The Connected and Tree Domination Number of $P(n,k)$ for $k = 4, 6, 8$ *

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Abstract. Let $\gamma_c(G)$ be the connected domination number of G and $\gamma_{tr}(G)$ be the tree domination number of G. In this paper,we study the connected domination number and tree domination of $P(n,k)$, and show that $\gamma_{tr}(P(n,4)) = \gamma_c(P(n,4)) = n-1$ for $n \geq 17$, $\gamma_{tr}(P(n,6)) = \gamma_c(P(n,6)) = n-1$ for $n \geq 25$ and $\gamma_{tr}(P(n,8)) = \gamma_c(P(n,8)) = n-1$ for $n \geq 33$.

Keywords. Generalized Petersen graph; tree domination number; connected domination number

1 Introduction

We only consider finite connected and undirected graphs without loops or multiple edges.

Let $G = (V(G), E(G))$ be a graph with $|V(G)| = p$ and $|E(G)| = q$.

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The open neighborhood and the closed neighborhood of a vertex $v \in V$ are denoted by $N(v) = \{u \in V(G) : vu \in E(G)\}$ and $N[v] = N(v) \cup \{v\}$, respectively. For a vertex set $S \subseteq V(G)$, $N(S) = \bigcup_{v \in S} N(v)$ and $N[S] = \bigcup_{v \in S} N[v]$. A set $S \subseteq V(G)$ is a dominating set if and only if $N[S] = V(G)$. The domination number $\gamma(G)$ is the minimum cardinalities of minimal dominating sets.

Sampathkumar and Walikar [5] defined a connected dominating set S to be a dominating set S whose induced subgraph $G[S]$ is connected. The minimum cardinality of a connected dominating set of G is the connected domination number $\gamma_c(G)$.

Chen et al.[1] etc. defined a tree dominating set S to be a dominating set S whose induced subgraph $G[S]$ is a tree. The minimum cardinality of a tree dominating set of G is the tree domination number $\gamma_{tr}(G)$. If there is no tree dominating set in G, then let $\gamma_{tr}(G) = 0$. They showed the exact values of the tree domination number for several classes of graphs, including $P_p, C_p, K_p, K_{1,p-1}, K_{r,s}$ and T, and gave several bounds for γ_{tr} and the relationship of γ_c and γ_{tr}.

Observation 1.1. If $\gamma_{tr}(G) > 0$, then $\gamma_c(G) \leq \gamma_{tr}(G)$.

Theorem 1.2. Let G be a connected graph with $\delta(G) \geq 2$. If $\gamma_{tr}(G) > 0$, then $\gamma_{tr}(G) \geq \frac{(\delta+1)p-2q-2}{\delta-1}$ and the bound is sharp.

Corollary 1.3. Let G be a connected k-regular graph and $k \geq 2$. If $\gamma_{tr}(G) > 0$, then $\gamma_{tr}(G) \geq \frac{p-2}{k-1}$ and the bound is sharp.

Theorem 1.4. Every connected graph G contains a spanning connected subgraph H such that $\gamma_{tr}(H) = \gamma_c(G)$.

The generalized Petersen graph $P(n,k)$ is defined to be a graph on $2n$ vertices with $V(P(n,k)) = \{v_i, u_i : 0 \leq i \leq n-1\}$ and $E(P(n,k)) = \{v_i v_{i+1}, v_i u_i, u_i u_{i+k} : 0 \leq i \leq n-1, \text{ subscripts module } n\}$. Let S be the minimum tree domination set of $P(n,k)$.

In [6], we studied the connected domination number and tree domination number of the generalized Petersen graph $P(n,k)$, and showed the following lemma and theorem.

Lemma 1.5. For $n \neq 2k$, $n-1 \leq \gamma_{tr}(P(n,k)) \leq n$.

Theorem 1.6. $\gamma_c(P(n,k)) = \gamma_{tr}(P(n,k))$.

In this paper,we study the connected domination number and tree domination of $P(n,k)$. In sections 2-4, we show that $\gamma_{tr}(P(n,4)) = \gamma_c(P(n,4)) = n-1$ for $n \geq 17$, $\gamma_{tr}(P(n,6)) = \gamma_c(P(n,6)) = n-1$ for $n \geq 25$ and $\gamma_{tr}(P(n,8)) = \gamma_c(P(n,8)) = n-1$ for $n \geq 33$, respectively. In section 5, we conjecture that $\gamma_{tr}(P(n,k)) = \gamma_c(P(n,k)) = n-1$ for even $k \geq 4$ and $n \geq 4k+1$.

2 The tree and connected domination number of $P(n,4)$

We left for reader to verify that $\gamma_{tr}(P(n,4)) = n$, for $n \in \{11, 12, 16\}$. In Figure 2.1(1)-(5), we show tree dominating sets S_n of $P(n,4)$ with $|S_n| = n-1$ for $n = 9, 10, 13, 14, 15$, where the vertices in S_n are in dark. Hence we have

Lemma 2.1. $\gamma_{tr}(P(n,4)) = n-1$, for $n \in \{9, 10, 13, 14, 15\}$.

Theorem 2.2. $\gamma_{tr}(P(n,4)) = n-1$, for $n \geq 17$.
Proof. In Figure 2.1(6), we show $\gamma_{tr}(P(20,4)) = 19$. For $n \geq 17$ and $n \neq 20$, let

$S_0^4 = \{u_0, u_1, v_3, u_3, u_5, v_7, u_7, v_{10}, u_{10}, v_{11}, u_{11}, v_{12}, u_{12}, v_{13}, v_{16}, u_{16}, v_{19}, v_{20}, u_{20}, v_{21}, u_{21}, v_{22}, u_{22}\}$ (see Figure 2.1(10): $P(24,4)$),

$S_1^4 = \{u_0, u_1, v_3, u_3, v_4, u_4, v_7, u_7, v_{10}, u_{10}, v_{11}, u_{11}\}$ (see Figure 2.1(7): $P(21,4)$),

$S_2^4 = \{u_0, u_1, v_3, u_3, u_5, v_7, u_7, v_{10}, u_{10}, v_{11}, u_{11}, v_{12}, u_{12}\}$ (see Figure 2.1(8): $P(22,4)$),

$S_3^4 = \{u_0, u_1, v_3, u_3, v_6, v_7, u_7, v_{10}, v_{11}, u_{11}, v_{12}, u_{12}, v_{13}, u_{13}\}$ (see Figure 2.1(9): $P(23,4)$).

Let $m = \lfloor n/4 \rfloor$, $t = n \bmod 4$, then $n = 4m + t$ ($n \geq 17$ and $n \neq 20$).

Let

$$S^4 = \begin{cases} S_0^4 \cup \bigcup_{i=0}^{m-7}\{u_{4i+j} : 24 \le j \le 27\}, & \text{if } t = 0, n \ne 20, \\ S_1^4 \cup \bigcup_{i=0}^{m-4}\{u_{4i+j} : 13 \le j \le 16\}, & \text{if } t = 1, \\ S_2^4 \cup \bigcup_{i=0}^{m-4}\{u_{4i+j} : 14 \le j \le 17\}, & \text{if } t = 2, \\ S_3^4 \cup \bigcup_{i=0}^{m-4}\{u_{4i+j} : 15 \le j \le 18\}, & \text{if } t = 3. \end{cases}$$

Then $G[S^4]$ is a tree and $N[S^4] = V(P(n,4))$. So S^4 is a tree dominating set of $P(n,4)$ with $|S^4| = n-1$ and $\gamma_{tr}(P(n,4)) \le n-1$. By Lemma 1.5, we have $\gamma_{tr}(P(n,4)) \ge n-1$. Hence $\gamma_{tr}(P(n,4)) = n-1$. □

By Theorem 1.6 and Theorem 2.2, we have

Corollary 2.3. For $n \ge 17$, $\gamma_c(P(n,4)) = n-1$.

3 The tree and connected domination number of $P(n,6)$

We left for reader to verify that $\gamma_{tr}(P(n,6)) = n$, for $n \in \{16, 18, 20, 24\}$. In Figure 3.1, we show tree dominating sets S_n of $P(n,6)$ with $|S_n| = n-1$ for $n = 13, 14, 15, 17, 19, 21, 22, 23$. Hence we have

Lemma 3.1. $\gamma_{tr}(P(n,6)) = n-1$, for $n \in \{13, 14, 15, 17, 19, 21, 22, 23\}$.

Theorem 3.2. $\gamma_{tr}(P(n,6)) = n-1$, for $n \ge 25$.
Proof. In Figure 3.2(1)(2), we show $\gamma_{tr}(P(26,6)) = 25$ and $\gamma_{tr}(P(28,6)) = 27$. Let

$S_1^6 = \{v_1, u_1, u_3, v_5, u_5, v_6, v_7, u_7, v_8, v_9, u_9, u_{11}, v_{13}, u_{13}\}$
(see Figure 3.2(4): $P(31,6)$),

$S_0^6 = \{u_0, u_1, u_2, u_3, v_5, u_5, v_8, u_8, v_{11}, u_{11}, v_{14}, u_{14}, v_{15}, v_{16}, u_{16}, v_{17}, u_{17}, v_{18}, u_{18}, v_{19}, u_{19}, v_{22}, u_{22}, u_{24}, u_{25}, v_{27}, u_{27}, v_{28}, u_{28}\}$
(see Figure 3.2(3): $P(30,6)$),

$S_2^6 = \{u_0, u_1, u_2, u_3, v_5, u_5, u_7, v_9, u_9, v_{10}, v_{11}, u_{11}, u_{13}, v_{15}, u_{15}, v_{18}, u_{18}, v_{19}, u_{19}, v_{20}, u_{20}, u_{22}, v_{24}, u_{24}, u_{26}, v_{28}, u_{28}, v_{29}, u_{29}, v_{30}, u_{30}\}$
(see Figure 3.2(5): $P(32,6)$),

$S_4^6 = \{u_0, u_1, u_2, u_3, v_5, u_5, u_7, v_9, u_9, v_{10}, v_{11}, u_{11}, u_{13}, v_{15}, u_{15}, v_{18}, u_{18}, v_{19},$

$u_{19}, v_{20}, u_{20}, v_{21}, u_{21}, v_{22}, u_{22}, u_{24}, v_{26}, u_{26}, u_{28}, u_{29}, u_{30}, v_{32}, u_{32}\}$ (see Figure 3.2(6): $P(34, 6)$).

Let $m = \lfloor n/6 \rfloor$, $t = n \bmod 6$, then $n = 6m + t$ ($n \geq 25$ and $n \neq 26, 28$). Let

$$S^6 = \begin{cases} S_1^6 \cup \bigcup_{i=0}^{(n-17)/2} \{u_{2i+j} : 15 \leq j \leq 16\}, & \text{if } t = 1, 3, 5, \\ S_0^6 \cup \bigcup_{i=0}^{m-6} \{u_{6i+j} : 30 \leq j \leq 35\}, & \text{if } t = 0, \\ S_2^6 \cup \bigcup_{i=0}^{m-6} \{u_{6i+j} : 32 \leq j \leq 37\}, & \text{if } t = 2, \\ S_4^6 \cup \bigcup_{i=0}^{m-6} \{u_{6i+j} : 34 \leq j \leq 39\}, & \text{if } t = 4. \end{cases}$$

Then $G[S^6]$ is a tree and $N[S^6] = V(P(n, 6))$. So S^6 is a tree dominating set of $P(n, 6)$ with $|S^6| = n - 1$ and $\gamma_{tr}(P(n, 6)) \leq n - 1$. By Lemma 1.5, we have $\gamma_{tr}(P(n, 6)) \geq n - 1$. Hence $\gamma_{tr}(P(n, 6)) = n - 1$. □

By Theorem 1.6 and Theorem 3.2, we have

Corollary 3.3. For $n \geq 25$, $\gamma_c(P(n, 6)) = n - 1$.

4 The tree and connected domination number of $P(n, 8)$

We left for reader to verify that $\gamma_{tr}(P(24, 8)) = 24$, $\gamma_{tr}(P(32, 8)) = 32$ and $\gamma_{tr}(P(n, 8)) = n - 1$ for $17 \leq n \leq 31$, $n \neq 24$. We have

Lemma 4.1. $\gamma_{tr}(P(n, 8)) = n - 1$ for $17 \leq n \leq 31$, $n \neq 24$.

Theorem 4.2. $\gamma_{tr}(P(n, 8)) = n - 1$, for $n \geq 33$.
Proof. Let

$S_0^8 = \{u_0, u_1, u_2, u_3, v_5, u_5, v_6, u_6, v_7, u_7, v_8, u_8, v_{11}, u_{11}, u_{13}, v_{15}, u_{15}, v_{18}, v_{19}, u_{19}, v_{20}, u_{20}, v_{23}, u_{23}, v_{24}, v_{25}, u_{25}, v_{28}, u_{28}, v_{29}, v_{30}, u_{30}, v_{33}, u_{33}, v_{34}, u_{34}, u_{36}, v_{38}, u_{38}\}$ (see Figure 4.1(8): $P(40, 8)$),

$S_1^8 = \{u_0, u_1, v_3, u_3, u_5, v_7, u_7, v_8, u_8, v_{11}, u_{11}, v_{12}, v_{13}, u_{13}, v_{14}, v_{15}, u_{15}, v_{18}, u_{18}, v_{19}, u_{19}, u_{21}, v_{23}, u_{23}\}$ (see Figure 4.1(1): $P(33, 8)$),

$S_2^8 = \{v_1, u_1, v_2, u_2, v_5, u_5, v_6, u_6, v_7, u_7, v_{10}, u_{10}, v_{11}, u_{13}, u_{14}, v_{16}, u_{16}, v_{17}, v_{18}, u_{18}, u_{20}, u_{21}, u_{22}, v_{24}, u_{24}\}$ (see Figure 4.1(2): $P(34, 8)$),

$S_3^8 = \{u_0, u_1, u_2, u_3, u_4, u_5, v_7, u_7, u_9, v_{11}, u_{11}, v_{12}, u_{12}, v_{13}, u_{13}, u_{15}, v_{17}, u_{17}\}$

(see Figure 4.1(3): $P(35,8)$),

$S_4^8 = \{u_0, v_2, u_2, v_3, u_3, v_4, u_4, v_5, u_5, u_7, v_9, v_{10}, u_{10}, u_{12}, u_{13}, v_{15}, u_{15}, v_{18}, u_{18}\}$ (see Figure 4.1(4): $P(36,8)$),

$S_5^8 = \{v_1, u_1, u_3, v_5, u_5, v_6, u_6, v_7, u_7, u_9, v_{11}, u_{11}\}$ (see Figure 4.1(5): $P(37,8)$),

$S_6^8 = \{u_0, v_2, u_2, v_5, u_5, v_6, u_6, v_7, u_7, v_{10}, u_{10}, v_{11}, u_{13}, u_{14}, u_{15}, v_{17}, u_{17}, v_{18}, u_{18}, v_{19}, v_{20}, u_{20}, u_{22}, u_{23}, u_{24}, u_{25}, u_{26}, v_{28}, u_{28}\}$ (see Figure 4.1(6): $P(38,8)$),

$S_7^8 = \{u_0, u_1, u_2, u_3, u_4, u_5, v_7, u_7, v_8, u_8, v_9, u_9, u_{11}, v_{13}, u_{13}, v_{14}, v_{17}, u_{17}, v_{18}, v_{19}, u_{19}, v_{20}, v_{21}, u_{21}, v_{22}, v_{25}, u_{25}, u_{27}, v_{29}, u_{29}\}$ (see Figure 4.1(7): $P(39,8)$).

Let $m = \lfloor n/8 \rfloor$, $t = n \bmod 8$, then $n = 8m + t$ ($n \geq 33$). Let

$$S^8 = \begin{cases} S_0^8 \cup \bigcup_{i=0}^{m-6}\{u_{8i+j} : 40 \leq j \leq 47\}, & \text{if } t = 0, \\ S_1^8 \cup \bigcup_{i=0}^{m-4}\{u_{8i+j} : 25 \leq j \leq 32\}, & \text{if } t = 1, \\ S_2^8 \cup \bigcup_{i=0}^{m-4}\{u_{8i+j} : 26 \leq j \leq 33\}, & \text{if } t = 2, \\ S_3^8 \cup \bigcup_{i=0}^{m-3}\{u_{8i+j} : 19 \leq j \leq 26\}, & \text{if } t = 3, \\ S_4^8 \cup \bigcup_{i=0}^{m-3}\{u_{8i+j} : 20 \leq j \leq 27\}, & \text{if } t = 4, \\ S_5^8 \cup \bigcup_{i=0}^{m-2}\{u_{8i+j} : 13 \leq j \leq 20\}, & \text{if } t = 5, \\ S_6^8 \cup \bigcup_{i=0}^{m-4}\{u_{8i+j} : 30 \leq j \leq 37\}, & \text{if } t = 6, \\ S_7^8 \cup \bigcup_{i=0}^{m-4}\{u_{8i+j} : 31 \leq j \leq 38\}, & \text{if } t = 7. \end{cases}$$

Then $G[S^8]$ is a tree and $N[S^8] = V(P(n,8))$, for $n \geq 33$. So S^8 is a tree dominating set of $P(n,8)$ with $|S^8| = n-1$ and $\gamma_{tr}(P(n,8)) \leq n-1$. By Lemma 1.5, we have $\gamma_{tr}(P(n,8)) \geq n-1$. Hence $\gamma_{tr}(P(n,8)) = n-1$. □

By Theorem 1.6 and Theorem 4.2, we have

Corollary 4.3. For $n \geq 33$, $\gamma_c(P(n,8)) = n-1$.

5 Conclusion

We have the following conjecture,

Conjecture 5.1. $\gamma_{tr}(P(n,k)) = \gamma_c(P(n,k)) = n-1$, for even $k \geq 4$ and $n \geq 4k+1$.

By Theorem 2.2,3.2,4.2 and Corollary 2.3,3.3,4.3, Conjecture 5.1 holds for $k = 4, 6, 8$.

References

[1] X. Chen, L. Sun and A. McRae, Tree domination in graphs, *ARS Combinatoria*, 73(2004)193-203.

[2] S. T. Hedetniemi and R. C. Lasker, Connected domination in graphs, In B.Bollobás, editor, *Graph Theory and Combinatorics*, Academic Press, London, 1984, 209-218.

[3] D. J. Kleitman and D. B. West, Spanning trees with many leaves, *SIAM J. Discrete Math.*, 4(1991), 99-106.

[4] J. Nieminen, Two bounds for the domination number of a graph, *J. Inst. Math. Applics.*, 14(1974) 183-187.

[5] E. Sampathkumar and H. B. Walikar. The connected domination number of a graph, *J. Math. Phys. Sci.*, 13(1979)607-613.

[6] Zhao Chengye, Yang Yuansheng, Sun Linlin. The connected domination and tree domination of $P(n,k)$ for $k = 1, 2, \lfloor n/2 \rfloor$, *ARS Combinatoria*, to appear.

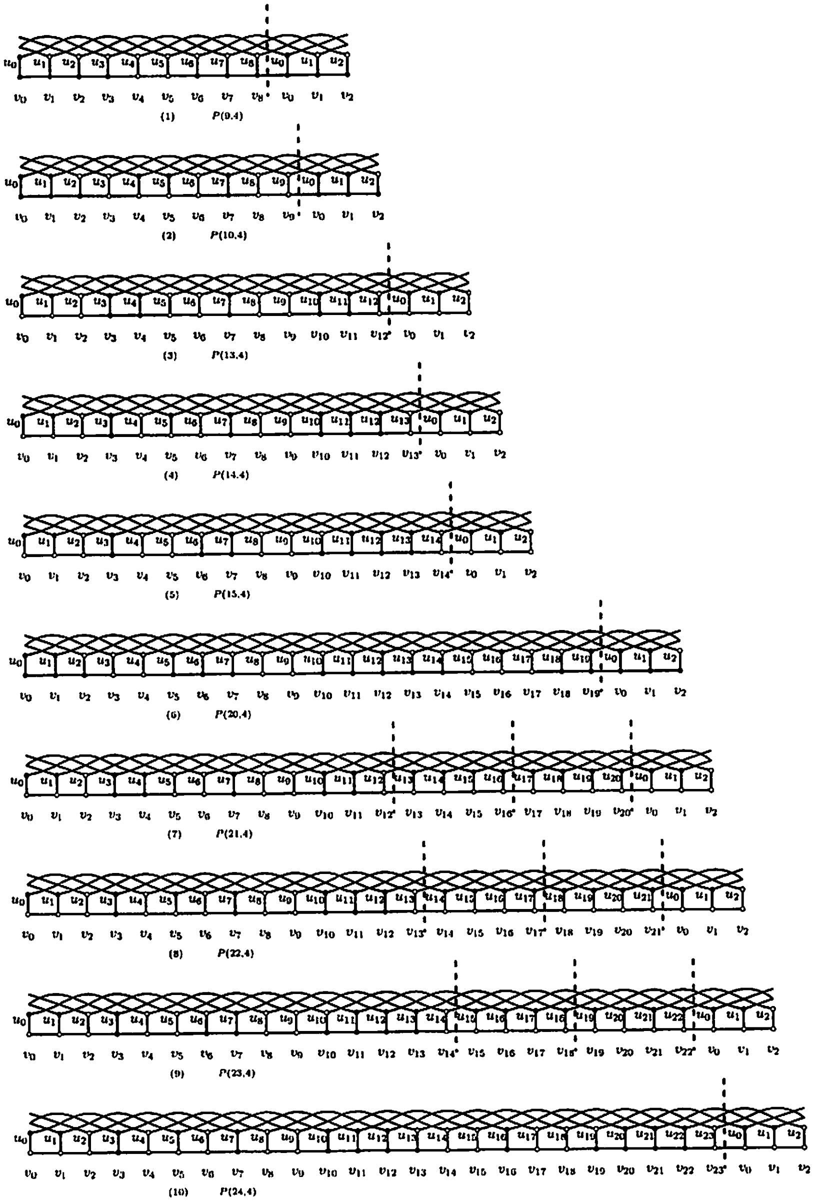

Figure 2.1: The tree dominating sets of $P(n, 4)$

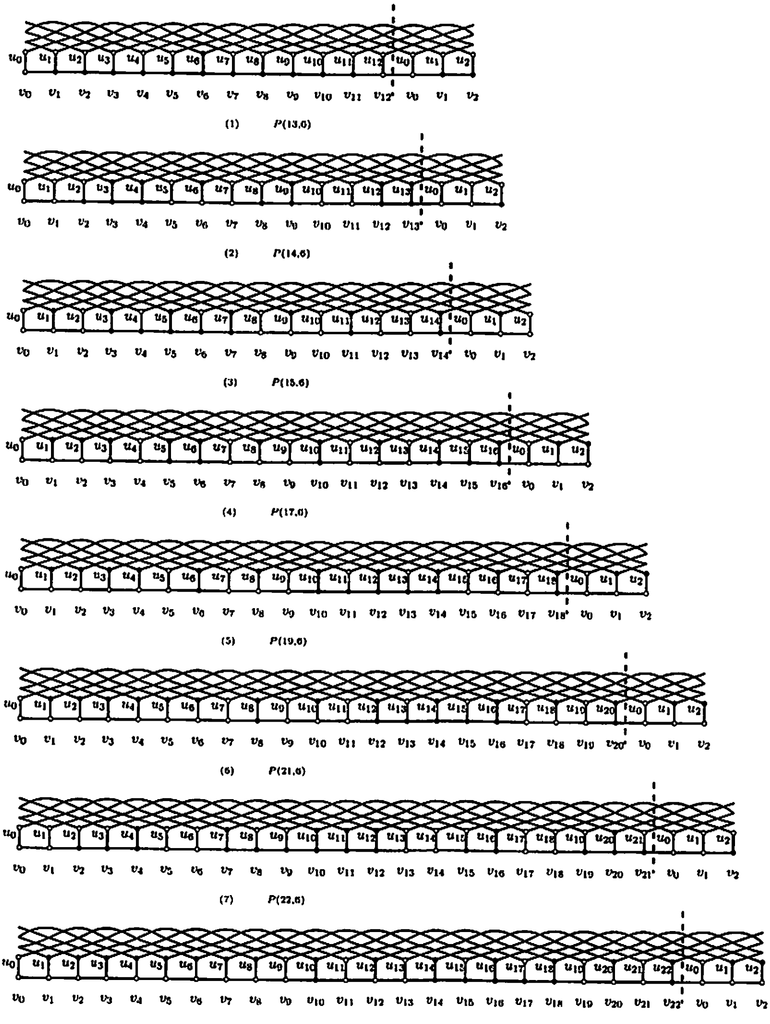

Figure 3.1: The tree dominating sets of $P(n, 6)$ (1)

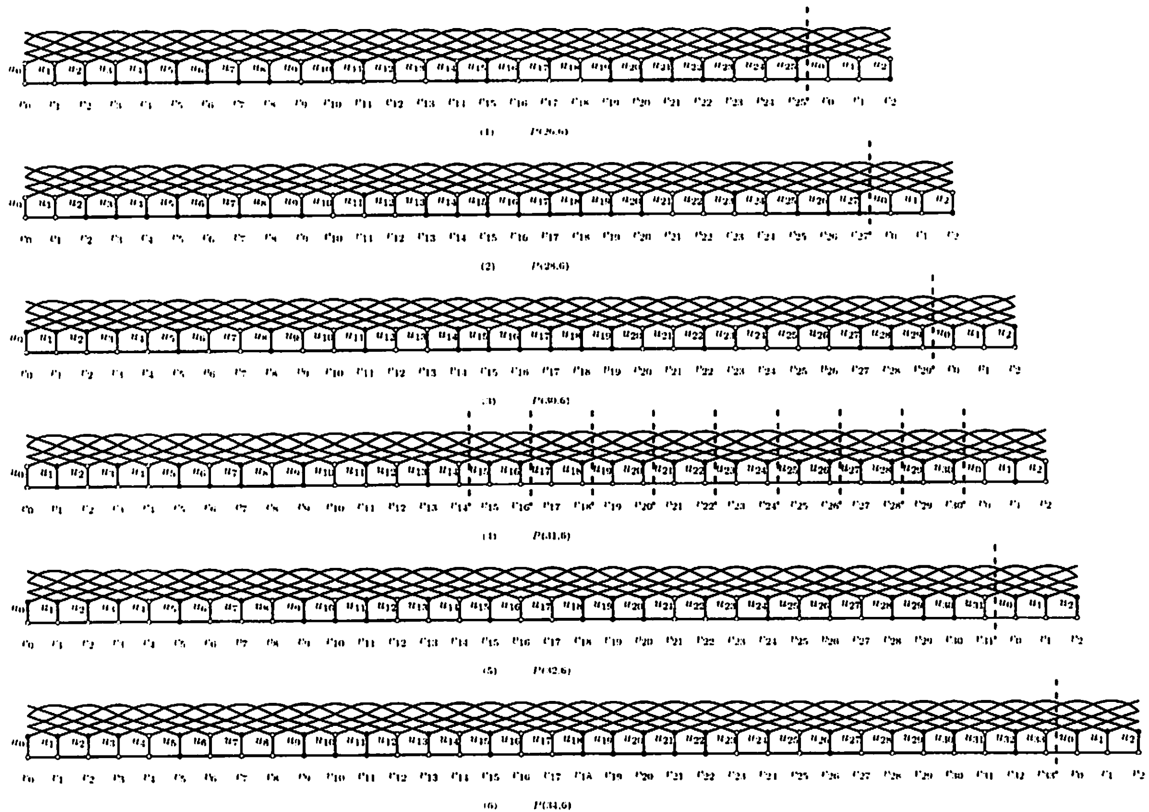

Figure 3.2: The tree dominating sets of $P(n, 6)$ (2)

Figure 4.1: The tree dominating sets of $P(n, 8)$