Some Algebraic Properties of Bi-Circulant Digraphs*

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Abstract

For a finite group G and subsets T_1, T_2 of G, the Bi-Cayley digraph $D = (V(D), E(D)) = D(G, T_1, T_2)$ of G with respect to T_1 and T_2 is defined as the bipartite digraph with vertex set $V(D) = G \times \{0, 1\}$, and for $g_1, g_2 \in G$, $((g_1, 0), (g_2, 1)) \in E(D)$ if and only if $g_2 = t_1 g_1$, for some $t_1 \in T_1$, and $((g_1, 1), (g_2, 0)) \in E(D)$ if and only if $g_1 = t_2 g_2$, for some $t_2 \in T_2$. If $|T_1| = |T_2| = k$, then D is k-regular. In this paper, the spectra of Bi-Circulant digraph are determined. In addition, some asymptotic enumeration theorem for the number of directed spanning trees in Bi-Circulant digraphs are presented.

Keywords: Bi-Circulant digraph; Spectra; Spanning trees.

1 Introduction

Let G be a finite group with identity 1 and S be the subset of $G\backslash 1$. The Cayley digraph $D = D(G, S)$ of G with respect S is a directed graph with vertex set G, and for $g_1, g_2 \in G$, there is an arc from g_1 to g_2 if and only if $g_2 g_1^{-1} \in S$. If S is inverse-closed, that is, $S = S^{-1}$, then $D(G, S)$ corresponds to an undirected graph which is called a Cayley graph of G with respect to S and is denoted by $C(G, S)$.

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It is well known that Cayley digraphs (graphs) are vertex transitive. In [1] and [5], the authors studied the spectra of Cayley graphs. In [3],[7],[9],[10], some asymptotic enumeration theorems for the number of spanning trees for circulant digraphs (graphs) are presented. Here, we will consider the same problems for a newly defined classes digraphs called Bi-Circulant digraphs.

To study semi-symmetric graphs (regular edge transitive but not vertex transitive graphs), Xu [6] defined the so called Bi-Cayley graphs. Let G be a finite group and S be the subset of G, the *Bi-Cayley graph* $BC(G,S)$ is a bipartite graph with vertex set $G \times \{0,1\}$ and edge set $\{\{(g,0),(sg,1)\}: g \in G, s \in S\}$. When G is a cyclic group, the Bi-Cayley graph $BC(G,S)$ is called Bi-Circulant graph. In [11], Zou and Meng derived the spectra of a Bi-Circulant graph and an asymptotic enumeration theorem for the number of spanning trees in Bi-Circulant graph.

Lemma 1.1. *[11] Let G be a cyclic group of order n, $S = \{s_1, s_2, \cdots, s_k\}$ be a subset of G.*
(1)If $S \neq S^{-1}$, then the eigenvalues of Bi-Circulant graph $BC(G,S)$ are $\pm k, \pm|\omega^{s_1 j} + \cdots + \omega^{s_k j}|(j = 1, 2, \cdots, n-1)$;
(2)If $S = S^{-1}$, then the eigenvalues of Bi-Circulant graph $BC(G,S)$ are $\pm k, \pm(\omega^{s_1 j} + \cdots + \omega^{s_k j})(j = 1, 2, \cdots, n-1)$.

Lemma 1.2. *[11] Let $BC(G,S)$ be a k-regular Bi-Circulant graph of order n and $T(G,S)$ be the number of spanning trees of connected Bi-Circulant graph $BC(G,S)$, then*

$$T(G,S) \sim \frac{k^{2n+1}}{f(1)}, n \longrightarrow \infty,$$

where $f(z) = \sum_{i=0}^{2s_1-1} z^i + \sum_{i=0}^{2s_2-1} z^i + \cdots + \sum_{i=0}^{2s_k-1} z^i + 2\sum_{l \neq m} \sum_{i=0}^{s_l+s_m-1} z^i$ *and* $f(1) = 2s_1 + 2s_2 + \cdots + 2s_k + 2\sum_{l \neq m}(s_l + s_m)$.

Now we generalize the Bi-Cayley graphs to Bi-Cayley digraphs. For a finite group G and subsets T_1, T_2 of G, the *Bi-Cayley digraph* $D = (V(D), E(D)) = D(G, T_1, T_2)$ of G with respect to T_1 and T_2 is defined as the bipartite digraph with vertex set $V(D) = G \times \{0,1\}$ and for $g_1, g_2 \in G$, $((g_1,0),(g_2,1)) \in E(D)$ if and only if $g_2 = t_1 g_1$, for some $t_1 \in T_1$, and

$((g_1,1),(g_2,0)) \in E(D)$ if and only if $g_1 = t_2 g_2$, for some $t_2 \in T_2$. If $|T_1| = |T_2| = k$, then D is k-regular. When G is a cyclic group, the Bi-Cayley digraph $D(G,T_1,T_2)$ is called a Bi-Circulant digraph.

In this paper, we investigate the spectra of Bi-Circulant digraphs and some asymptotic enumeration theorem for the number of directed spanning trees in Bi-Circulant digraphs are presented.

In the following, we cite some known results which will be used in the next section.

Lemma 1.3 (Horn [4]). *Let A,B,C,D be $n \times n$ matrices, and $|A| \neq 0$, $AC = CA$, then*

$$\begin{vmatrix} A & B \\ C & D \end{vmatrix} = |AD - CB|.$$

Let W denote the circulant matrix whose first row is $[0,1,0,\cdots,0]$, and Z_n be the cyclic group of integers modulo n.

Lemma 1.4 (Biggs [2]). *Let $X = X(Z_n,S)$ be a circulant graph. Then the adjacency matrix of X is $A = \sum_{s\in S} W^s$ and the eigenvalues of X are $\lambda_r = \sum_{s\in S} \omega^{sr}, r = 0,1,\cdots,n-1$, where $\omega = exp(2\pi i/n)$.*

Lemma 1.5 (Zhang [8]). *Let D be a k-regular digraph, $\chi(D,\lambda)$ be the characteristic polynomial of D, then the number of direct spanning trees of D is*

$$T(D) = \chi'(D,\lambda) = \frac{d}{d\lambda}\chi(D,\lambda)|_{\lambda=k}.$$

For concepts used but not defined here we refer to [2].

2 Main results

We denote the number of directed spanning trees of Bi-Circulant digraph $D(Z_n,T_1,T_2)$ by $T(Z_n,T_1,T_2)$.

The following theorem discuss the spectra of the Bi-Circulant digraph $D(Z_n,T_1,T_2)$.

Theorem 2.1. *The eigenvalues of Bi-Circulant digraph $D(Z_n,T_1,T_2)$ are*

$$\lambda_r = \pm[\sum_{\substack{i=1,2,\cdots,k \\ j=1,2,\cdots,l}} (\omega^{t_{2j}+t_{1i}})^r]^{1/2}, r = 0,1,\cdots,n-1,$$

where $T_1 = \{t_{11}, t_{12}, \cdots, t_{1k}\}, T_2 = \{t_{21}, t_{22}, \cdots, t_{2l}\}$.

Proof. Let $D(Z_n, T_1, T_2)$ be a Bi-Circulant digraph and B be the adjacency matrix of $D(Z_n, T_1, T_2)$, A_1 be the adjacency matrix of the circulant digraph $D(Z_n, T_1)$ and A_2 be the adjacency matrix of the circulant digraph $D(Z_n, -T_2)$. By the definition of Bi-Circulant digraph, it is easy to see that

$$B = \begin{pmatrix} 0 & A_1 \\ A_2 & 0 \end{pmatrix}.$$

Therefore, we have

$$|\lambda I - B| = \begin{vmatrix} \lambda I & -A_1 \\ -A_2 & \lambda I \end{vmatrix} = |\lambda^2 I - A_2 A_1|. \qquad (1)$$

Since A_1 and A_2 are circulant matrix, we have

$$A_1 = \sum_{t_{1i} \in T_1} W^{t_{1i}}, A_2 = \sum_{t_{2j} \in T_2} W^{t_{2j}}$$

and

$$\begin{aligned} A_2 A_1 &= \sum_{t_{2j} \in T_2} W^{t_{2j}} \sum_{t_{1i} \in T_1} W^{t_{1i}} \\ &= (W^{t_{21}} + W^{t_{22}} + \cdots + W^{t_{2l}})(W^{t_{11}} + W^{t_{12}} + \cdots + W^{t_{1k}}) \\ &= \sum_{\substack{i=1,\cdots,k \\ j=1,\cdots,l}} W^{t_{2j}+t_{1i}}. \end{aligned}$$

Thus, the eigenvalues of $A_2 A_1$ are $\mu_r = \sum_{\substack{i=1,\cdots,k \\ j=1,\cdots,l}} (\omega^{t_{2j}+t_{1i}})^r$. By (1) we see that the eigenvalues of B are $\lambda_r = \pm \mu_r^{1/2} = \pm [\sum_{\substack{i=1,\cdots,k \\ j=1,\cdots,l}} (\omega^{t_{2j}+t_{1i}})^r]^{1/2}, r = 0, 1, \cdots, n-1$. □

For the $k-$regular Bi-Circulant digraph, we have

Theorem 2.2. *If the Bi-Circulant digraph* $D(Z_n, T_1, T_2)$ *is k-regular, i.e.,* $|T_1| = |T_2| = k$, *then the eigenvalues of Bi-Circulant digraph* $D(Z_n, T_1, T_2)$ *are*

$$\pm k, \pm [\sum_{\substack{i=1,\cdots,k \\ j=1,\cdots,k}} (\omega^{t_{2j}+t_{1i}})]^{1/2}, \cdots, \pm [\sum_{\substack{i=1,\cdots,k \\ j=1,\cdots,k}} (\omega^{t_{2j}+t_{1i}})^{n-1}]^{1/2}.$$

Lemma 2.3. *Let G be a cyclic group of order n, $T_1 = \{t_{11}, t_{12}, \cdots, t_{1k}\}$ $(1 \leq t_{11} < t_{12} < \cdots < t_{1k})$ and $T_2 = \{t_{21}, t_{22}, \cdots, t_{2k}\}$ $(1 \leq t_{21} < t_{22} < \cdots < t_{2k})$ are the subsets of G. If the roots of the polynomial*

$$f(z) = \sum_{m=0}^{t_{11}+t_{21}-1} z^m + \cdots + \sum_{m=0}^{t_{1i}+t_{2j}-1} z^m + \cdots + \sum_{m=0}^{t_{1k}+t_{2k}-1} z^m$$

are $\alpha_1, \alpha_2, \cdots, \alpha_{t_{1k}+t_{2k}-1}$, then

$$T(Z_n, T_1, T_2) = \frac{(-1)^{(n-1)(t_{1k}+t_{2k}-1)} 2nk \prod_{t=1}^{t_{1k}+t_{2k}-1}(1-\alpha_t^n)}{f(1)},$$

where $f(1) = kt_{11} + \cdots + kt_{1k} + kt_{21} + \cdots + kt_{2k}$.

Proof. By Lemma 1.5 and Theorem 2.2, we have

$$\begin{aligned}
T(Z_n, T_1, T_2) &= 2k \prod_{r=1}^{n-1} [k \pm (\sum_{i,j=1,\cdots,k} (\omega^{t_{1i}+t_{2j}})^r)^{1/2}] \\
&= 2k \prod_{r=1}^{n-1} [k^2 - \sum_{i,j=1,\cdots,k} (\omega^{t_{1i}+t_{2j}})^r] \\
&= 2k \prod_{r=1}^{n-1} [1 - (\omega^{t_{11}+t_{21}})^r + 1 - (\omega^{t_{11}+t_{22}})^r + \cdots \\
&\quad +1 - (\omega^{t_{1k}+t_{2k}})^r] \\
&= 2k \prod_{r=1}^{n-1} (1-\omega^r)(\sum_{m=0}^{t_{11}+t_{21}-1} \omega^{mr} + \sum_{m=0}^{t_{11}+t_{22}-1} \omega^{mr} + \cdots \\
&\quad + \sum_{m=0}^{t_{1k}+t_{2k}-1} \omega^{mr}) \\
&= 2k \prod_{r=1}^{n-1} (1-\omega^r) f(\omega^r).
\end{aligned}$$

Since

$$\prod_{r=1}^{n-1} (x - \omega^r) = \sum_{l=0}^{n-1} x^l, \tag{2}$$

when $x = 1$, we have $\prod_{r=1}^{n-1}(1-\omega^r) = n$.

By the definition of $f(z)$ and using the equation (2), we can get that

$$\begin{aligned} T(Z_n, T_1, T_2) &= 2k\prod_{r=1}^{n-1}(1-\omega^r)\prod_{r=1}^{n-1}(\omega^r-\alpha_1)\cdots\prod_{r=1}^{n-1}(\omega^r-\alpha_{t_{1k}+t_{2k}-1}) \\ &= 2nk\prod_{r=1}^{n-1}(\omega^r-\alpha_1)\cdots\prod_{r=1}^{n-1}(\omega^r-\alpha_{t_{1k}+t_{2k}-1}) \\ &= (-1)^{(n-1)(t_{1k}+t_{2k}-1)}2nk(\sum_{l=0}^{n-1}\alpha_1^l)\cdots(\sum_{l=0}^{n-1}\alpha_{t_{1k}+t_{2k}-1}^l) \\ &= (-1)^{(n-1)(t_{1k}+t_{2k}-1)}2nk\prod_{t=1}^{t_{1k}+t_{2k}-1}\frac{1-\alpha_t^n}{1-\alpha_t} \\ &= (-1)^{(n-1)(t_{1k}+t_{2k}-1)}2nk\frac{\prod_{t=1}^{t_{1k}+t_{2k}-1}(1-\alpha_t^n)}{f(1)}. \end{aligned}$$

□

Lemma 2.4. *Let*

$$f(z) = \sum_{m=0}^{t_{11}+t_{21}-1} z^m + \cdots + \sum_{m=0}^{t_{1i}+t_{2j}-1} z^m + \cdots + \sum_{m=0}^{t_{1k}+t_{2k}-1} z^m,$$

where $1 \le t_{11} < t_{12}\cdots < t_{1k}$ *and* $1 \le t_{21} < t_{22}\cdots < t_{2k}$. *If* $gcd(t_{11}+t_{21},\cdots,t_{1i}+t_{2j},\cdots,t_{1k}+t_{2k}) = 1$, *then the roots of* $f(z)$ *satisfy*

$$|\alpha_t| > 1, t = 1, 2, \cdots, t_{1k}+t_{2k}-1.$$

Proof. It is easy to see that $f(1) \neq 0$ and

$$(z-1)f(z) = z^{t_{11}+t_{21}} + \cdots + z^{t_{1i}+t_{2j}} + \cdots + z^{t_{1k}+t_{2k}} - k^2.$$

For $\alpha_t \neq 1$, we have

$$\alpha_t^{t_{11}+t_{21}} + \cdots + \alpha_t^{t_{1i}+t_{2j}} + \cdots + \alpha_t^{t_{1k}+t_{2k}} = k^2. \tag{3}$$

If $|\alpha_t| < 1$, then

$$\begin{aligned} &|\alpha_t^{t_{11}+t_{21}} + \cdots + \alpha_t^{t_{1i}+t_{2j}} + \cdots + \alpha_t^{t_{1k}+t_{2k}}| \\ \le\ & |\alpha_t^{t_{11}+t_{21}}| + \cdots + |\alpha_t^{t_{1i}+t_{2j}}| + \cdots + |\alpha_t^{t_{1k}+t_{2k}}| < k^2, \end{aligned}$$

which contradicts to the equation (3), and hence $|\alpha_t| \ge 1$.

Since $\alpha_t \neq 1$, if $|\alpha_t| = 1$, then we have

$$\begin{cases} \alpha_t = \cos\phi_t + \sqrt{-1}\sin\phi_t, & \sin\phi_t \neq 0, t \in \{1, 2, \cdots, t_{1k} + t_{2k} - 1\}; \\ \alpha_t = -1, & \sin\phi_t = 0, t \in \{1, 2, \cdots, t_{1k} + t_{2k} - 1\}. \end{cases}$$

By (3), when $\alpha_t = \cos\phi_t + \sqrt{-1}\sin\phi_t$, we can get that

$$\cos(t_{11} + t_{21})\phi_t + \cdots + \cos(t_{1i} + t_{2j})\phi_t + \cdots + \cos(t_{1k} + t_{2k})\phi_t = k^2.$$

Thus, $\cos(t_{1i} + t_{2j})\phi_t = 1$, it is contradicts to the assumption $gcd(t_{11} + t_{21}, \cdots, t_{1i} + t_{2j}, \cdots, t_{1k} + t_{2k}) = 1$. When $\alpha_t = -1$, we can get that $(-1)^{t_{11}+t_{21}} + \cdots + (-1)^{t_{1i}+t_{2j}} + \cdots + (-1)^{t_{1k}+t_{2k}} = k^2$, it is also contradicts to the assumption $gcd(t_{11} + t_{21}, \cdots, t_{1i} + t_{2j}, \cdots, t_{1k} + t_{2k}) = 1$. Therefore, we have $|\alpha_t| > 1, t = 1, 2, \cdots, t_{1k} + t_{2k} - 1$, the lemma follows. □

Theorem 2.5. *Let $D(Z_n, T_1, T_2)$ be a k-regular Bi-Circulant digraph of order $2n$, then*

$$T(Z_n, T_1, T_2) \sim \frac{2nk^{2n+1}}{f(1)}, n \longrightarrow \infty,$$

where $T_1 = \{t_{11}, t_{12}, \cdots, t_{1k}\}, T_2 = \{t_{21}, t_{22}, \cdots, t_{2k}\}, gcd(t_{11}+t_{21}, \cdots, t_{1i}+t_{2j}, \cdots, t_{1k} + t_{2k}) = 1.$

Proof. Let

$$\begin{aligned} \sigma_1(n) &= \sum_{j=1}^{t_{1k}+t_{2k}-1} \alpha_j^n, \qquad \sigma_2(n) = \sum_{1 \le i < j \le t_{1k}+t_{2k}-1} \alpha_i^n \alpha_j^n, \\ \sigma_3(n) &= \sum_{1 \le i < j < r \le t_{1k}+t_{2k}-1} \alpha_i^n \alpha_j^n \alpha_r^n, \\ &\vdots \\ \sigma_{t_{1k}+t_{2k}-1}(n) &= \prod_{j=1}^{t_{1k}+t_{2k}-1} \alpha_j^n. \end{aligned}$$

Then

$$\begin{aligned} \prod_{t=1}^{t_{1k}+t_{2k}-1} (z - \alpha_t^n) &= z^{t_{1k}+t_{2k}-1} - \sigma_1(n) z^{t_{1k}+t_{2k}-2} + \cdots \\ &\quad + (-1)^{t_{1k}+t_{2k}-1} \sigma_{t_{1k}+t_{2k}-1}(n). \end{aligned}$$

Let $z = 1$, we have

$$\prod_{t=1}^{t_{1k}+t_{2k}-1} (1-\alpha_t^n) = 1 - \sigma_1(n) + \cdots + (-1)^{t_{1k}+t_{2k}-1}\sigma_{t_{1k}+t_{2k}-1}(n).$$

By the definition of $f(z)$, let $z = 0$, then

$$\sigma_{t_{1k}+t_{2k}-1}(1) = \alpha_1\alpha_2\cdots\alpha_{t_{1k}+t_{2k}-1} = (-1)^{t_{1k}+t_{2k}-1}k^2.$$

Thus, $\sigma_{t_{1k}+t_{2k}-1}(n) = (-1)^{(t_{1k}+t_{2k}-1)n}k^{2n}$. Since $gcd(t_{11}+t_{21}, \cdots, t_{1i}+t_{2j}, \cdots, t_{1k}+t_{2k}) = 1$, by Lemma 2.4, $|\alpha_t| > 1, t = 1, 2, \cdots, t_{1k}+t_{2k}-1$, then, we have

$$\frac{\sigma_i(n)}{\sigma_{t_{1k}+t_{2k}-1}(n)} \longrightarrow 0, n \longrightarrow \infty,$$

for any $i < t_{1k} + t_{2k} - 1$.

By Lemma 2.3, we can get that

$$\begin{aligned} &\frac{T(Z_n, T_1, T_2)}{\frac{2nk^{2n+1}}{f(1)}} \\ =\ & \frac{(-1)^{t_{1k}+t_{2k}-1}(1-\sigma_1(n)+\cdots+(-1)^{t_{1k}+t_{2k}-1}\sigma_{t_{1k}+t_{2k}-1}(n))}{\sigma_{t_{1k}+t_{2k}-1}(n)} \\ =\ & \frac{(-1)^{t_{1k}+t_{2k}-1}}{\sigma_{t_{1k}+t_{2k}-1}(n)} - \frac{(-1)^{t_{1k}+t_{2k}-1}\sigma_1(n)}{\sigma_{t_{1k}+t_{2k}-1}(n)} + \cdots + \frac{\sigma_{t_{1k}+t_{2k}-1}(n)}{\sigma_{t_{1k}+t_{2k}-1}(n)} \\ \longrightarrow\ & 1, n \longrightarrow \infty. \end{aligned}$$

The theorem follows. □

Theorem 2.6. *Let $D(Z_n, T_1, T_2)$ be a k-regular Bi-Circulant digraph of order $2n$, then*

$$\lim_{n\longrightarrow\infty} \frac{1}{k}\{T(D, T_1, T_2)\}^{\frac{1}{2n}} = 1.$$

Proof. Obviously, $f(1) = kt_{11} + kt_{12} + \cdots + kt_{1k} + kt_{21} + \cdots + kt_{2k}$, when k is fixed, we have $(f(1))^{\frac{1}{2n}} \longrightarrow 1$, $k^{\frac{1}{2n}} \longrightarrow 1$ and $(2n)^{\frac{1}{2n}} \longrightarrow 1$, $n \longrightarrow \infty$. By Theorem 2.5, the proof is completed. □

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