

Brother Avoiding Round Robin Doubles Tournaments II

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January 7, 2009

Abstract

We investigate brother avoiding round robin doubles tournaments and construct several infinite families. We show that there is a $\text{BARRDT}(n)$ that is not a $\text{SAMDRR}(n)$ for all $n > 4$.

Key words and phrases: brother avoiding round robin doubles tournament (BARRDT), spouse avoiding mixed doubles round robin tournament (SAMDRR), Z -cyclic whist tournament, patterned starter, incomplete self orthogonal latin square (ISOL)

AMS subject classifications: Primary 05B30; Secondary 05B05, 05B10

1 Introduction

Berman and Smith [5] introduced the concept of a brother avoiding round robin doubles tournament as an intermediate between spouse avoiding mixed doubles tournaments and whist tournaments in the context of embedding the former in the later. We begin with definitions. The reader is referred to Anderson [1] for background and basic results on SAMDRR s and whist tournaments.

Definition 1.1 *A spouse avoiding mixed doubles round robin tournament, $\text{SAMDRR}(n)$, for n couples is a schedule of games for n male-female couples. In each game two players of opposite sex compete against two other players of opposite sex, and*

- *each pair of players of the same sex are opponents exactly once*
- *each pair of players of the opposite sex, except spouses, are opponents exactly once and partners exactly once*

Definition 1.2 *A brother avoiding round robin doubles tournament, $BARRDT(n)$, for n pairs of brothers consists of games for teams of two players such that*

- *each player has exactly one brother*
- *brothers never play in the same game as partners nor as opponents*
- *each pair of players who are not brothers are opponents exactly once and partners at most once.*

We denote the game in which partners a and b oppose partners c and d as $a, b \vee c, d$. The number of games in a $BARRDT(n)$ and in a $SAMDRR(n)$ is $n(n-1)/2$.

As described in [5], these conditions are motivated by consideration of the properties required of the set of games that would have to be added to a $SAMDRR(2n)$ to form a whist tournament for the $4n$ players involved.

The original definition of a $BARRDT$ allowed the possibility of brothers partnering in a game. Lee Leonard [8] noticed that every $SAMDRR(n)$ for n couples is a $BARRDT(n)$ in which spouses in the $SAMDRR$ are identified as brothers in the $BARRDT$, and brothers are never partners. For this reason we consider in this paper only $BARRDT$ s in the stricter sense of Definition 1.2.

It is easy to see that a $BARRDT(n)$ does not exist when $n < 4$. The only $BARRDT(4)$ is also a $SAMDRR(4)$. In this paper we show that there is a $BARRDT(n)$ [having no pair of brothers as partners] that is not a $SAMDRR(n)$ for all $n > 4$.

Every $SAMDRR(n)$ is a $BARRDT(n)$. Under what conditions is a $BARRDT(n)$ a $SAMDRR(n)$? The set of $BARRDT$ players must be partitioned into two sets of n players, the males and females, in such a way that every partnership consists of one male and one female, and players who were called "brothers" in the $BARRDT$ are now recognized as spouses. The existence of a partition for a given tournament is easy to determine in practice. Start with any brother pair x, y and arbitrarily assign x to be male and y female. Then the partners of x must be female and the partners of y must be male. Continuation of this process will either demonstrate that the given $BARRDT$ is a $SAMDRR$ or exhibit a pair of partners of the same sex. If x is a player in a $BARRDT$, let $P(x)$ denote the set of partners of x . It follows from the definition of a $SAMDRR$ that the following conditions are necessary for a given $BARRDT$ to be a $SAMDRR$.

1. For every brother pair x and y , $P(x) \cap P(y) = \emptyset$.
2. For every brother pair x and y , if $w \in P(x)$ and $z \in P(y)$, then $w \in P(z)$ or w and z are brothers.
3. The graph determined by the partner relation is bipartite.
4. For all players a, b, x , if $a, b \in P(x)$ then $a \notin P(b)$.

Condition 1 is not sufficient for a BARRDT to be a SAMDRR. The cyclic BARRDT(13) of the next section with generating games $0,4 \vee 5,12$; $0,1 \vee 3,11$; and $0,2 \vee 6,9$ satisfies condition 1 but is not a SAMDRR because it does not satisfy condition 3. Players 1 and 2 must have the same sex since they partner player 0, but players 1 and 2 are partners in another game.

Theorem 1.3 *A BARRDT(n) that satisfies condition 2 is a SAMDRR(n).*

Proof: Assume condition 2 holds. Let x and y be brothers. If z were in $P(x) \cap P(y)$, then z would either be a partner of x or a brother of x , which is impossible. Thus condition 1 holds.

Choose any brother pair x and y . Define the set of males and females by letting the set of males be $\{x\} \cup P(y)$ and the set of females be $\{y\} \cup P(x)$. There are n males and n females and the two sets are disjoint by condition 1, so every player has been uniquely assigned a sex.

It is not possible for two males (or females) to be brothers because x has only one brother y , and if $c, d \in P(y)$ are brothers then $y \in P(c) \cap P(d)$, which contradicts condition 1.

By condition 2, every male and female who are not brothers (spouses) are partners. ■

Theorem 1.4 *A BARRDT(n) that satisfies condition 3 is a SAMDRR(n).*

Proof: Suppose two brothers x and y are in the same part of the bipartite partner graph. Then x opposes every other player exactly once, but there are n players in the other part of the graph and only $n - 2$ in the part with x and y . ■

Theorem 1.5 *A BARRDT(n) that satisfies condition 4 is a SAMDRR(n).*

Proof: Assume condition 4 holds in a BARRDT(n). Then the partner graph contains no triangles and each vertex has degree $n - 1$. If $n = 5$ it is easy to see that the BARRDT(n) is a SAMDRR(n). If $n > 5$ the partner graph is bipartite (see [3]) and the BARRDT(n) is a SAMDRR(n). ■

2 Constructing BARRDTs from Patterned Starters

We first recall the definitions of a whist tournament and a patterned starter.

Definition 2.1 For $n = 4k$ (or $n = 4k + 1$), a whist tournament, $\text{Wh}(n)$, for n players is a schedule of games, each pairing two players against two others such that

- games are played in rounds, each of k games,
- each player plays in one game in each round (or sits out exactly one round),
- each player partners every other player exactly once,
- each player opposes every other exactly twice

A $\text{Wh}(4k + 1)$ is **Z-cyclic** if the set of players is Z_{4k+1} and an initial round determines the tournament. Subsequent rounds are generated by adding 1 (mod $4k + 1$) to the previous round. A $\text{Wh}(4k)$ is **Z-cyclic** if the set of players is $Z_{4k-1} \cup \{\infty\}$ and an initial round determines the tournament. Subsequent rounds are generated by adding 1 (mod $4k - 1$) to the previous round with the convention that $\infty + 1 = \infty$.

Definition 2.2 Let n be an odd integer. The patterned starter for Z_n is the set $S_n = \{\{x, -x\} : x \in Z_n, x \neq 0\}$. A Z-cyclic $\text{Wh}(4k)$ such that the set of initial round partner pairs is the set $S_{4k-1} \cup \{\{\infty, 0\}\}$ is called a **Z-cyclic patterned starter whist tournament for $4k$ players, ZCPSWh($4k$)**. A Z-cyclic $\text{Wh}(4k + 1)$ such that the set of initial round partner pairs is the set S_{4k+1} is called a **Z-cyclic patterned starter whist tournament for $4k + 1$ players, ZCPSWh($4k + 1$)**.

Berman and Smith [5] gave three constructions for BARRDTs. Each of these constructions converted a ZCPSWh to a BARRDT. We now summarize these constructions and show that each of these BARRDTs is not a SAMDRR and that brothers are never partners.

Theorem 2.3 Let p be a prime of the form $4k + 1$. Then there is a Z-cyclic BARRDT(p) that is not a SAMDRR.

Proof: Let θ be a primitive element in Z_p . By Baker's construction [4] there is a ZCPSWh(p) with first round games $\theta^i, \theta^{2k+i} \vee \theta^{k+i}, \theta^{3k+i}$ for $i = 0, 1, 2, \dots, k - 1$. Each first round game can be written in the form $r, u \vee s, t$ where $r < s < t < u$. The game $0, u - t \vee s, u$ is a generating

game for a Z-cyclic BARRDT(p) (mod $2p$). Brother pairs are $x, x + p$ for $0 \leq x < p$. Partner pairs have a difference less than p so brothers are never partners.

Suppose the BARRDT is a SAMDRR. Then the set of players can be partitioned into two sets of p players, the males and females, in such a way that every partnership consists of one male and one female, and players who are “brothers” in the BARRDT are now recognized as spouses.

For convenience, label player 0 as male. Then every partner of 0 is female. Let $0, u - t \vee s, u$ be any generating game of the BARRDT. Then exactly one of the partner differences $u - t$ or $u - s$ is even, because the players in the pairs $\{r, u\}$ and $\{s, t\}$ have opposite parity ($r + u = s + t = p$). One of the games generated from $0, u - t \vee s, u$ has the form $0, x \vee -, -$, where x is even. Consider games formed by successively adding x . Since x is female, $2x$ must be male, and in general, every even multiple of x is male and every odd multiple of x is female. But the odd multiple px is 0, which is a contradiction. ■

Theorem 2.4 *Let p be a prime of the form $4k + 1$ and $n > 1$. There there is a Z-cyclic BARRDT(p^n) that is not a SAMDRR.*

Proof: Begin with a Wh(p^n) as constructed in Lemma 3.2 of [5]. Construct a BARRDT(p^n) with generating games as in the previous theorem. Players who differ by p^n are brothers and by construction are never partners. The same argument as in the previous theorem shows that the BARRDT is not a SAMDRR. ■

Theorem 2.5 *If there is a ZCPSWh($4k$), then there is a BARRDT($4k$) that is not a SAMDRR.*

Proof: Let $m = 4k - 1$. The initial round of a ZCPSWh($4k$) has one game $\infty, 0 \vee a, b$ and all other games of the form $r, u \vee s, t$ where $a + b = r + u = s + t = m$, and without loss of generality $r < s < t < u$. A BARRDT($4k$) is constructed on players $Z_{2m} \cup \{\infty_1, \infty_2\}$ and consists of the following games:

- (i) generating games $\infty_1, 0 \vee a, b$ and $\infty_2, 1 \vee a + 1, b + 1$,
- (ii) all games developed from the games in (i) by repeatedly adding 2 (mod $2m$). [short cycles]
- (iii) generating games $r, u + m \vee s, t$ for each of the other first round games in the ZCPSWh($4k$),
- (iv) all games developed from the games in (iii) by repeatedly adding 1 (mod $2m$). [long cycles]

The brother pairs are $\{\infty_1, \infty_2\}$ and pairs in Z_{2m} that differ by m . Observe that no partner pairs have a difference of m and that players ∞_1 and ∞_2 never partner.

Suppose that the BARRDT is a SAMDRR. Assume ∞_1 is female and ∞_2 is male. Then all evens are male and all odds are female. However r and $u + m$ have the same parity because $r + u + m = 2m$. Thus r and $u + m$ are partners of the same sex which contradicts the SAMDRR requirements. ■

3 Constructing BARRDTs from Family Tournaments

A BARRDT(n) has players from n families, each consisting of two brothers. For two given families $A = \{A_1, A_2\}$ and $B = \{B_1, B_2\}$, four oppositions must occur: $A_1 \vee B_1$, $A_1 \vee B_2$, $A_2 \vee B_1$, and $A_2 \vee B_2$. Brothers never partner so these oppositions must occur in different games.

If we take a BARRDT(n) and replace each player with his family label the result is a tournament on n families in which pairs of families oppose other pairs of families and every pair of families oppose exactly four times. We can sometimes reverse this process to construct a BARRDT by starting with such a family tournament. For each family label we must then select one of the two brothers as the player representing that family. We call this selection process *orientation*. The orientation must be done so that for every pair of families all four oppositions occur, and partner pairs occur at most once. We may simplify the situation by assuming that pairs of families partner exactly twice as well as oppose exactly four times. This is a generalized whist tournament.

Lemma 3.1 *Let $k = 2d + 1$, $k = 1$ or $5 \pmod{6}$, and $k \geq 5$. Then there is a family tournament with k families in which each pair of families partner twice and oppose four times.*

Proof: Suppose the set of families is Z_k . Define generating games $0, 2x \vee x, 3x \pmod{k}$ for $1 \leq x \leq d$. Every element of Z_k occurs twice as a partner difference and four times as an opponent difference. Thus we get the required tournament when the games are cyclically developed mod k . ■

Given a game in a family tournament we denote an orientation by a, b o c, d where $a, b, c, d \in \{1, 2\}$. For example, if the game $w, x \vee y, z$ is oriented $1, 2$ o $2, 1$ then replace the family labels w, x, y, z with respectively the first, second, second, and first member of the family. The family tournament in Lemma 3.1 may be oriented by orienting each generating game and then

using the same orientation on the developed games. This procedure can be used to construct many BARRDTs, some of which are not SAMDRRs.

Theorem 3.2 *Let $k = 2d + 1$, $k = 1$ or $5 \pmod{6}$, and $k \geq 5$. Then there is a BARRDT(k) that is not a SAMDRR.*

Proof: Begin with the family tournament in Lemma 3.1. The possible orientations are summarized in the table below. The column labeled “ x ” indicates the oppositions that have occurred for difference x with the given orientation. For example, ~ 11 means that of the four required oppositions for difference x , the only one that has not occurred is when the first family member opposes the first family member. The column labeled “ $3x$ ” indicates the single opposition that has occurred for difference $3x$. Picture each generating game as a “domino” with an x side on the left and a $3x$ side on the right. Constructing a BARRDT is then a game of matching the dominos into cycles. The right side of each domino must match the left side of the next domino in order for the orientations to work out (cancel). This insures that all oppositions have occurred. So given a generating game $0, 2a \vee a, 3a$ we take the sequence of games $0, 6a \vee 3a, 9a$; $0, 18a \vee 9a, 27a$; \dots ; $0, 2 \cdot 3^i a \vee 3^i a, 3^{i+1} a$ where i is defined by $3^{i+1} a = a$. The opponent differences form the sequence: $\pm a, \pm a, \pm a, \pm 3a$; $\pm 3a, \pm 3a, \pm 3a, \pm 9a$; $\pm 9a, \pm 9a, \pm 9a, \pm 27a$; \dots ; $\pm 3^i a, \pm 3^i a, \pm 3^i a, \pm a$. The process terminates because 3 is not a divisor of k . If there are any remaining generating games that have not been used we start with one of those to construct another cycle as above. Continue until every generating game is in a cycle.

orientation	x	$3x$	domino name	next domino	SAMDRR?
$1,2 \circ 2,1$	~ 11	11	A	A,C, or F	
$1,1 \circ 2,1$	~ 22	11	B	A,C, or F	no
$2,1 \circ 2,2$	~ 11	22	C	B,D, or E	no
$1,2 \circ 1,1$	~ 22	11	D	A,C, or F	no
$2,1 \circ 1,2$	~ 22	22	E	B,D, or E	
$2,2 \circ 1,2$	~ 11	22	F	B,D, or E	no

All partner pairs are distinct because the left and right side of each orientation are different.

The column labeled “SAMDRR?” indicates those dominos that when used will result in a BARRDT that is not a SAMDRR. To see this take the game developed from the generating game $0, 2x \vee x, 3x$ by subtracting x (i.e., $-x, x \vee 0, 2x$). Then in every case we will observe a player who has partnered two brothers. For a given k the dominos will form one or more cycles. To obtain a BARRDT that is not a SAMDRR orient the even cycles $BCBC \dots$, and the odd cycles $BCBC \dots BCE$.

If all dominos used come from rows 1 and 5 then the resulting BARRDT will be a SAMDRR. Males can be chosen as the first member of each family and females as the second member. ■

Example 3.3 A BARRDT(35) that is not a SAMDRR is summarized in the following table. The set of families is Z_{35} . Develop each generator cyclically mod 35. Then for each of the developed games, take the first or second player of each family as specified by the orientation for the generator.

generator	orientation	partner differences	opponent differences
0,2 v 1,3	1,1 o 2,1 (B)	2,33	$\pm 1,1,1,3$
0,6 v 3,9	2,1 o 2,2 (C)	6,29	$\pm 3,3,3,9$
0,18 v 9,27	1,1 o 2,1 (B)	17,18	$\pm 9,9,9,8$
0,19 v 27,11	2,1 o 2,2 (C)	16,19	$\pm 8,8,8,11$
0,22 v 11,33	1,1 o 2,1 (B)	13,22	$\pm 11,11,11,2$
0,31 v 33,29	2,1 o 2,2 (C)	4,31	$\pm 2,2,2,6$
0,23 v 29,17	1,1 o 2,1 (B)	12,23	$\pm 6,6,6,17$
0,34 v 17,16	2,1 o 2,2 (C)	1,34	$\pm 17,17,17,16$
0,32 v 16,13	1,1 o 2,1 (B)	3,32	$\pm 16,16,16,13$
0,26 v 13,4	2,1 o 2,2 (C)	9,26	$\pm 13,13,13,4$
0,8 v 4,12	1,1 o 2,1 (B)	8,27	$\pm 4,4,4,12$
0,24 v 12,1	2,1 o 2,2 (C)	11,24	$\pm 12,12,12,1$
0,10 v 5,15	1,1 o 2,1 (B)	10,25	$\pm 5,5,5,15$
0,30 v 15,10	2,1 o 2,2 (C)	5,30	$\pm 15,15,15,10$
0,20 v 10,30	2,1 o 1,2 (E)	15,20	$\pm 10,10,10,5$
0,14 v 7,21	1,1 o 2,1 (B)	14,21	$\pm 7,7,7,14$
0,7 v 21,28	2,1 o 1,2 (C)	7,28	$\pm 14,14,14,7$

4 Constructing BARRDT(4k)s from Wh(4k)s

The construction of a BARRDT(4k) from a ZCPSWh(4k) of Theorem 2.5 can be extended to construct a BARRDT(4k) that is not a SAMDRR from any Z-cyclic Wh(4k). To illustrate the technique we begin with a small example.

Example 4.1 The Z-cyclic Wh(8) has a generating game $\infty, 0 v 4, 5$ and one other generating game $1, 3 v 2, 6$. In what follows, we label opponent and partner differences with respect to generating games (and only with respect to generating games) using the positive differences (and only the positive differences) between opponents and between partners. For example, the game $1, 3 v 2, 6$ has opponent differences 1, 1, 3, and 5 and partner differences 2 and 4.

Our BARRDT(8) will have players from the set $\{\infty_1, \infty_2\} \cup Z_{14}$. The two infinite players will be brothers as will x and $x + 7 \pmod{14}$. The infinite game is modified to form two generating games from which games are developed cyclically by successively adding 2. These two short cycles are

generated by $\infty_1, 0 \vee 4, 5$ and $\infty_2, 1 \vee 5, 6$. With these games we have all oppositions involving the infinite players, all partner differences of 1, and all opponent differences of 4 and 5. The other game $1, 3 \vee 2, 6$, if used as a generating game, would produce all partner differences of 2 and 4 (mod 14) and all opponent differences of 1, 1, 3, and 5 (mod 14). To avoid repeating the opponent differences 1 and 5 (mod 14), we replace the game $1, 3 \vee 2, 6$ with the game $8, 3 \vee 2, 6$, and use it to generate a long cycle. The result is that the opponent differences are 1, 2, 3, 6 in addition to 4 and 5 (mod 14). The three generating games produce the distinct partner differences 1, 4, and 5 (mod 14). Thus every pair of players who are not brothers are opponents exactly once and partners at most once.

The obstacle faced in this example was the fact that the opponent difference blocks $\{4, 5\}$ and $\{1, 1, 3, 5\}$ when viewed modulo 14 contain repeated instances of 1 and 5. The situation was remedied by "flipping" one of the 1 differences and the 5 difference from the block $\{1, 1, 3, 5\}$ so that 1 becomes 6 (mod 14) and 5 becomes 2 (mod 14). This "flip" was accomplished by substituting player 8 for player 1 in the game $1, 3 \vee 2, 6$ so that the set of opponent differences becomes 1, 2, 3, 4, 5, 6 modulo 14.

The procedure for forming a BARRDT($4k$) from a given Z -cyclic $\text{Wh}(4k)$ with players in $\{\infty\} \cup Z_m$, where $m = 4k - 1$, is outlined as follows. Let $x \in \{1, 2, \dots, m - 1\}$. Then either x and $m - x$ both occur as an opponent difference in the generating games, or one of x or $m - x$ occurs twice. We form the BARRDT($4k$) on the set $\{\infty_1, \infty_2\} \cup Z_{2m}$ by expanding one infinite game to two infinite games as generators of short cycles and by substituting modified generating games as necessary. This modification is done so as to flip repeated opponent differences.

The purpose of modifying generating games is to eliminate all repeated opponent differences, with the eventual result that for every $i \in \{1, 2, \dots, m - 1\}$, exactly one of i or $-i \pmod{2m}$ occurs once as an opponent difference, and there are no repeated partner differences.

Let i and j be two players. The positive difference between the players is $|i - j|$. If x is a positive difference between players, then $-x$ is determined using the modulus and thus depends on whether the players are in a game from the original $\text{Wh}(4k)$ or from the desired BARRDT($4k$). In a $\text{Wh}(4k)$, $\pm x \pmod{m} = \{x, m - x\}$. In a BARRDT($4k$), $\pm x \pmod{2m} = \{x, 2m - x\}$ and $\pm(m - x) \pmod{2m} = \{m - x, m + x\}$. The numbers $x, 2m - x, m - x$, and $m + x$ are all distinct because m is odd and $x \neq 0$.

In the lemma below, the object is to show that if x is an opponent difference in a given game on the set $\{\infty_1, \infty_2\} \cup Z_{2m}$, then there is an alternative game that may be substituted for the given game, such that the opponent difference x has been replaced by either $m + x$ or $m - x$. The actual replacement will vary from case to case, and will not affect the final

outcome. To simplify the discussion we say x has been flipped in either case and denote the flip of x as x' . In practice, one would want to flip x when x is repeated (possibly in some other game) as an opponent difference, with the eventual goal of obtaining a set of games so that no opponent difference is repeated. That is, for every opponent difference x , there is to be exactly one other associated opponent difference, x' .

We first consider the opponent differences from the two infinite games $\infty_1, 0 \vee a, b$ and $\infty_2, 1 \vee a + 1, b + 1$ where $\infty, 0 \vee a, b$ is the generating infinite game from the given $\text{Wh}(4k)$, and these two games are developed cyclically by successively adding $2 \pmod{2m}$.

Lemma 4.2 *Of the opponent differences a and b in the infinite games, either a or b or both can be flipped to a' , to b' , or to a' and b' by substituting alternate infinite games.*

Proof: Substituting the games $\infty_1, 0 \vee a + m, b$ and $\infty_2, 1 \vee a + m + 1, b + 1$ yields games with opponent differences a' and b . Similarly adding m to b or m to both a and b produces games in which the differences are a and b' or a' and b' . ■

Lemma 4.3 *Let $a, b \vee c, d$ be any game with players in Z_{2m} . For the given game, let $w = |a - d|$, $x = |a - c|$, $y = |b - c|$, and $z = |b - d|$. Then any two of the four opponent differences can be flipped by substituting an alternative game.*

Proof: To obtain the opponent difference block $\{w', x', y, z\}$ use the alternate game $a + m, b \vee c, d$.

For $\{w, x, y', z'\}$ use the game $a, b + m \vee c, d$.

To flip x and y , use $a, b \vee c + m, d$.

To flip w and z , use $a, b \vee c, d + m$.

To flip x and z , use $a + m, b \vee c, d + m$.

To flip w and y , use $a + m, b \vee c + m, d$. ■

We turn now to the problem of determining which opponent differences are to be flipped to avoid repeated instances of the same difference. We show first that the generating games of the given $\text{Wh}(4k)$ can, via the lemmas, be replaced by games involving players from the set $\{\infty_1, \infty_2\} \cup Z_{2m}$ such that no opponent difference is repeated in the difference block of a single game.

Case 1 If the difference block is of the form $\{x, x, y, y\}$, then there is a game with difference block $\{x, x', y, y'\}$.

Case 2 If the difference block has the form $\{x, x, s, t\}$ where neither of s, t is repeated in some other game, then there is a game with difference block $\{x, x', s', t\}$. If the (repeated) difference s' occurs in a game with difference block $\{y, y, s', r\}$, that game can be replaced by a game having difference

block $\{y, y', s, r\}$; otherwise s' occurs in a difference block in which there are no repeated differences from a single game.

Case 3 If the difference block has the form $\{x, x, y, t\}$ where the difference y is repeated in some other game, then there is a game with difference block $\{x, x', y', t\}$.

Suppose now that we have a set of games on $\{\infty_1, \infty_2\} \cup Z_{2m}$ with no repeated opponent differences in a single game. If there is a game such that all four of its opponent differences are repeated in other games, this game can be replaced by a game in which all four differences have been flipped. This process may be repeated as necessary, each time reducing the number of repeated differences in other games as well, until there are no games with four repeated differences. If any game now has exactly three differences that are repeated in other games, this game can be replaced by a game that has exactly one repeated difference, and this process may be repeated until all remaining games have no more than two repeated differences. Similarly games with exactly two repeated differences may be replaced, one at a time, until all games have no more than one repeated difference.

Let V be the set of all opponent difference blocks of modified games, including the two-element difference block associated with the infinite games. Define an adjacency relation R on V as follows: for $v_1 \neq v_2$ in V , v_1 is adjacent to v_2 iff for some opponent difference x in v_1 , either x or the unique associated difference x' is in v_2 . In the graph $G = (V, R)$ every vertex has even degree. For each $v \in V$, let $C(v)$ be the connected component of v . We claim that repeated differences can be eliminated in each component $C(v)$.

Case 1 Suppose the difference block from the infinite game is not in $C(v)$. We claim that for each v_1 in V , if v_1 has exactly one repeated opponent difference x , then some v_2 in $C(v)$ contains a repeated difference y , where $x \neq y$. Suppose x is the only repeated difference occurring in two distinct difference blocks in $C(v)$. Then the sum of all differences in $C(v)$ is even because the sum of differences in each block is even, but the sum of all opponent differences is $2x$ plus an odd number of odd sums $y + y'$, so the sum is odd. This is impossible.

Now assume v is of the form $\{x, y, z, w\}$, with difference x repeated in some other difference block, and there is a difference $y \neq x$ that occurs in a difference block v^* in $C(v)$. Then there is a path from v to v^* consisting of difference blocks of the form

$$v = \{x, a_1, -, -\}, \{a'_1, a_2, -, -\}, \{a'_2, a_3, -, -\}, \dots, \{a'_i, y, -, -\} = v^*.$$

The associated games can be replaced with games having difference blocks

$$\{x', a'_1, -, -\}, \{a_1, a'_2, -, -\}, \dots, \{a_i, y', -, -\},$$

so that neither x nor y is a repeated difference in $C(v)$. This process may be repeated until there are no repeated differences in $C(v)$.

Case 2 Suppose the difference block $\{a, b\}$ from the infinite game is in $C(v)$.

If $C(v)$ consists of only $\{a, b\}$ then neither a nor b is repeated.

If either a or b is repeated in another difference block in $C(v)$ then flip either a or b to eliminate the repeat.

If $v = \{x, y, z, w\}$, with difference x repeated in some other difference block, then take a path from v to $\{a, b\}$.

$$v = \{x, c_1, -, -\}, \{c'_1, c_2, -, -\}, \dots, \{c'_i, a', -, -\}, \{a, b\}$$

Replace the associated games with games having difference blocks

$$v = \{x', c'_1, -, -\}, \{c_1, c'_2, -, -\}, \dots, \{c_i, a, -, -\}, \{a', b\}.$$

This eliminates the repeated difference x . This process may be repeated until there are no repeated differences in $C(v)$.

By construction, for x in Z_{2m} the players x and $x+m \pmod{2m}$ and the players ∞_1 and ∞_2 never appear in the same game, so we may take these pairs to be brothers. Also since every i in $Z_m - \{0\}$ is a partner difference in the $\text{Wh}(4k)$ and partner differences in the resulting tournament can only be one of i or i' , no partner differences are repeated. Thus the resulting tournament is a $\text{BARRDT}(4k)$.

We now consider the possibility that the resulting $\text{BARRDT}(4k)$ is also a $\text{SAMDRR}(4k)$. Then the players can be partitioned in two sets, males and females. If we arbitrarily call ∞_1 a male, then all even players are female because they partner ∞_1 , ∞_2 must be female, and all odds are male. Furthermore, in every game the partners must have opposite parity. Let x be any player in a generating game other than the infinite games. If x and x' are differences in the same game, then flip both x and x' using Lemma 4.3. In the resulting game, the only change is that one pair of partners have the same parity, so substituting the resulting game yields a BARRDT that is not a SAMDRR . Alternatively, if x and x' are opponent differences in different generating games, then as above we consider, in the adjacency graph, the component for the difference blocks that contain x and x' . Since these two difference blocks have even degree at least 2, there is a path consisting of difference blocks of the form

$$v = \{x, a_1, -, -\}, \{a'_1, a_2, -, -\}, \{a'_2, a_3, -, -\}, \dots, \{a'_i, x', -, -\}.$$

We replace all the associated games, as above. Again using Lemma 4.3, in each of these games there is a partner pair having the same parity, so the BARRDT is not a SAMDRR .

We have now established our theorem. For the existence of Z -cyclic $\text{Wh}(4k)$, see Theorem 64.12 of [2].

Theorem 4.4 *If $k > 1$ and there is a Z -cyclic $Wh(4k)$ then there is a $BARRDT(4k)$ that is not a $SAMDRR(4k)$.*

5 Completing the Spectrum

There is no $BARRDT(n)$ for $n < 4$. The only $BARRDT(4)$ is also a $SAMDRR$. The spectrum for $BARRDTs$ that are not $SAMDRRs$ can be completed using well known results for $ISOLS$. First we need five sizes not covered by earlier constructions.

Example 5.1 *A $BARRDT(6)$ that is not a $SAMDRR$. The brother pairs are 0, 1; 2, 3; 4, 5; 6, 7; 8, 9; and 10, 11.*

0	2	v	4	6	1	7	v	2	11	2	5	v	9	10
0	3	v	9	11	1	8	v	3	10	2	6	v	8	11
0	4	v	8	10	1	9	v	4	7	3	7	v	4	10
0	5	v	2	7	1	10	v	6	9	3	8	v	5	7
0	6	v	3	5	1	11	v	5	8	4	9	v	6	11

Note that brothers 2 and 3 both have 0 as a partner, so this tournament is not a $SAMDRR$.

Example 5.2 *The two generating games 9, 17 v 1, 16 and 0, 17 v 3, 5 developed modulo 18 form a Z -cyclic $BARRDT(9)$ that is not a $SAMDRR$. Players who differ by 9 are brothers. Note that brothers 0 and 9 both have 17 as a partner.*

Example 5.3 *A $BARRDT(10)$ that is not a $SAMDRR$. The brother pairs are 0, 6; 1, 7; 2, 8; 3, 9; 4, 10; 5, 11; A, B; C, D; E, F; and G, H.*

0	1	v	2	3	0	2	v	9	11	0	3	v	5	10
4	5	v	6	7	4	6	v	1	3	4	7	v	9	2
8	9	v	10	11	8	10	v	5	7	8	11	v	1	6
A	C	v	0	4	B	D	v	3	7	C	E	v	6	10
A	2	v	C	10	B	8	v	C	0	C	9	v	E	1
A	4	v	D	11	B	0	v	D	1	C	7	v	F	3
A	E	v	1	5	B	F	v	4	8	C	G	v	7	11
A	11	v	E	3	B	3	v	E	2	C	1	v	G	5
A	5	v	F	9	B	6	v	F	10	C	8	v	H	9
A	G	v	2	6	B	H	v	5	9	D	F	v	1	9
A	3	v	G	8	B	10	v	G	11	D	7	v	E	11
A	0	v	H	7	B	7	v	H	6	D	2	v	F	5
E	G	v	0	8	F	H	v	3	11	D	H	v	2	10
E	5	v	G	4	F	4	v	G	0	D	9	v	G	6
E	6	v	H	2	F	1	v	H	10	D	4	v	H	8

Note that brothers A and B both have 0 as a partner.

Example 5.4 A $BARRDT(14)$ that is not a $SAMDRR$. The set of players is $\{a, b, c, d\} \times Z_7$. The following games are generators and are developed modulo 7 on the indices, where we write player (x, i) as x_i .

$$a_0b_1 \ v \ c_0d_1, \ a_0b_2 \ v \ c_2d_0, \ a_0b_3 \ v \ c_1d_4$$

$$a_0c_0 \ v \ b_4d_6, \ a_0c_1 \ v \ b_6d_5,$$

$$a_0d_1 \ v \ b_5c_6, \ a_0d_6 \ v \ b_2c_5,$$

$$a_0b_5 \ v \ a_2b_1, \ a_0c_5 \ v \ a_1c_3, \ a_0d_0 \ v \ a_4d_2, \ c_0d_6 \ v \ c_1d_3, \ b_0c_0 \ v \ b_6c_4, \ b_0d_0 \ v \ b_5d_6.$$

Brother pairs are generated by a_0, b_0 and c_0, d_0 . Brothers c_0 and d_0 both partner b_0 , so this tournament is not a $SAMDRR$.

Example 5.5 A $BARRDT(15)$ that is not a $SAMDRR$. The set of players is $\{a, b\} \times Z_{15}$. The following games are generators and are developed modulo 15 on the indices.

$$a_0b_2 \ v \ a_1a_7, \ a_2b_0 \ v \ b_1b_7,$$

$$a_0b_{14} \ v \ a_2b_3, \ a_0b_{12} \ v \ a_3b_6, \ a_0b_{11} \ v \ a_4b_8, \ a_0b_9 \ v \ a_5b_{11}, \ a_0b_8 \ v \ a_6b_{13}.$$

Brother pairs are generated by a_0, b_0 . Brothers a_0 and b_0 both partner a_6 .

Lemma 5.6 A $BARRDT(n)$ that is not a $SAMDRR(n)$ exists for all $n > 15$.

Proof: There is a standard construction of $SAMDRR(n)$ s from self orthogonal latin squares of order n , i.e. from $SOLS(n)$ s (see e.g., [6, p. 601]). An $ISOLS(n, k)$ is an incomplete $SOLS(n)$ missing a $SOLS(k)$ sub-square, which we could take to be in the upper left position and using the first k symbols. There exists an $ISOLS(n, 5)$ for all $n > 15$ (see [7, p. 214]). Using this in the standard construction yields an incomplete $SAMDRR(n)$, which in turn yields a $BARRDT(n)$ missing a $BARRDT(5)$ on the first five pairs of brothers. We supply the missing $BARRDT(5)$ as one that is not a $SAMDRR(5)$ to achieve our result. ■

Theorem 5.7 A $BARRDT(n)$ that is not a $SAMDRR(n)$ exists for all $n > 4$.

Proof: We have constructed a $BARRDT(n)$ that is not a $SAMDRR(n)$ for $n = 5$ (Theorem 2.3 or Theorem 3.2), 7 (Theorem 3.2), 8 (Theorem 4.4). A $BARRDT(n)$ that is not a $SAMDRR(n)$ exists for $n = 11$ and 13 by Theorem 3.2, and exists for $n = 12$ by Theorem 4.4. Examples 5.1– 5.5 cover the cases $n=6, 9, 10, 14,$ and 15. A $BARRDT(n)$ that is not a $SAMDRR(n)$ exists for $n > 15$ by Lemma 5.6. This completes the proof. ■

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