2}} |B_{2j}| + \sum_{j=1}^{\frac{n-3}{2}} |B_{n-2j}| = \sum_{j=2}^{n-3} |B_j| \\ &= \sum_{j=2}^{n-k}(k-2) + \sum_{j=n-k+1}^{n-3}(n-2-j) \\ &= (k-2)(n-k-1) + \tfrac{(k-2)(k-3)}{2} \\ &= (k-2)n - \tfrac{(k-2)(k+5)}{2}. \qquad (3.7)\end{aligned}$$

From equations (3.2), (3.3), (3.5), (3.6) and (3.7) we have:

(1) $cr(P_n^k) \leq \frac{k^3-9k^2+32k-36}{6}n - \frac{3k^4-22k^3+45k^2+82k-216}{24}$ for even n and even k.

(2) $cr(P_n^k) \leq \frac{k^3-9k^2+32k-36}{6}n - \frac{3k^4-22k^3+45k^2+82k-204}{24}$ for even n and odd k.

(3) $cr(P_n^k) \leq \frac{k^3-9k^2+32k-36}{6}n - \frac{3k^4-22k^3+45k^2+94k-240}{24}$ for odd n. □

Theorem 3.2. (1) $cr(P_n^6) \leq 8n - 51$ for $n \geq 10$.

(2) $cr(P_n^7) \leq 15n - 109$ for $n \geq 12$.

Proof. By Lemma 3.1, $\nu(ND_{n,6}) = 8n - 43$ for even n, $\nu(ND_{n,6}) = 8n - 45$ for odd n. The drawing $ND_{n,6}$ can be adjusted slightly to provide a drawing $ND'_{n,6}$ with fewer crossings as follows: for $n \geq 10$, erase all the edges in $\{v_1v_3, v_3v_5, v_{2\lfloor\frac{n}{2}\rfloor-2}v_{2\lfloor\frac{n}{2}\rfloor}, v_{2\lfloor\frac{n}{2}\rfloor-4}v_{2\lfloor\frac{n}{2}\rfloor-2}\}$, then, redraw v_1v_3 and v_3v_5 from right-wall to right-wall, and redraw $v_{2\lfloor\frac{n}{2}\rfloor-2}v_{2\lfloor\frac{n}{2}\rfloor}$ and $v_{2\lfloor\frac{n}{2}\rfloor-4}v_{2\lfloor\frac{n}{2}\rfloor-2}$ from left-wall to left-wall (see Figure 3.2, where i stands for v_i). Thus, we reduce 8 crossings for even n and can reduce 6 crossings for odd n. Hence, $\nu(ND'_{n,6}) = 8n - 51$ for $n \geq 10$.

By Lemma 3.1, $\nu(ND_{n,7}) = 15n - 93$ for even n, and $\nu(ND_{n,7}) = 15n - 95$ for odd n. The drawing $ND_{n,7}$ of P_n^7 can be adjusted slightly to provide a drawing $ND'_{n,7}$ with fewer crossings as follows: for $n \geq 12$, erase all the edges in $\{v_iv_{i+2} | \ i \in \{1, 3, 2\lfloor\frac{n}{2}\rfloor - 4, 2\lfloor\frac{n}{2}\rfloor - 2\} \cup \{v_iv_{i+3} | \ i \in \{2, 3, n-5, n-4\}\}$, then, redraw v_1v_3, v_2v_5, v_3v_5 and v_3v_6 from right-wall to right-wall, and redraw $v_{2\lfloor\frac{n}{2}\rfloor-4}v_{2\lfloor\frac{n}{2}\rfloor-2}$, $v_{2\lfloor\frac{n}{2}\rfloor-2}v_{2\lfloor\frac{n}{2}\rfloor}$, $v_{n-5}v_{n-2}$ and $v_{n-4}v_{n-1}$ from left-wall to left-wall (see Figure 3.3, where i stands for v_i). Thus, we reduce 16 crossings for even n and can reduce 14 crossings for odd n. By counting, $\nu(ND'_{n,7}) = 15n - 109$ for $n \geq 12$. □

Figure 3.4(a) shows a drawing of P_7^6 with 9 crossings, Figure 3.4(b) shows a drawing of P_8^6 with 15 crossings and Figure 3.4(c) shows a drawing of P_9^6 with 22 crossings. Figure 3.5(a) shows a drawing of P_8^7 with 18 crossings, Figure 3.5(b) shows a drawing of P_9^7 with 30 crossings and Figure 3.5(c) shows a drawing of P_{10}^7 with 42 crossings and Figure 3.5(d) shows a drawing of P_{11}^7 with 57 crossings. Hence, we have Lemma 3.3.

Lemma 3.3 $cr(P_9^6) \leq 22$, $cr(P_9^7) \leq 30$, $cr(P_{10}^7) \leq 42$ and $cr(P_{11}^7) \leq 57$. □

4 Concluding remarks

We show the values of $cr(P_n^k)$ for $n \leq 15, k \leq 7$ in Table 4.1.

Table 4.1. The crossing numbers of P_n^k for $n \leq 15$, $k \leq 7$

n / k	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1		0	0	0	0	0	0	0	0	0	0	0	0	0	0
2			0	0	0	0	0	0	0	0	0	0	0	0	0
3				0	0	0	0	0	0	0	0	0	0	0	0
4					1[19]	2	3	4	5	6	7	8	9	10	11
5						3[19]	6[17]	9	≤ 13	≤ 17	≤ 21	≤ 25	≤ 29	≤ 33	≤ 37
6							9[19]	15[16]	≤ 22	≤ 29	≤ 37	≤ 45	≤ 53	≤ 61	≤ 69
7								18[19]	≤ 30	≤ 42	≤ 57	≤ 71	≤ 86	≤ 101	≤ 116

By Theorems 2.2, 2.4, 3.2 and Lemmas 2.3, 2.5, 3.3, we have

(1) $cr(P_n^k) = 0$ for $k \leq 3$,

(2) $cr(P_n^4) = n - 4$,

(3) $cr(P_6^5) = 3$, $cr(P_7^5) = 6$, $cr(P_8^5) = 9$, $cr(P_n^5) \leq 4n - 23$ for $n \geq 9$,

(4) $cr(P_7^6) = 9$, $cr(P_8^6) = 15$, $cr(P_9^6) \leq 22$, $cr(P_n^6) \leq 8n - 51$ for $n \geq 10$,

(5) $cr(P_8^7) = 18$, $cr(P_9^7) \leq 30$, $cr(P_{10}^7) \leq 42$, $cr(P_{11}^7) \leq 57$, $cr(P_n^7) \leq$

$15n-109$ for $n \geq 12$.

Furthermore, we have the following Conjectures:

Conjecture 4.1. $cr(P_n^5) = 4n - 23$ for $n \geq 8$.

Conjecture 4.2. $cr(P_9^6) = 22$, $cr(P_n^6) = 8n - 51$ for $n \geq 10$.

Conjecture 4.3. $cr(P_9^7) = 30$, $cr(P_{10}^7) = 42$, $cr(P_{11}^7) = 57$, $cr(P_n^7) = 15n - 109$ for $n \geq 12$.

By Lemma 2.5, Conjecture 4.1 holds for $n = 8$.

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