A GENUS INEQUALITY OF THE UNION GRAPHS

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ABSTRACT. The genus of a graph G, denoted by $\gamma(G)$, is the minimum genus of an orientable surface in which the graph can be embedded. In the paper,we use the Joint Tree Model to immerse a graph on the plane and obtain an associate polyhegon of the graph. Along the way, we construct genus embedding of the edge disjoint union of K and H, and solve Michael Stiebitz. etc. proposed conjecture: Let G be the edge disjoint union of a complete graph K and an arbitrary graph H. Let H' be the graph obtained from H by contracting the set $V(K)$ to a single vertex. Then

$$\gamma(K) + \gamma(H') \le \gamma(G).$$

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1. Introduction

All graphs considered,in this paper,are connected and its embeddings are orientable. The vertex set and edge set of a graph G are denoted by $V(G)$ and $E(G)$,respectively. A surface is a compact closed 2-dimensional manifold without boundary. It can be represented as a polygon of even edges in the plane,whose edges are pairwise and directed clockwise or counterclockwise. Such a polygon is called a polyhegon in terminology[2]or[3]. Further,an orientable polyhegon S can be represented as a cyclic sequence of letters in parentheses,where each letter appears exactly twice and the two occurrences of each letter have distinct indices +(always omitted) and

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$-$. Let A and B be sections of successive letters in cyclic orders(linear sequences) and $\emptyset$ be the empty. The elementary(or topological) equivalence $\sim$ is obtained by the following three operations on a set of surfaces (or polyhegons) $\mathcal{S}$[3]:

Op.1 $\forall S \in \mathcal{S}$, $S = (Aaa^{-}B)$, $A \neq \emptyset$, or $B \neq \emptyset \Longleftrightarrow S = (AB)$.

Op.2 $\forall S \in \mathcal{S}$, $S = (AabBb^{-}a^{-}C) \Longleftrightarrow S = (AaBa^{-}C)$.

Op.3 $\forall S \in \mathcal{S}$, $S = (AaBCa^{-}D) \Longleftrightarrow S = (BaADa^{-}C)$.

On the basics of these operations, the following lemmas can be done.

Lemma 1.1[3] Polyhegons are classified by the following transformations,$\forall S \in \mathcal{S}$,

$$S = (AxByCx^{-}Dy^{-}E) \sim (ADCBExyx^{-}y^{-}).$$

where A, B, C, D and E are linear sequences, $x, y, x^{-}, y^{-} \notin ABCDE$.

From the lemma1.1, each orientable polyhegonis equivalent to one, and only one of the following canonical forms:

$$S = \begin{cases} (a_0a_0^{-}), & \text{if the surface is sphere;} \\ (\prod_{k=1}^{p} a_kb_ka_k^{-}b_k^{-}), & \text{if the genus of a surface is } p. \end{cases}$$

A topological representation of a graph on a surface as a point set without interior point in common between two edges as curve segments is called an embedding. If an intersection at an interior point between two curve segments is permitted,then what obtained is an immersion.

An embedding (or cellular embedding) of a graph G into a surface S is a homeomorphism i: $G \rightarrow S$, such that each component of $S - i(G)$ is homeomorphic to an open disc. Two embeddings f: $G \rightarrow S$ and g: $G \rightarrow S$ are the same if there is a homeomorphism h: $S \rightarrow S$ such that $f = gh$. An embedding is called orientable if S is orientable. Let $g = g(S)$ denote a genus of an orientable surface S. The genus of a graph G,denoted by $\gamma(G)$,is the minimum genus of an orientable surface in which the graph can be embedded. $\gamma_M(G) = \max\limits_{S \in A(G)} g(S)$ is called the maximum orientable genus of G,where $A(G)$ is the set of all associate surfaces of a graph G.

Lemma 1.2[8] Let G be a connected graph.Then the genus range $GR(G)$ is an unbroken interval of integers$[\gamma(G), \gamma_M(G)]$.

Lemma1.3[5] A connected graph G has embeddings in all surfaces S_g,where,

$$\gamma(G) \leq g \leq \gamma_M(G).$$

Let G be a graph,T be a spanning tree of G. Firstly, to immerse G on the plane;Secondly,for each nontree edge e of G, to split e into two semiedges e^{+}($+$ often omitted),e^{-}. It is obvious that a new graph obtained by splitting of each cotree edge is a tree,which is called a joint tree,denoted $\tilde{T}$.

For $v \in V(G)$,let ρ_v be a rotation at v,ρ_v is a cyclic permutation of semiedges incident with v. $\rho_G = \{\rho_v | \forall v \in V(G)\}$ is the rotation system of G. Then ρ_G induces a unique embedding of G into an orientable surface.Conversely,every embedding of a graph G into an orientable surface induces a unique rotation system for G.

For a joint tree $\tilde{T}$ of G,a cyclic sequence of all letters of semi-edges along the clockwise(or anti-clockwise)rotation is the embedded polyhegon of G. It is also called an associate polyhegon of G.

So an embedding of a graph G into a surface S can be represented by a joint tree of G,i.e. by an associate polyhegon of G.

For a fixed spanning tree T of G,there is a 1-to-1 correspondence between the set of the associte polyhegons and the set of the embeddings of G.

Lemma 1.4[2] The number of distinct embeddings of G on a surface of orientable genus p is independent of the choice of tree T on G.

Lemma 1.5[2] If $S_1 = (a_1, a_2, \dots, a_{2s})$,$S_2 = (b_1, b_2, \dots, b_{2t})$ are two orientable polyhegons, $\{a_1, a_2, \dots, a_{2s}\} \bigcap \{b_1, b_2, \dots, b_{2t}\} = \emptyset$ then

$$S = (a_1, a_2, \dots, a_{2s}, b_1, b_2, \dots, b_{2t})$$

is an orientable polyhegon,and $g(S) = g(S_1) + g(S_2)$.Where $g(S)$ is the genus of S.

Lemma 1.6 [2] If $S = (a_1, a_2, \dots, a_{2s})$,is an orientable polyhegon , $t \notin \{a_1, a_2, \dots, a_{2s}\}$,then $Q = (a_1, \dots, a_{i_1}, t, a_{i_1+1}, \dots, a_{i_2}, t^-, a_{i_2+1}, \dots, a_{2s})$ is an orientable polyhegon,and $g(Q) \geq g(S)$.Where $g(S)$ is the genus of S.

Proof: By the definition of an orientable associate polyhegon,the first part of the lemma is true.By lemma 1.1,let $(\prod_{i=1}^{k} x_i y_i x_i^- y_i^-)$ be a canonical form of S,then

(i). if $k = \frac{s}{2}$,then,

$$Q \sim ((\prod_{i=1}^{k} x_i y_i x_i^- y_i^-)tt^-) \sim (\prod_{i=1}^{k} x_i y_i x_i^- y_i^-),$$

$$g(Q) = k = g(S).$$

(ii). if $k < \frac{s}{2}$,then,

$$S \sim ((\prod_{i=1}^{k} x_i y_i x_i^- y_i^-)zz^-).$$

Where $z = x_{2k+1} \dots x_s$.$Q \sim ((\prod_{i=1}^{k} x_i y_i x_i^- y_i^-)tzz^-t^-)$,$g(Q) = g(S) = k$,or $Q \sim ((\prod_{i=1}^{k} x_i y_i x_i^- y_i^-)tzt^-z^-) \sim (\prod_{i=1}^{k+1} x_i y_i x_i^- y_i^-)$,$g(Q) = k + 1 > g(S)$.

Therefore

$$g(Q) \geq g(S).$$

Lemma 1.7[2] If G_1 and G are two graphs,G_1 is a subgraph of G,then,

$$\gamma(G_1) \leq \gamma(G).$$

Proof:By the definition of the genus of a graph ,it is obvious.

2. Main results

Michael Stiebitz.,etc.,searched a Heawood type formula for the edge disjoint union of two graphs that are embedded on a given surface Σ.They proposed the following conjecture in[4].

Conjecture: Let G be the edge disjoint union of a complete graph K and an arbitrary graph H. Let H'be the graph obtained from H by contracting the set $V(K)$ to a single vertex.Then

$$\gamma(K) + \gamma(H') \leq \gamma(G).$$

Following ,we use the Joint Tree Model to prove that the conjecture is true.

Lemma2.1 [5] Let e be a cut edge of a graph G,$G-e$ has two components H and K,then

$$\gamma(K) + \gamma(H) = \gamma(G).$$

Corollary2.1 Let K be a complete graph of order n,H be an arbitrary graph of order m,$V(K) = \{v_1, v_2, \dots, v_n\}$ is the vertex set of K,$V(H) = \{u_1, u_2, \dots, u_m\}$ is the vertex set of H,$V(K) \cap V(H) = \emptyset$, $V(G) = V(H) \bigcup V(K)$, $E(G) = E(H) \bigcup E(K) \bigcup \{u_1 v_1\}$. Let H' be the graph obtained from H by contracting the set $V(K)$ to a single vertex.Then

$$\gamma(H') = \gamma(H), \gamma(H') + \gamma(K) \leq \gamma(G).$$

If a graph G is the union of two graphs H and K,and $V(H) \cap V(K) = v, E(H) \cap E(K) = \emptyset$,then one says that the graph G is an amalgamation of two graphs H and K at avertex v,denoted $G = H *_v K$.Where v is the vertex of amalgamation.

Lemma2.2 [6] If a graph G is an amalgamation of two graphs H and K at a vertex v,then,

$$\gamma(G) = \gamma(H) + \gamma(K).$$

Corollary2.2 Let K be a complete graph of order n,H be an arbitrary graph of order m,$V(K) = \{v, v_2, \dots, v_n\}$ is the vertex set of K,$V(H) =$

$\{v, u_2, \dots, u_m\}$ is the vertex set of H,$G = H *_v K$.Let H' be the graph obtained from H by contracting the set $V(K)$ to a single vertex. Then,

$$\gamma(H') + \gamma(K) \leq \gamma(G).$$

Theorem 2.1 (main theorem) Let K be a complete graph of order n,H be an arbitrary graph of order m.G is obtained from H and K by adding some edges between $V(H)$ and $V(K)$.Let H' be a graph obtained from G by contracting the set $V(K)$ to a single vertex.Then

$$\gamma(H') + \gamma(K) \leq \gamma(G).$$

Proof: Let $V(K) = \{v_1, v_2, \dots, v_n\}$ be the vertex set of K,$V(H) = \{u_1, u_2, \dots, u_m\}$ be the vertex set of H.Let G be obtained by adding the edges $e_1, e_2, e_3, \dots, e_p$ $(1 \leq p)$ on $V(K)$ and $V(H)$.

When $p = 1$,by Corollary 2.1,the conclusion is true.

When $2 \leq p$,one discusses the conclusion in two cases,

(1). If the p edges join a vertex v_1 of the graph K and different vertices of the graph H.Then $G = K *_{v_1} H'$, by Corollary 2.2 ,the conclusion is true.

If the p edges join a vertex u_1 of the graph H and different vertices of the graph K.By contracting the set $V(K)$ to a single vertex v_1,we obtain the graph H'.$H' = H *_{u_1} H_1$.Where $V(H_1) = \{u_1, v_1\}$,$E(H_1) = \{e_1, e_2, \dots, e_p\}$.

Therefore,

$$\gamma(H_1) = 0, \gamma(H') = \gamma(H) + \gamma(H_1) = \gamma(H).$$

By contracting the set $V(H)$ to a single vertex u_1,we obtain the graph K'.It is obvious that $G = K' *_{u_1} H$.By lemma 2.2,we obtain,

$$\gamma(G) = \gamma(K') + \gamma(H).$$

But K is a subgraph of K',by lemma 1.7 ,we have,

$$\gamma(K') \geq \gamma(K).$$

Therefore,

$$\gamma(G) \geq \gamma(H') + \gamma(K).$$

(2).If the p edges join different vertices of the graph K and different vertices of the graph $H(p \geq 2)$.

When $p = 2$,one selects a edge $e_1 = v_1 u_1$ as a tree edge of G, $e_2 = v_i u_j$ as a cotree edge of G.Let T_1, T_2 be two trees of K, H respectively,$\bar{T}_1, \bar{T}_2$ be two cotrees of K,H correspondingly,then $T = T_1 + e_1 + T_2$ is a tree of G,$\bar{T} = \bar{T}_1 + e_2 + \bar{T}_2$ is a cotree of G.Contracting K to a single vertex v_1,then $e_1 + T_2$ is a tree of H',$v_1 u_j + \bar{T}_2$ is a cotree of H'.One labels the semiedges of $\bar{T}_1$ by $a_1, a_2, \dots, a_{2s-1}, a_{2s}$,where $\{a_1, a_2, \dots, a_{2s-1}, a_{2s}\} =$

$\{a_1, a_1^-, \ldots, a_s, a_s^-\}, s = \beta(K)$,$a_i$,$a_i^-$ are two semiedges obtained by splitting the edge a_i of $\bar{T}_1 (i = 1, 2, \ldots, s)$.One Labels the semiedges of $\bar{T}_2$ by $b_1, b_2, \ldots, b_{2t-1}, b_{2t}$,where $\{b_1, b_2, \ldots, b_{2t-1}, b_{2t}\} = \{b_1, b_1^-, \ldots, b_t, b_t^-\}$,$t = \beta(H)$,$b_i, b_i^-$ are two semiedges obtained by splitting the edge b_i of $\bar{T}_2(i = 1, 2, \ldots, t)$.$c$,$c^-$ are two semiedges obtained by splitting the edge $e_2 = v_i u_j$. d,d^- are two semiedges obtained by splitting the edge $v_1 u_j$.$\{a_1, a_1^-, \ldots, a_s, a_s^-\} \bigcap \{b_1, b_2, \ldots, b_{2t-1}, b_{2t}\} = \emptyset$.

Let $S_1 = (a_1 a_2 \ldots a_{2s-1} a_{2s})$ be an associate polyhegon of K who has the minimum genus $\gamma(K)$, $g(S_1) = \gamma(K)$,$S_2 = (b_1 b_2 \ldots b_{2t-1} b_{2t})$ be an associate polyhegon of H , $S_4 = (a_1 \ldots a_i c a_{i+1} \ldots a_{2s} z_1 \ldots z_j c^- z_{j+1} \ldots z_{2t})$ $(\{z_1, \ldots, z_j, z_{j+1}, \ldots, z_{2t}\} = \{b_1, \ldots, b_j, b_{j+1}, \ldots, b_{2t}\})$ be an associate polyhegon of G who has the minimum genus $\gamma(G)$, $\gamma(G) = g(S_4)$.Then $Q_2 = (c z_1 \ldots z_j c^- z_{j+1} \ldots z_{2t})$ is an associate polyhegon of H',$g(Q_2) \geq \gamma(H')$. By lemma1.1,

$S_4 \sim ((\prod_{i=1}^{k} x_i y_i x_i^- y_i^-) x_{2k+1} \ldots x_h c x_h^- \ldots x_{2k+1}^- x_{h+1} \ldots x_s x_s^- \ldots x_{h+1}^-$ $z_1 \ldots z_j c^- z_{j+1} \ldots z_{2t}) \sim ((\prod_{i=1}^{k} x_i y_i x_i^- y_i^-) a c a^- z_1 \ldots z_j c^- z_{j+1} \ldots z_{2t})$. where,$(a_1 a_2 \ldots a_{2s-1} a_{2s}) \sim (\prod_{i=1}^{k} x_i y_i x_i^- y_i^-)$, $\{x_1, x_1^-, \ldots, x_s, x_s^-\} = \{a_1, a_2, \ldots, a_{2s-1}, a_{2s}\}$,$a = x_{2k+1} \ldots x_h \notin \{z_1, z_2, \ldots, z_{2t-1}, z_{2t}, c, c^-\}$.
Denote

$$Q = (\prod_{i=1}^{k} x_i y_i x_i^- y_i^-), Q_1 = (a c a^- z_1 \ldots z_j c^- z_{j+1} \ldots z_{2t}).$$

Because,

$\{x_1, y_1, x_1^-, y_1^-, \ldots, x_k, y_k, x_k^-, y_k^-\} \bigcap \{a, a^-, c, c^-, z_1, \ldots, z_{2t}\} = \emptyset$. By lemma 1.5 ,

$$g(S_4) = g(Q) + g(Q_1), g(Q) = \gamma(K) = k.$$

By lemma 1.7 ,$g(Q_1) = g((a c a^- z_1 \ldots z_j c^- z_{j+1} \ldots z_{2t})) \geq g((c z_1 \ldots z_j c^- z_{j+1} \ldots z_{2t})) = g(Q_2) \geq \gamma(H')$. Therefore

$$g(Q_1) \geq \gamma(H'),$$

$$g(s_4) = g(Q) + g(Q_1) \geq \gamma(K) + \gamma(H'),$$

$$\gamma(G) \geq \gamma(K) + \gamma(H').$$

When $p \geq 3$,let $E_1 = \{c_2, c_2^-, \ldots, c_p, c_p^-\}$ be the semiedge set (where each edge e_i is split into two semi-edgesc_i, c_i^-,$i = 2, \ldots, p$),$e_1 = v_1 u_1$ be a tree edge of G, $S_6 = (a_1 \ldots a_{i_2} c_2 a_{i_2+1} \ldots a_{i_p} c_p a_{i_p+1} \ldots a_{2s} z_1 \ldots z_{m_2} c_{l_2}^- z_{m_2+1} \ldots z_{m_p} c_{l_p}^- z_{m_p+1} \ldots z_{2t})$ be an associate polyhegon of G who has the minimum $\gamma(G)$,$g(S_6) = \gamma(G)$.

Where $i_2, \ldots, i_p \in \{1, 2, \ldots, 2s\}, m_2, \ldots, m_p \in \{1, 2, \ldots, 2t\}. \{z_1, \ldots, z_{2t}\} = \{b_1, \ldots, b_{2t}\}$. By lemma 1.1

$S_6 \sim ((\prod_{i=1}^{k} x_i y_i x_i^- y_i^-) x_{h_1} \ldots x_{h_2} c_{k_2} x_{h_2}^- \ldots x_{h_1}^- \ldots x_{h_{2p-3}} \ldots x_{h_{2p-1}} c_{k_p} x_{h_{2p-1}}^- \ldots x_{h_{2p-3}}^- x_{h_{2p-1}} \ldots x_{h_{2p}} x_{h_{2p}}^- \ldots x_{h_{2p-1}}^- z_1 \ldots z_{m_2} c_{l_2}^- z_{m_2+1} \ldots z_{m_p} c_{l_p}^- z_{m_p+1} \ldots z_{2t}) \sim ((\prod_{i=1}^{k} x_i y_i x_i^- y_i^-)(t_2 c_{k_2} t_2^- \ldots t_p c_{k_p} t_p^- t_1 t_1^- z_1 \ldots z_{m_2} c_{l_2}^- z_{m_2+1} \ldots z_{m_p} c_{l_p}^- z_{m_p+1} \ldots z_{2t})$. Denote
$Q = (\prod_{i=1}^{k} x_i y_i x_i^- y_i^-)$, $Q_0 = (x_{h_1} \ldots x_{h_2} c_{k_2} x_{h_2}^- \ldots x_{h_1}^- \ldots x_{h_{2p-3}} \ldots x_{h_{2p-2}} c_{k_p} x_{h_{2p-2}}^- \ldots x_{h_{2p-3}}^- x_{h_{2p-1}} \ldots x_{h_{2p}} x_{h_{2p}}^- \ldots x_{h_{2p-1}}^- z_1 \ldots z_{m_2} c_{l_2}^- z_{m_2+1} \ldots z_{m_p} c_{l_p}^- z_{m_p+1} \ldots z_{2t})$, $Q_1 = (t_2 c_{k_2} t_2^- \ldots t_p c_{k_p} t_p^- t_1 t_1^- z_1 \ldots z_{m_2} c_{l_2}^- z_{m_2+1} \ldots z_{m_p} c_{l_p}^- z_{m_p+1} \ldots z_{2t})$, $Q_2 = (c_{k_2} \ldots c_{k_p} z_1 \ldots z_{m_2} c_{l_2}^- z_{m_2+1} \ldots z_{m_p} c_{l_p}^- z_{m_p+1} \ldots z_{2t})$.
Then Q_2 is an associate polyhegon of H', $g(Q_2) \geq \gamma(H')$,and $g(Q) = \gamma(K) = k$. Where,
$\{x_1, x_1^-, y_1, y_1^-, \ldots, x_k, x_k^-, y_k, y_k^-, x_{h_1}, x_{h_1}^-, \ldots, x_{h_2}, x_{h_2}^-, \ldots, x_{h_{2p-3}}, x_{h_{2p-3}}^-, \ldots, x_{h_{2p-1}}, x_{h_{2p-1}}^-, \ldots, x_{h_{2p}}, x_{h_{2p}}^-\} = \{a_1, a_1, \ldots, a_s, a_s^-\}$, $\{b_1, b_2, \ldots, b_{2t-1}, b_{2t}\} = \{z_1, z_2, \ldots, z_{2t-1}, z_{2t}\}$, $\{x_1, x_1^-, y_1, y_1^-, \ldots, x_k, x_k^-, y_k, y_k^-\} \bigcap \{x_{h_1}, x_{h_1}^-, \ldots, x_{h_2}, x_{h_2}^-, \ldots, x_{h_{2p-3}}, x_{h_{2p-3}}^-, \ldots, x_{h_{2p-1}}, x_{h_{2p-1}}^-, \ldots, x_{h_{2p}}, x_{h_{2p}}^-, c_{l_2}, c_{l_2}^-, \ldots, c_{l_p}, c_{l_p}^-, z_1, z_2, \ldots, z_{2t-1}, z_{2t}\} = \emptyset$. $t_2 = x_{h_1} \ldots x_{h_2}, \ldots, t_p = x_{h_{2p-3}} \ldots x_{h_{2p-2}}, t_1 = x_{h_{2p-1}} \ldots x_{h_{2p}}, t_2, \ldots, t_p \notin \{b_1, \ldots, b_{2t}, c_2, \ldots, c_p\}$. $\{k_2, \ldots, k_p\} = \{l_2, \ldots, l_p\} = \{2, \ldots, p\}$.
By lemma 1.6,

$$g(Q_0) = g(Q_1) \geq g(Q_2) \geq \gamma(H').$$

By lemma 1.5,

$$g(S_6) = g(Q) + g(Q_0).$$

Therefore,

$$g(S_6) \geq \gamma(K) + \gamma(H'), g(S_6) = \gamma(G).$$
$$\gamma(G) \geq \gamma(K) + \gamma(H').$$

The proof is completed.

Theorem 2.2 Let K be a complete graph of order n,H be an arbitrary graph of order m.G is the edge disjoint union of H and K.Let H' be a graph obtained from H by contracting the set $V(K)$ to a single vertex.Then

$$\gamma(H') + \gamma(K) \leq \gamma(G))$$

Proof: Let $V(K) = \{v_1, v_2, \dots, v_n\}$ be the vertex set of K,$V(H) = \{u_1, u_2, \dots, u_m\}$ be the vertex set of H.$V(K) \cap V(H) = \{u_1, \dots, u_r\}, 1 \leq r \leq min\{m, n\}$.

Denotes $H_1 = G[V(H) - \{u_1, \dots, u_r\}]$,then G is the vertex disjoint union of a complete graph K and an arbitrary graph H_1.$E(G) = E(K) \bigcup E(H_1) \bigcup E_1$,where $E_1$is the edge set adding from the vertex set $V(K)$to the vertex set $V(H_1)$.Let H_1' be a graph obtained from H_1 by contracting the set $V(K)$ to a single vertex.It is obvious that $H_1' = H'$.By theorem 2.1,the conclusion is true.The proof is completed.

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