On the Gracefulness of the Digraphs $n - \vec{C}_m$ *

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Abstract

A digraph $D(V, E)$ is said to be graceful if there exists an injection $f : V(G) \to \{0, 1, \cdots, |E|\}$ such that the induced function $f' : E(G) \to \{1, 2, \cdots, |E|\}$ which is defined by $f'(u, v) = [f(v) - f(u)] \pmod{|E| + 1}$ for every directed edge (u, v) is a bijection. Here, f is called a graceful labeling (graceful numbering) of $D(V, E)$, while f' is called the induced edge's graceful labeling of D. In this paper we discuss the gracefulness of the digraph $n - \vec{C}_m$ and prove that $n - \vec{C}_m$ is a graceful digraph for $m = 4, 6, 8, 10$ and even n.

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1 Introduction

A graph $G(V, E)$ is said to be graceful if there exists an injection $f: V(G) \rightarrow \{0, 1, \cdots, |E|\}$ such that the induced function $f^{'}: E(G) \rightarrow \{1, 2, \cdots, |E|\}$ which is defined by $f^{'}(u, v) = |f(u) - f(v)|$ for every edge (u, v) is a bijection. Here, f is called a graceful labeling(graceful numbering) of G, while $f^{'}$ is called the induced edge's graceful labeling of G. A digraph $D(V, E)$ is said to be graceful if there exists an injection $f: V(G) \rightarrow \{0, 1, \cdots, |E|\}$ such that the induced function $f^{'}: E(G) \rightarrow \{1, 2, \cdots, |E|\}$ which is defined by $f^{'}(u, v) = [f(v) - f(u)] \pmod{|E|+1}$ for every directed edge (u, v) is a bijection, where $[v] \pmod{u}$ denotes the least positive residue of v modulo n. In this case, f is called a graceful labeling (graceful numbering) of D and $f^{'}$ is called the induced edge's graceful labeling of D ([3]).

Let C_m and $\vec{C}_m$ denote the cycle and directed cycle on m vertices, respectively, $n \cdot C_n$ and $n - C_m$ denote the graphs obtained from any n copies of C_m which have just one common vertex and have just one common edge, respectively. At the same time, let $n \cdot \vec{C}_m$ and $n - \vec{C}_m$ denote the digraphs obtained from any n copies of the directed cycle $\vec{C}_m$ which have just one common vertex and have just one common edge, respectively.

As to the gracefulness of $n \cdot \vec{C}_m$ we know the following results: Ma proved in [3] that the gracefulness of $n \cdot \vec{C}_3$ implies that n is even, at same times he conjectured that the condition that n is even was also sufficient for $n \cdot \vec{C}_3$ to be graceful. Jirimutu has showed this conjecture in [5]. It was showed that $n \cdot \vec{C}_{2k}$ is graceful for every integer $n \geq 1$ and $k \geq 1$ in [6], and $n \cdot \vec{C}_{2k+1}$ is graceful for n even and $k = 2, 3$ in [7].

About the gracefulness of $n - \vec{C}_m$, to our knowledge, there are no so much result: It was showed in [3] that $n - \vec{C}_3$ is graceful when n is even, and it was proved in [6] that the necessary condition for $n - \vec{C}_m$ to be graceful is $mn \equiv 0 \pmod{2}$.

In this paper, we will discuss the gracefulness of the digraph $n - \vec{C}_m$ and prove the digraph $n - \vec{C}_m$ is graceful if $m = 4, 6, 8, 10$ and n is even.

2 Main Results

Let $\vec{C}_m^1, \vec{C}_m^2, \cdots, \vec{C}_m^n$ denote the n directed cycles in $n - \vec{C}_m$. The two vertices of the common edge of $\vec{C}_m^i$'s are denoted by v_0 and v_{m-1}, and other

$m-2$ vertices of the $\vec{C}^i_m$ are denoted by v^i_j $(j = 1, \dots, m-2; i = 1, 2, \cdots, n)$, respectively. For convenience, we put $v^1_0 = v^2_0 = \cdots = v^n_0 = v_0, v^1_{m-1} = v^2_{m-1} = \cdots = v^n_{m-1} = v_{m-1}$, and take subscripts j's modulo m. Obviously, $|E(n - \vec{C}_m)| = (m-1)n + 1$.

Suppose that $n - \vec{C}_m$ is graceful and f and f' are its graceful labeling and the induced edge's graceful labeling, respectively.

For every i, it is easy to see that

$$\sum_{j=0}^{m-1} [f(v^i_j) - f(v^i_{j-1})] \equiv \sum_{j=0}^{m-1} f(v^i_j) - \sum_{j=0}^{m-1} f(v^i_{j-1}) = 0 \pmod{((m-1)n+2)},$$

which means that there is an integer k_i such that

$$\sum_{j=0}^{m-1} [f(v^i_j) - f(v^i_{j-1})] = k_i((m-1)n+2), \ (1 \le i \le n). \tag{1}$$

This implies that there is an integer k such that

$$\sum_{i=1}^{n} \sum_{j=0}^{m-1} [f(v^i_j) - f(v^i_{j-1})] = k((m-1)n+2). \tag{2}$$

On the other hand, setting $q = |E(n - \vec{C}_m)| = (m-1)n + 1$ and $d = [f(v_0) - f(v_{m-1})]$, by definition we have

$$\sum_{i=1}^{n} \sum_{j=0}^{m-1} [f(v^i_j) - f(v^i_{j-1})] = (n-1)d + \frac{1}{2}q(q+1) = k(q+1). \tag{3}$$

From the above discussion we obtain the necessary condition as follows.

$$(n-1)d \equiv 0 \pmod{\frac{q+1}{2}}. \tag{4}$$

In the argument below we always take $f(v_0) = 0$ and $f(v_{m-1}) = \frac{q+1}{2}$. Thus, $d = [f(v_0) - f(v_{m-1})] = [-\frac{q+1}{2}] \equiv \frac{q+1}{2} \pmod{q+1}$, which satisfies the condition given in (4).

Theorem 1 *For every even integer n, the digraph $n - \vec{C}_4$ is graceful.*

Proof. We have had $f(v_0) = 0$ and $f(v_3) = \frac{3n}{2} + 1$. For other vertices, define:

$$f(v^i_1) = i, \quad 1 \le i \le n,$$

$$f(v^i_2) = \begin{cases} \frac{3n}{2} + 1 + 2i, & 1 \le i \le \frac{n}{2}, \\ \frac{n}{2} + 2i, & \frac{n}{2} + 1 \le i \le n. \end{cases}$$

Firstly, we show that f is an injective mapping from $V(n-\vec{C}_4)$ into $\{0,1,\cdots,3n+1\}$.
Put $S_j=\{f(v_j^i)|1\le i\le n\}$, $0\le j\le 3$. Then

$$\begin{aligned}
S_0 &= \{f(v_0)\}=\{0\},\\
S_1 &= \{f(v_1^i)|1\le i\le n\}=\{i|1\le i\le n\}=\{1,2,\cdots n\},\\
S_2 &= \{f(v_2^i)|1\le i\le n\}=\{\tfrac{3n}{2}+1+2i|1\le i\le \tfrac{n}{2}\}\cup\{\tfrac{n}{2}+2i|\tfrac{n}{2}+1\le i\le n\},\\
&= \{\tfrac{3n}{2}+3,\tfrac{3n}{2}+5,\cdots,\tfrac{3n}{2}+n+1\}\cup\{\tfrac{n}{2}+n+2,\tfrac{n}{2}+n+4,\cdots,\tfrac{n}{2}+2n\}\\
S_3 &= \{f(v_3)\}=\{\tfrac{3n}{2}+1\}.
\end{aligned}$$

Hence, $S_i\cap S_j=\varnothing$ for $i\neq j, i,j\in\{0,1,2,3\}$, which yields that f is an injection from $V(n-\vec{C}_4)$ into $\{0,1,\cdots,3n+1\}$.

Secondly, we show the induced edges labeling f' is a bijective mapping from $E(n-\vec{C}_4)$ onto $\{1,2,\cdots,3n+1\}$.

Set $B_j=\{[f(v_{j+1}^i)-f(v_j^i)] \pmod{3n+2}|1\le i\le n\}$, $0\le j\le 3$, and $B=B_0\bigcup B_1\bigcup B_2\bigcup B_3$. Then

$$\begin{aligned}
B_0 &= \{[f(v_1^i)-f(v_0)] \pmod{3n+2}|1\le i\le n\}=\{1,2,\cdots,n\},\\
B_1 &= \{[f(v_2^i)-f(v_1^i)] \pmod{3n+2}|1\le i\le n\}\\
&= B_{11}\cup B_{12},\ \text{where}\\
&\quad B_{11}=\{\tfrac{3n}{2}+1+2i-i|1\le i\le\tfrac{n}{2}\}=\{\tfrac{3n}{2}+2,\tfrac{3n}{2}+3,\cdots,2n+1\},\\
&\quad B_{12}=\{\tfrac{n}{2}+2i-i|\tfrac{n}{2}+1\le i\le n\}=\{n+1,n+2,\cdots,n+\tfrac{n}{2}\},\\
B_2 &= \{[f(v_3)-f(v_2^i)] \pmod{3n+2}|1\le i\le n\}\\
&= B_{21}\bigcup B_{22},\ \text{where}\\
&\quad B_{21}=\{[\tfrac{3n}{2}+1-(\tfrac{3n}{2}+1+2i)] \pmod{3n+2}|1\le i\le\tfrac{n}{2}\}\\
&\qquad =\{3n,3n-2,\cdots,2n+2\},\\
&\quad B_{22}=\{[\tfrac{3n}{2}+1-(\tfrac{n}{2}+2i)] \pmod{(3n+2)}|\tfrac{n}{2}+1\le i\le n\}\\
&\qquad =\{3n+1,3n-1,\cdots,2n+3\},\\
B_3 &= \{[f(v_0)-f(v_3)] \pmod{3n+2}\}=\{\tfrac{3n}{2}+1\}.
\end{aligned}$$

Hence,

$$\begin{aligned}
B &= B_0\bigcup B_1\bigcup B_2\bigcup B_3=B_0\bigcup B_{12}\bigcup B_3\bigcup B_{11}\bigcup B_{21}\bigcup B_{22}\\
&= \{1,2,\cdots,n\}\bigcup\{n+1,n+2,\cdots,n+\tfrac{n}{2}\}\bigcup\{\tfrac{3n}{2}+1\}\\
&\quad \bigcup\{\tfrac{3n}{2}+2,\tfrac{3n}{2}+3,\cdots,2n+1\}\bigcup\{2n+2,2n+4,\cdots,3n\}\\
&\quad \bigcup\{2n+3,2n+5,\cdots,3n+1\}\\
&= \{1,2,\cdots,3n+1\},
\end{aligned}$$

which implies that f' is surjective, hence, bijective. So we prove that $n-\vec{C}_4$ is a graceful digraph for even n. □

Theorem 2 *For every even integer n, the digraph $n-\vec{C}_6$ is graceful.*

Proof. Define
$$f(v_0)=0, f(v_5)=\frac{5n}{2}+1$$
and
$$f(v_j^i)=\begin{cases} \frac{j-1}{2}n+i, & j=1,3,\ 1\le i\le \frac{n}{2};\\ 4n+1-\frac{j-1}{2}(\frac{3n}{2}+1)+i, & j=1,3,\ \frac{n}{2}+1\le i\le n;\\ nj-\frac{n-j}{2}-1+2i, & j=2,4,\ 1\le i\le \frac{n}{2};\\ n(j-1)-\frac{n-j}{2}-2+2i, & j=2,4,\ \frac{n}{2}+1\le i\le n.\end{cases}$$

Firstly, we show that f is an injective mapping from $V(n-\vec{C}_6)$ into $\{0,1,\cdots,5n+1\}$.

Set $S_j=\{f(v_j^i)|1\le i\le n\}$, $0\le j\le 5$ and $S=S_0\bigcup S_1\bigcup S_2\bigcup S_3\bigcup S_4\bigcup S_5$. Then

$$\begin{aligned}
S_0 &= \{f(v_0)\}=\{0\}\\
S_1 &= \{f(v_1^i)|1\le i\le n\}\\
&= S_{11}\bigcup S_{12}, \text{ where}\\
&\quad S_{11}=\{f(v_1^i)|\tfrac{n}{2}+1\le i\le n\}=\{\tfrac{9n}{2}+2,\tfrac{9n}{2}+3,\cdots,5n+1\},\\
&\quad S_{12}=\{f(v_1^i)|1\le i\le \tfrac{n}{2}\}=\{1,2,\cdots,\tfrac{n}{2}\}\\
S_2 &= \{f(v_2^i)|1\le i\le n\}\\
&= S_{21}\cup S_{22}, \text{ where}\\
&\quad S_{21}=\{f(v_2^i)|1\le i\le \tfrac{n}{2}\}=\{2n-\tfrac{n}{2}+2,2n-\tfrac{n}{2}+4,\ldots,3n-\tfrac{n}{2}\},\\
&\quad S_{22}=\{f(v_2^i)|\tfrac{n}{2}+1\le i\le n\}\\
&\quad\quad =\{2n-\tfrac{n}{2}+1,2n-\tfrac{n}{2}+3,\ldots,3n-\tfrac{n}{2}-1\},\\
S_3 &= \{f(v_3^i)|1\le i\le n\}\\
&= S_{31}\bigcup S_{32}, \text{ where}\\
&\quad S_{31}=\{f(v_3^i)|1\le i\le \tfrac{n}{2}\}=\{n+1,n+2,\ldots,n+\tfrac{n}{2}\},\\
&\quad S_{32}=\{f(v_3^i)|\tfrac{n}{2}+1\le i\le n\}=\{3n+1,3n+2,\cdots,4n-\tfrac{n}{2}\},\\
S_4 &= \{f(v_4^i)|1\le i\le n\}\\
&= S_{41}\cup S_{42}, \text{ where}\\
&\quad S_{41}=\{f(v_4^i)|1\le i\le \tfrac{n}{2}\}=\{4n-\tfrac{n}{2}+3,4n-\tfrac{n}{2}+5,\ldots,5n-\tfrac{n}{2}+1\},\\
&\quad S_{42}=\{f(v_4^i)|\tfrac{n}{2}+1\le i\le n\}=\{4n-\tfrac{n}{2}+2,4n-\tfrac{n}{2}+4,\ldots,5n-\tfrac{n}{2}\},\\
S_5 &= \{f(v_5)\}=\{\tfrac{5n}{2}+1\}.
\end{aligned}$$

Hence,

$$\begin{aligned}
& S_0 \bigcup S_1 \bigcup S_2 \bigcup S_3 \\
& = S_0 \bigcup S_{12} \bigcup S_{31} \bigcup S_{22} \bigcup S_{21} \bigcup S_5 \bigcup S_{32} \bigcup S_{42} \bigcup S_{41} \bigcup S_{11} \\
& = \{0\} \bigcup \{1, 2, \cdots, \tfrac{n}{2}\} \bigcup \{n+1, n+2, \cdots, n+\tfrac{n}{2}\} \\
& \quad \bigcup \{2n - \tfrac{n}{2} + 1, 2n - \tfrac{n}{2} + 3, \ldots, 3n - \tfrac{n}{2} - 1\} \\
& \quad \bigcup \{2n - \tfrac{n}{2} + 2, 2n - \tfrac{n}{2} + 4, \ldots, 3n - \tfrac{n}{2}\} \\
& \quad \bigcup \{\tfrac{5n}{2} + 1\} \bigcup \{3n+1, 3n+2, \cdots, 4n - \tfrac{n}{2}\} \\
& \quad \bigcup \{4n - \tfrac{n}{2} + 2, 4n - \tfrac{n}{2} + 4, \ldots, 5n - \tfrac{n}{2}\} \\
& \quad \bigcup \{4n - \tfrac{n}{2} + 3, 4n - \tfrac{n}{2} + 5, \ldots, 5n - \tfrac{n}{2} + 1\} \\
& \quad \bigcup \{\tfrac{9n}{2} + 2, \tfrac{9n}{2} + 3, \cdots, 5n+1\} \\
& \quad \subseteq \{1, 2, \cdots, 5n+1\}.
\end{aligned}$$

It is clear that the labels of each vertices are different. So, f is a injection from $V(n - \vec{C}_6)$ into $\{0, 1, \cdots, 5n+1\}$.

Secondly, we show the induced edges labeling f' is a bijective mapping from $E(n - \vec{C}_6)$ onto $\{1, 2, \cdots, 5n+1\}$.

Set $B_j = \{[f(v_{j+1}^i) - f(v_j^i)] \pmod{3n+2} | 1 \le i \le n\}$ $(0 \le j \le 5)$ and let $B = B_0 \cup B_1 \bigcup B_2 \bigcup B_3 \bigcup B_4 \bigcup B_5$. Then

$$\begin{aligned}
B_0 &= \{[f(v_1^i) - f(v_0)] \pmod{5n+2} | 1 \le i \le n\} \\
&= B_{01} \bigcup B_{02}, \text{ where} \\
&\quad B_{01} = \{[i - 0] \pmod{5n+2} | 1 \le i \le \tfrac{n}{2}\} = \{1, 2, \cdots \tfrac{n}{2}\}, \\
&\quad B_{02} = \{[4n + 1 + i - 0] \pmod{5n+2} | \tfrac{n}{2} + 1 \le i \le n\} \\
&\quad\quad = \{\tfrac{9n}{2} + 2, \tfrac{9n}{2} + 3, \cdots, 5n+1\}, \\
B_1 &= \{[f(v_2^i) - f(v_1^i)] \pmod{5n+2} | 1 \le i \le n\} \\
&= B_{11} \bigcup B_{12}, \text{ where} \\
&\quad B_{11} = \{[2n - \tfrac{n-2}{2} - 1 + 2i - i] \pmod{5n+2} | 1 \le i \le \tfrac{n}{2}\} \\
&\quad\quad = \{\tfrac{3n}{2} + 1, \tfrac{3n}{2} + 2, \cdots, 2n\}, \\
&\quad B_{12} = \{[n - \tfrac{n-2}{2} - 2 + 2i - (4n + 1 + i)] \pmod{5n+2} | \tfrac{n}{2} + 1 \le i \le n\} \\
&\quad\quad = \{2n+1, 2n+2, \cdots, \tfrac{5n}{2}\}, \\
B_2 &= \{[f(v_3^i) - f(v_2^i)] \pmod{5n+2} | 1 \le i \le n\} \\
&= B_{21} \bigcup B_{22}, \text{ where} \\
&\quad B_{21} = \{[n + i - (2n - \tfrac{n}{2} + 2i)] \pmod{5n+2} | 1 \le i \le \tfrac{n}{2}\} \\
&\quad\quad = \{\tfrac{9n}{2} + 1, \tfrac{9n}{2}, \cdots, 4n+2\}, \\
&\quad B_{22} = \{[3n - \tfrac{n}{2} + i - (n - \tfrac{n}{2} - 1 + 2i)] \pmod{5n+2} | \tfrac{n}{2} + 1 \le i \le n\} \\
&\quad\quad = \{\tfrac{3n}{2}, \tfrac{3n}{2} - 1, \cdots, n+1\},
\end{aligned}$$

$$
\begin{aligned}
B_3 &= \{[f(v_4^i) - f(v_3^i)] \pmod{5n+2} | 1 \le i \le n\} \\
&= B_{31} \bigcup B_{32}, \text{ where} \\
&\quad B_{31} = \{[4n - \tfrac{n}{2} + 1 + 2i - (n+i)] \pmod{5n+2} | 1 \le i \le \tfrac{n}{2}\} \\
&\quad\quad = \{\tfrac{5n}{2} + 2, \tfrac{5n}{2} + 3, \cdots, 3n+1\}, \\
&\quad B_{32} = \{[3n - \tfrac{n}{2} + 2i - (3n - \tfrac{n}{2} + i)] \pmod{5n+2} | \tfrac{n}{2} + 1 \le i \le n\} \\
&\quad\quad = \{\tfrac{n}{2} + 1, \tfrac{n}{2} + 2, \cdots, n\}, \\
B_4 &= \{[f(v_5^i) - f(v_4^i)] \pmod{5n+2} | 1 \le i \le n\} \\
&= B_{41} \bigcup B_{42}, \text{ where} \\
&\quad B_{41} = \{[\tfrac{5n}{2} + 1 - (4n - \tfrac{n}{2} + 1 + 2i)] \pmod{5n+2} | 1 \le i \le \tfrac{n}{2}\} \\
&\quad\quad = \{4n, 4n-2, \cdots, 3n+2\}, \\
&\quad B_{42} = \{[\tfrac{5n}{2} + 1 - (3n - \tfrac{n}{2} + 2i)] \pmod{5n+2} | \tfrac{n}{2} + 1 \le i \le n\} \\
&\quad\quad = \{4n+1, 4n-1, \cdots, 3n+3\}, \\
B_5 &= \{[f(v_0^i) - f(v_5^i)] \pmod{5n+2} | 1 \le i \le n\} = \{\tfrac{5n}{2} + 1\}.
\end{aligned}
$$

Hence, $B = B_0 \bigcup B_1 \bigcup B_2 \bigcup B_3 \bigcup B_4 \bigcup B_5$ is the set of labels of all edges, and

$$
\begin{aligned}
& B_0 \bigcup B_1 \bigcup B_2 \bigcup B_3 \bigcup B_4 \bigcup B_5 \\
&= B_{01} \bigcup B_{32} \bigcup B_{22} \bigcup B_{11} \bigcup B_{12} \bigcup B_5 \bigcup B_{31} \bigcup B_{41} \bigcup B_{42} \bigcup B_{21} \bigcup B_{02} \\
&= \{1, 2, \cdots, \tfrac{n}{2}\} \bigcup \{\tfrac{n}{2} + 1, \tfrac{n}{2} + 2, \cdots, n\} \bigcup \{n+1, n+2, \cdots, \tfrac{3n}{2}\} \\
&\quad \bigcup \{\tfrac{3n}{2} + 1, \tfrac{3n}{2} + 2, \cdots, 2n\} \bigcup \{2n+1, 2n+2, \cdots, 2n + \tfrac{n}{2}\} \\
&\quad \bigcup \{\tfrac{5n}{2} + 1\} \bigcup \{\tfrac{5n}{2} + 2, \tfrac{5n}{2} + 3, \cdots, 3n+1\} \bigcup \{3n+2, 3n+4, \cdots, 4n\} \\
&\quad \bigcup \{3n+3, 3n+5, \cdots, 4n+1\} \bigcup \{4n+2, 4n+3, \cdots, \tfrac{9n}{2} + 1\} \\
&\quad \bigcup \{\tfrac{9n}{2} + 2, \tfrac{9n}{2} + 3, \cdots, 5n+1\} \\
&= \{1, 2, \ldots, 5n+1\}.
\end{aligned}
$$

It shows that f' is a bijection from $E(n - \vec{C}_6)$ onto $\{1, 2, \cdots, |E(n - \vec{C}_6)|\}$. So we conclude that $n - \vec{C}_6$ is graceful for even n. □

Theorem 3 *For every even integer n, the digraph $n - \vec{C}_8$ is graceful.*

Proof. Define

$$f(v_0) = 0, \; f(v_7) = \frac{7n}{2} + 1$$

and

$$f(v_j^i) = \frac{n}{2}(j-1) + i, \; j = 1, 5 \text{ and } 1 \le i \le n,$$

$$f(v_2^i) = \begin{cases} 2n + 1 - i, & 1 \le i \le \frac{n}{2}, \\ 6n + 3 - i, & \frac{n}{2} + 1 \le i \le n, \end{cases}$$

$$f(v_3^i) = \begin{cases} \frac{9n}{2} + i, & 1 \le i \le \frac{n}{2}, \\ 3n + 1 + i, & \frac{n}{2} + 1 \le i \le n, \end{cases}$$

$$f(v_4^i) = \begin{cases} \frac{13n}{2} + 2 - i, & 1 \le i \le \frac{n}{2}, \\ 4n + 1 - i, & \frac{n}{2} + 1 \le i \le n, \end{cases}$$

$$f(v_6^i) = \begin{cases} 7n + 2 - i, & 1 \le i \le \frac{n}{2}, \\ 2n + 1 - i, & \frac{n}{2} + 1 \le i \le n. \end{cases}$$

Similar to the proof of Theorem 2 or Theorem 3, it can be shown that this assignment provides a graceful labeling of $n - \vec{C}_8$ for even n. Hence $n - \vec{C}_8$ is graceful for even n. □

Theorem 4 *For every even integer n, the digraph $n - \vec{C}_{10}$ is graceful.*

Proof. Define

$$f(v_0) = 0, \; f(v_9) = \frac{9n}{2} + 1$$

and

$$\begin{aligned} f(v_1^i) &= i, \quad i = 1, 2, \cdots, n, \\ f(v_2^i) &= \begin{cases} 2n + 1 - i, & 1 \le i \le \frac{n}{2}, \\ 3n + 2 - i, & \frac{n}{2} + 1 \le i \le n, \end{cases} \\ f(v_3^i) &= \begin{cases} 4n + i, & 1 \le i \le \frac{n}{2} \\ \frac{11n}{2} + 1 + i, & \frac{n}{2} + 1 \le i \le n \end{cases} \\ f(v_4^i) &= \begin{cases} 7n + 2 - i, & 1 \le i \le \frac{n}{2} \\ 4n + 1 - i, & \frac{n}{2} + 1 \le i \le n \end{cases} \\ f(v_5^i) &= \begin{cases} \frac{11n}{2} + 1 + i, & 1 \le i \le \frac{n}{2}, \\ \frac{15n}{2} + 2 + i, & \frac{n}{2} + 1 \le i \le n, \end{cases} \\ f(v_6^i) &= \begin{cases} 4n + 1 - i, & 1 \le i \le \frac{n}{2}, \\ 8n + 2 - i, & \frac{n}{2} + 1 \le i \le n, \end{cases} \\ f(v_7^i) &= \begin{cases} \frac{15n}{2} + 1 + i, & 1 \le i \le \frac{n}{2}, \\ \frac{9n}{2} + 1 + i, & \frac{n}{2} + 1 \le i \le n, \end{cases} \\ f(v_8^i) &= \begin{cases} 5n + 2 - i, & 1 \le i \le \frac{n}{2}, \\ 2n + 1 - i, & \frac{n}{2} + 1 \le i \le n. \end{cases} \end{aligned}$$

Similar to the proof of Theorem 2 or Theorem 3, it can be shown that this assignment provides a graceful labeling of $n - \vec{C}_{10}$ for even n. Hence $n - \vec{C}_{10}$ is graceful for even n. □

In Figure 1, we give graceful labelings of $8-\vec{C}_4$,$8-\vec{C}_6$,$8-\vec{C}_8$ and $8-\vec{C}_{10}$.

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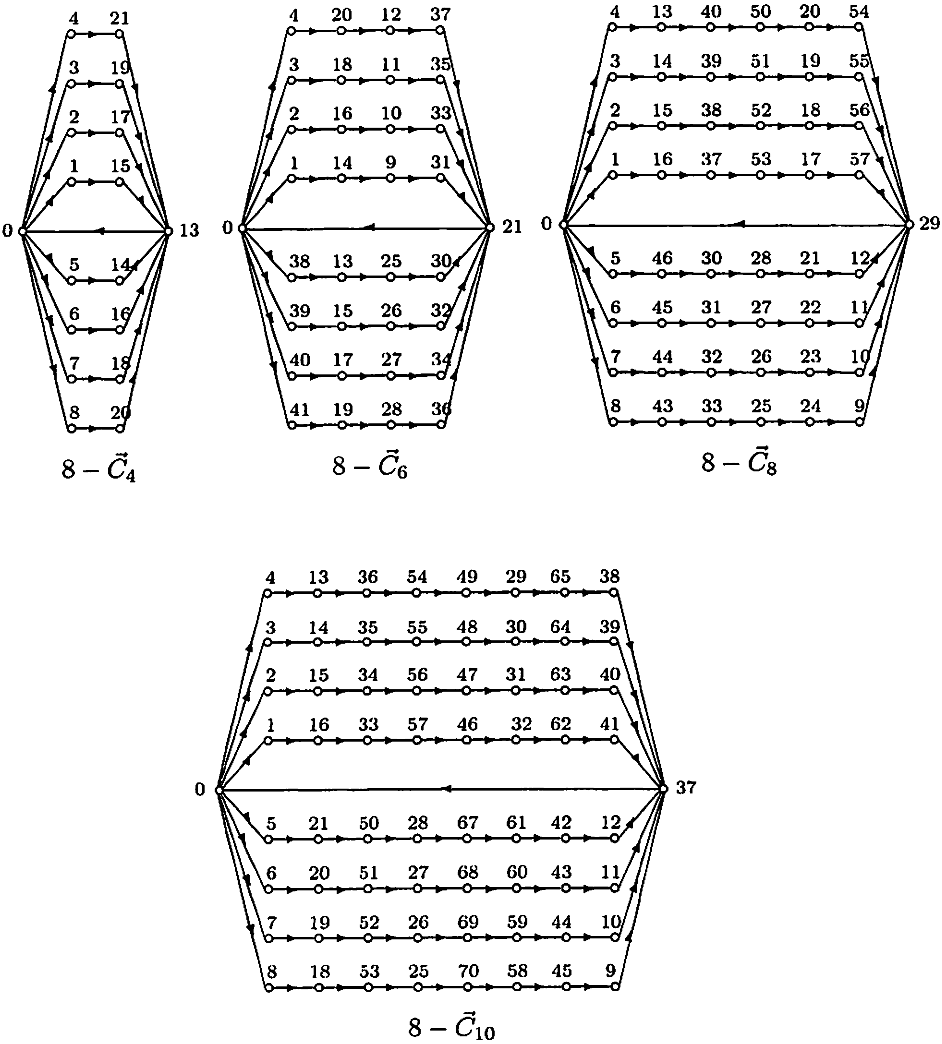

Figure 1: Graceful labelings of $8-\vec{C}_4$, $8-\vec{C}_6$, $8-\vec{C}_8$ and $8-\vec{C}_{10}$.