Skewness, splitting number and vertex deletion of some toroidal meshes

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Abstract

The skewness $sk(G)$ of a graph $G = (V, E)$ is the smallest integer $sk(G) \geq 0$ such that a planar graph can be obtained from G by the removal of $sk(G)$ edges. The splitting number $sp(G)$ of G is the smallest integer $sp(G) \geq 0$ such that a planar graph can be obtained from G by $sp(G)$ vertex splitting operations. The vertex deletion $vd(G)$ of G is the smallest integer $vd(G) \geq 0$ such that a planar graph can be obtained from G by the removal of $vd(G)$ vertices. Regular toroidal meshes are popular topologies for the connection networks of SIMD parallel machines. The best known of these meshes is the rectangular toroidal mesh $C_m \times C_n$, for which is known the skewness, the splitting number and the vertex deletion. In this work we consider two related families: a triangulation $\mathcal{T}_{C_m \times C_n}$ of $C_m \times C_n$ in the torus, and an hexagonal mesh $\mathcal{H}_{C_m \times C_n}$, the dual of $\mathcal{T}_{C_m \times C_n}$ in the torus. It is established that $sp(\mathcal{T}_{C_m \times C_n}) = vd(\mathcal{T}_{C_m \times C_n}) = sk(\mathcal{H}_{C_m \times C_n}) = sp(\mathcal{H}_{C_m \times C_n}) = vd(\mathcal{H}_{C_m \times C_n}) = \min\{m, n\}$ and that $sk(\mathcal{T}_{C_m \times C_n}) = 2\min\{m, n\}$.

keywords: *topological graph theory, graph drawing, toroidal mesh, planarity*

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1 Introduction

Because of their symmetry and regularity, toroidal meshes are a well known topology for the connection networks of Single Instruction and Multiple Data Stream parallel machines.They occur as interconnection diagrams of multiprocessor computers and cellular automata, and so results on nonplanarity parameters are relevant to the physical design of such machines [6, 7].

Let $C_m \times C_n$, $n \geq m \geq 3$, be the graph theoretical product of the two cycles C_m and C_n with set of vertices $V(C_m \times C_n) = \{v_{i,j} : i \in \{1, 2, 3, \ldots, m\}, j \in \{1, 2, 3, \ldots, n\}\}$, and set of edges $E(C_m \times C_n) = \{v_{i,j}v_{i,j+1}, v_{i,j}v_{i+1,j}$, where the first subscript is read modulo m and the second subscript is read modulo n, $i \in \{1, 2, 3, \ldots, m\}$, $j \in \{1, 2, 3, \ldots, n\}\}$. Graph $C_m \times C_n$ or *rectangular mesh* in the torus is the most popular of the regular toroidal meshes in the torus. The *skewness* $sk(G)$ of a graph $G = (V, E)$ is the smallest integer $sk(G) \geq 0$ such that a planar graph can be obtained from G by the removal of $sk(G)$ edges. A *splitting operation* replaces a vertex v of G by two new and nonadjacent vertices v_1 and v_2 and attaches each neighbour of v either to v_1 or to v_2. The *splitting number* $sp(G)$ of G is the smallest integer $sp(G) \geq 0$ such that a planar graph can be obtained from G by $sp(G)$ splitting operations. The *nonplanar vertex deletion* $vd(G)$, or *vertex deletion*, of G is the smallest integer $vd(G) \geq 0$ such that a planar graph can be obtained from G by the removal of $vd(G)$ vertices. The *crossing number* of a graph $G = (V, E)$ is the minimum number of crossings in a drawing of G in the plane. If $D(G)$ is such a drawing, then it is said that $D(G)$ is an *optimum drawing* of G. The crossing number of $C_m \times C_n$ is still a much studied open problem, conjectured [7] to be $(m-2)n, n \geq m$. Just recently [5], the conjecture has been proved for $n \geq m(m+1)$.

In this work we consider two families of regular graphs, related to the rectangular mesh. Interest in such special classes is justified by the fact that most of the nonplanar parameters, in special the crossing number, are not even known for graphs as regular as the complete graphs, so that every single step into this direction is noteworthy.

The graph considered first in this paper is $\mathcal{T}_{C_m \times C_n}$ a *triangulation* of $C_m \times C_n$ in the torus shown in Figure 1(a), with set of vertices $V(\mathcal{T}_{C_m \times C_n}) = V(C_m \times C_n)$ and set of edges $E(\mathcal{T}_{C_m \times C_n}) = E(C_m \times C_n) \cup \{v_{i,j}v_{i+1,j+1}$, where the first subscript is read modulo m and the second subscript is read modulo n, $i \in \{1, 2, 3, \ldots, m\}$, $j \in \{1, 2, 3, \ldots, n\}\}$.

In this work we prove that the vertex deletion and the splitting number of $\mathcal{T}_{C_m \times C_n}$ are equal to $\min\{m, n\}$, and that the skewness of $\mathcal{T}_{C_m \times C_n}$ is $2\min\{m, n\}$. The second graph considered is $\mathcal{H}_{C_m \times C_n}$, a *toroidal hexagonal mesh* shown in Figure 1(b), with set of vertices $V(\mathcal{H}_{C_m \times C_n}) = V(C_{2m} \times C_n)$, and set of edges $E(\mathcal{H}_{C_m \times C_n}) = \{v_{2i-1,j}v_{2i,j+1}, v_{i,j}v_{i+1,j}$, where the first subscript is read modulo $2m$ and the second subscript is read modulo

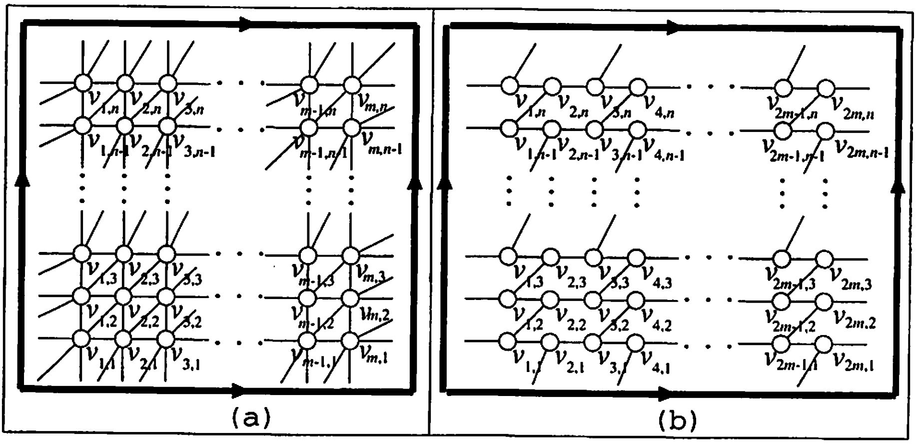

Figure 1: A triangular (a) and the dual hexagonal (b) mesh in the torus.

n, $i \in \{1, 2, 3, \ldots, 2m\}, j \in \{1, 2, 3, \ldots, n\}\}$. We prove that the vertex deletion, the splitting number and the skewness of $\mathcal{H}_{C_m \times C_n}$ are all equal to $\min\{m, n\}$.

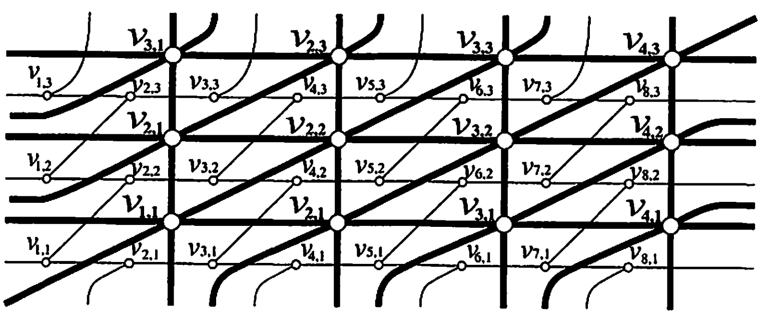

Figure 2: $\mathcal{T}_{C_3 \times C_4}$ in large vertices and bold edges, and $\mathcal{H}_{C_3 \times C_4}$ in small vertices and thin edges drawn as dual graphs in the torus.

The *dual* of an embedding $D(G)$ of a graph G in a surface S is the graph F, such that the vertices of F are the faces of D, and for each edge e of $D(G)$ there is an edge of F connecting the vertices of F on each side of e. A graph G is *auto-dual* in a surface S if there is an embedding $D(G)$ of G in S, such that G is the dual graph of $D(G)$ in S. The graph $C_m \times C_n$ is auto-dual in the torus. It is observed that $\mathcal{H}_{C_m \times C_n}$ is the dual graph of $\mathcal{T}_{C_m \times C_n}$ in the torus. For the convenience of the reader it is offered in Figure 2 an example showing graph $\mathcal{T}_{C_3 \times C_4}$ in white large vertices and bold edges, and the corresponding embedding of $\mathcal{H}_{C_3 \times C_4}$ in white tiny vertices drawn as dual graphs in the torus.

This paper presents the evaluation of the three nonplanarity parameters splitting number, skewness, and vertex deletion for dual classes of graphs in the torus. Note that $\mathcal{H}_{C_m \times C_n}$ is a cubic graph. For general cubic graphs, the NP-completeness of splitting number, skewness, and vertex deletion has been established [3, 4].

2 Basic properties

The results of this paper depend heavily on the following characterization by Kuratowski[8]: a graph is planar if and only if it does not contain a subgraph contractible to K_5 or $K_{3,3}$. If G has a subgraph H contractible to a graph F, then [1, 2] $sk(G) \geq sk(F)$, $sp(G) \geq sp(F)$ and $vd(G) \geq vd(F)$. The crossing number, skewness, splitting number and vertex deletion of a general graph $G = (V, E)$ are related [1, 2] by the following inequalities:

Fact 1 *For every graph G, $cr(G) \geq sk(G) \geq sp(G) \geq vd(G)$.*

Given $i \in \{1, 2, 3, \ldots, m\}$ the *meridian* M^i of $\mathcal{T}_{C_m \times C_n}$ is the subset of vertices $\{v_{i,j} : j \in \{1, 2, 3, \ldots, n\}$ of $V(\mathcal{T}_{C_m \times C_n})\}$; the *parallel* P^j of $\mathcal{T}_{C_m \times C_n}$ is the subset of vertices $\{v_{i,j} : i \in \{1, 2, 3, \ldots, m\}$ of $V(\mathcal{T}_{C_m \times C_n})\}$. The *diagonal* D^i of $\mathcal{T}_{C_m \times C_m}$, $i \in \{1, 2, 3, \ldots, m\}$, is the subset of vertices $\{v_{i+k,k+1} : k \in \{0, 1, 2, \ldots, m-1\}$ of $V(\mathcal{T}_{C_m \times C_m})$. We call the *maximum circles* of $\mathcal{T}_{C_m \times C_n}$ the subgraphs of $\mathcal{T}_{C_m \times C_n}$ induced by M^i, P^j or D^i. Two meridians (respectively, parallels, or diagonals) are said to belong to the *same class* of maximum circles.

An *automorphism* α of a graph G is a bijective function $\alpha : V \to V$, such that $uv \in E$ if and only if $\alpha(u)\alpha(v) \in E$. Given a graph G and a subgraph S of G, we say that G is *S-transitive* if for each pair F, H of subgraphs of G, where F and H are isomorphic to S, there is an automorphism α of G such that if $v \in V(F)$, then $\alpha(v) \in V(H)$. By using a suitable automorphism we can establish that $\mathcal{T}_{C_m \times C_n}$ is vertex transitive, and that is edge transitive if $m = n$. We can also establish that $\mathcal{H}_{C_m \times C_n}$ is vertex transitive.

3 Results on $\mathcal{T}_{C_m \times C_n}$

3.1 The splitting number and vertex deletion of $\mathcal{T}_{C_m \times C_n}$

Our strategy to prove that the vertex deletion and the splitting number of $\mathcal{T}_{C_m \times C_n}$ are both $\min\{m, n\}$ is as follows. In Lemma 2 we show the upper bound $\min\{m, n\}$ for the splitting number of $\mathcal{T}_{C_m \times C_n}$. In Lemma 3 we show the lower bound $\min\{m, n\}$ for the vertex deletion of $\mathcal{T}_{C_m \times C_n}$, which together with the inequalities from Fact 1 imply the claimed equality.

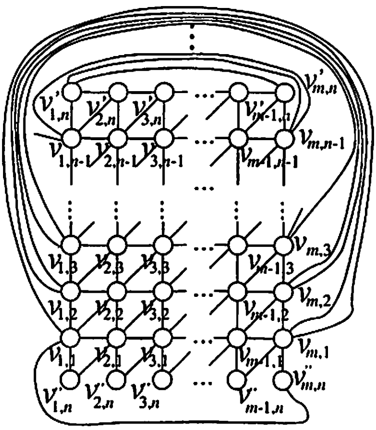

Figure 3: Planar embedding of a graph obtained from $\mathcal{T}_{C_m \times C_n}$, with m splittings. Hence, $sp(\mathcal{T}_{C_m \times C_n}) \leq m$.

Lemma 2 *$sp(\mathcal{T}_{C_m \times C_n})$ is at most* $\min\{m, n\}$.

Proof. Figure 3 displays a planar drawing of the graph obtained with m splitting operations in a set of vertices of $\mathcal{T}_{C_m \times C_n}$: $v_{i,n}$ into $v'_{i,n}$ and $v''_{i,n}$, such that $N(v''_{i,n}) = \{v_{i,1}, v_{i+1,1}\}$ and $N(v'_{i,n}) = N(v_{i,n}) \setminus N(v''_{i,n})$, $i \in \{1, 2, 3, \ldots, m\}$. Therefore, $sp(\mathcal{T}_{C_m \times C_n}) \leq \min\{m, n\}$. □

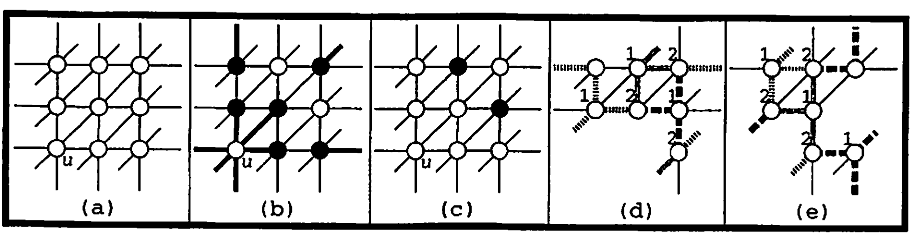

Figure 4: (a) $\mathcal{T}_{C_3 \times C_3}$ in the torus; (b) vertices of $\mathcal{T}_{C_3 \times C_3}$ (depicted in black) at distance 1 from vertex vertex $u = v_{1,1}$; (c) vertices of $\mathcal{T}_{C_3 \times C_3}$ at distance 2 from u; (d) and (e) subdivisions of $K_{3,3}$ as a subgraph of the graph obtained from $\mathcal{T}_{C_3 \times C_3}$ by the removal of u and v, respectively, at distance 1 and 2 from u.

Lemma 3 *$vd(\mathcal{T}_{C_3 \times C_3})$ is at least* 3.

Proof. We prove that the removal of a pair of vertices from $\mathcal{T}_{C_3 \times C_3}$, shown in Figure 4(a), does not yield a planar graph. Let u and v be two vertices of $\mathcal{T}_{C_3 \times C_3}$. As $\mathcal{T}_{C_3 \times C_3}$ is vertex transitive, u is assumed to be the vertex $u = v_{1,1}$ of $\mathcal{T}_{C_3 \times C_3}$. As the diameter of $\mathcal{T}_{C_3 \times C_3}$ is 2, we consider two cases:

vertices v at distance 1 from u, and vertices v at distance 2 from u. For the convenience of the reader it is shown in Figures 4(b) and (c) depicted in black vertices, respectively, the vertices of $\mathcal{T}_{C_3 \times C_3}$ whose distance from u are 1 and 2.

Figures 4 (d) and (e) consider the drawing of two subdivisions of $K_{3,3}$ as subgraphs of $\mathcal{T}_{C_3 \times C_3}$, each one of them with parts labelled with 1 and 2. It is shown next that $G \setminus \{u, v\}$ contains at least one of the graphs depicted in Figures 4 (d) or (e) as a subgraph. If the distance from u to v is 1, as $\mathcal{T}_{C_3 \times C_3}$ is edge transitive, we assume $v = v_{2,1}$ and then the resulting graph obtained from the removal of u and v from $\mathcal{T}_{C_3 \times C_3}$ contains the graph in Figure 4(d) as a subgraph. If the distance from u to v is 2, then the existence of the automorphism $\phi(v_{x,y}) = v_{y,x}$, provides that v can be assumed to be $v = v_{3,2}$. Hence, the remaining graph obtained from the removal of u and v from $\mathcal{T}_{C_3 \times C_3}$ contains the graph in Figure 4(e) as a subgraph. □

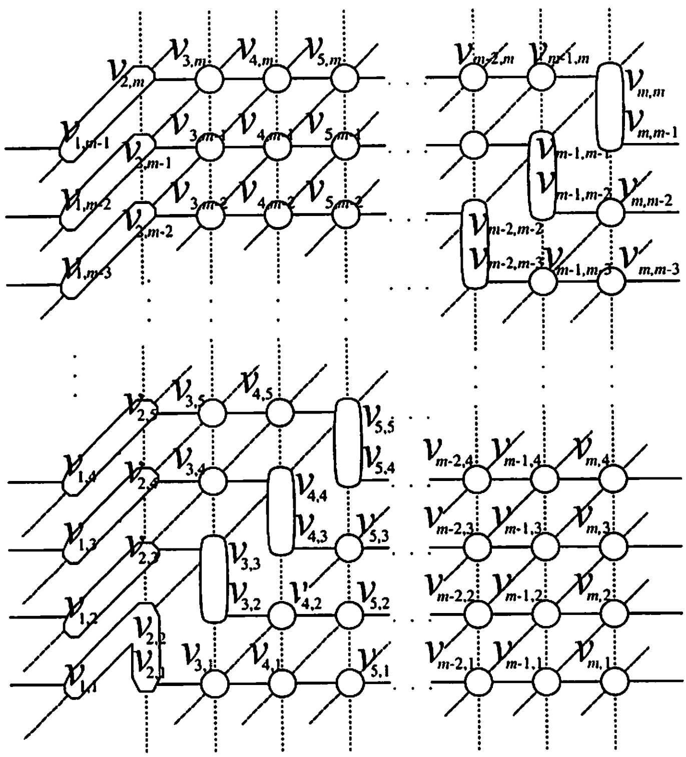

Figure 5: Subgraph F of $\mathcal{T}_{C_m \times C_m}$ contractible to $\mathcal{T}_{C_{m-1} \times C_{m-1}}$.

Lemma 4 *$vd(\mathcal{T}_{C_m \times C_m})$ is at least m.*

Proof. We argue by induction. From Lemma 3, it is valid for $k = 3$. Assume it is valid for $k < m$ with $m > 3$ and consider v a vertex of $\mathcal{T}_{C_m \times C_m}$. We prove that for every vertex v of $\mathcal{T}_{C_m \times C_m}$ the graph $\mathcal{T}_{C_m \times C_m} - v$ has a subdivision of $\mathcal{T}_{C_{m-1} \times C_{m-1}}$ as a subgraph. Since graph $\mathcal{T}_{C_m \times C_m}$ is vertex transitive, v is assumed to be vertex $v_{1,m}$. Figure 5 shows how to define a subgraph F contractible to $\mathcal{T}_{C_{m-1} \times C_{m-1}}$ as a subgraph of $\mathcal{T}_{C_m \times C_m} - v_{1,m}$. In Figure 5 horizontal full edges represent edges in a parallel of $\mathcal{T}_{C_{m-1} \times C_{m-1}}$, vertical clear dashed edges represent edges in a meridian of $\mathcal{T}_{C_{m-1} \times C_{m-1}}$, and diagonal dark dashed edges represent diagonal edges of $\mathcal{T}_{C_{m-1} \times C_{m-1}}$. In order to obtain the subgraph contractible to $\mathcal{T}_{C_{m-1} \times C_{m-1}}$ the edges between each consecutive pair of adjacent vertices in each of the following m sets of vertices are sequentially contracted: $\{v_{11}, v_{22}, v_{21}\}$, $\{v_{12}, v_{23}\}$, $\{v_{13}, v_{24}\}$, $\{v_{14}, v_{25}\}, \ldots, \{v_{1,m-1}, v_{2,m}\}$, and $\{v_{32}, v_{33}\}$, $\{v_{43}, v_{44}\}$, $\{v_{54}, v_{55}\}$, $\ldots$, $\{v_{m,m-1}, v_{m,m}\}$. □

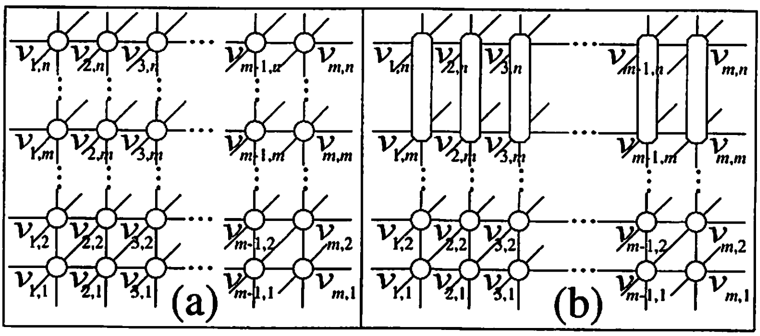

Figure 6: (a) Drawing of $\mathcal{T}_{C_m \times C_n}$ and (b) drawing of the corresponding graph contractible to $\mathcal{T}_{C_m \times C_m}$ as a subgraph of $\mathcal{T}_{C_m \times C_n}$, $m \leq n$.

Lemma 5 *$\mathcal{T}_{C_m \times C_n}$ contains a graph contractible to $\mathcal{T}_{C_m \times C_m}$ as a subgraph.*

Proof. Figure 6 shows how to define a graph contractible to $\mathcal{T}_{C_m \times C_m}$ as a subgraph of $\mathcal{T}_{C_m \times C_n}$, where the edges between each consecutive pair of adjacent vertices in each of the following m sets of vertices are sequentially contracted: $\{v_{1,m}, v_{1,m+1}, v_{1,m+2}, \ldots, v_{1,n}\}$, $\{v_{2,m}, v_{2,m+1}, v_{2,m+2}, \ldots, v_{2,n}\}$, $\{v_{3,m}, v_{3,m+1}, v_{3,m+2}, \ldots, v_{3,n}\}$, $\ldots$, $\{v_{m,m}, v_{m,m+1}, v_{m,m+2}, \ldots, v_{m,n}\}$. The contracted edges are depicted by white long vertical vertices. □

Theorem 6 *$vd(\mathcal{T}_{C_m \times C_n}) = sp(\mathcal{T}_{C_m \times C_n}) = \min\{m, n\}$.*

Proof. It follows from Fact 1, Lemma 2, Lemma 4, and Lemma 5. □

3.2 The skewness of $\mathcal{T}_{C_m \times C_n}$

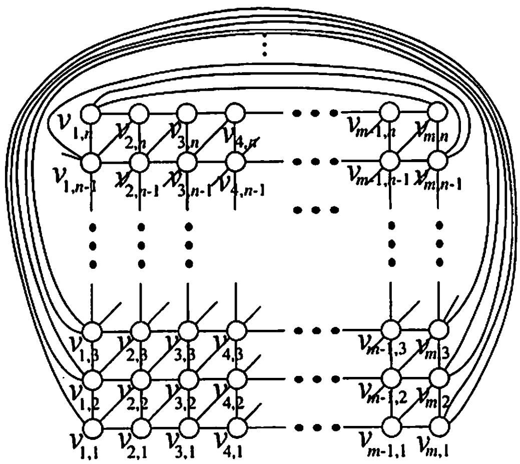

Figure 7: Planar embedding of a graph obtained from $\mathcal{T}_{C_m \times C_n}$ by removal of a set with $2m$ edges. Hence, $sk(\mathcal{T}_{C_m \times C_n}) \leq 2m$.

Lemma 7 *$sk(\mathcal{T}_{C_m \times C_n})$ is at most* $2\min\{m, n\}$.

Proof. Figure 7 shows a drawing of a planar graph obtained from $sk(\mathcal{T}_{C_m \times C_n})$ by removal of the set with $2m$ edges: $\{v_{1,1}v_{1,n}, v_{1,1}v_{m,n}, v_{2,1}v_{2,n}, v_{2,1}v_{1,n}, v_{3,1}v_{3,n}, v_{3,1}v_{2,n}, \ldots, v_{m,1}v_{m,n}, v_{m,1}v_{m-1,n}\}$. □

Lemma 8 $sk(\mathcal{T}_{C_3 \times C_n}) = 6$.

Proof. It follows from a consequence of the Euler's theorem, which says that every planar graph $G = (V, E)$ with $|V| \geq 3$ satisfies $|E| \leq 3|V| - 6$, from Lemma 7, that $|E(\mathcal{T}_{C_3 \times C_n})| = 3.(3n)$ and $|V(\mathcal{T}_{C_3 \times C_n})| = 3n$. □

Theorem 9 $sk(\mathcal{T}_{C_m \times C_n}) = 2\min\{m, n\}$.

Proof. From Lemma 7 and Lemma 5, it is enough to prove that $\mathcal{T}_{C_m \times C_m} \geq 2m$. We argue by induction. From Lemma 8 the theorem is valid if $m = 3$. Assume the theorem valid for every $k < n$ with $n > 3$. We prove that the removal of 2 edges from $\mathcal{T}_{C_m \times C_m}$ yields a graph G', which has a contractible graph to $\mathcal{T}_{C_{m-1} \times C_{m-1}}$, as a subgraph. Let e_1 and e_2 be the 2 edges removed from $E(\mathcal{T}_{C_m \times C_m})$ in order to yield G'. Consider the graph F contractible to $\mathcal{T}_{C_{m-1} \times C_{m-1}}$ defined in Figure 5. We proved that G' contains a graph isomorphic to F as a subgraph. We have 2 cases to analyse.

First case is when e_1 and e_2 do not belong to the same class of maximum circles. We may assume e_1 to belong to a meridian and e_2 to belong to a parallel of $\mathcal{T}_{C_m \times C_m}$. First of all note that, for all $r \in \{1, 2, 3, \ldots, m\}$ the edges

$v_{1,r}v_{1,r+1} \notin S$; and for all $s \in \{1,2,3,\ldots,m\}$ the edges $v_{s,s}v_{s+1,s} \notin S$. Our main goal is to prove that there is a pair r, s which defines an automorphism which maps e_1 and e_2, respectively, to $v_{1,r}v_{1,r+1}$ and $v_{s,s}v_{s+1,s}$ establishing that G' contains F as a subgraph. Let $e_1 = v_{i,j}v_{i,j+1}$ and $e_2 = v_{k,\ell}v_{k+1,\ell}$. We exhibit the subgraph of G' isomorphic to F by composing 2 automorphisms. The first automorphism α moves vertex e_1 to meridian M_1: $\alpha(v_{x,y}) = v_{x-i+1,y}$. The second automorphism β moves vertex $\alpha(v_{k,\ell}) = v_{(k-i)+1,\ell}$ to the vertex $v_{(k-i)+1,(k-i)+1}$, $\beta(v_{x,y}) = v_{x,(y-\ell+k-i-1)+1}$. Finally, the composition automorphism of α and β gives automorphism ϕ: $\phi(v_{x,y}) = \beta(\alpha(v_{x,y})) = v_{(x-i)+1,(y-\ell+k-i-1)+1}$.

Second case is when e_1 and e_2 belong to the same class of maximum circles. We may assume e_1 and e_2 to belong to parallels. Because no edge $v_{1,r}v_{2,r} : r \in \{1,2,3,\ldots,m\}$ belongs to S, the same automorphism $\phi(v_{x,y}) = \beta(\alpha(v_{x,y})) = v_{(x-i)+1,(y-\ell+k-i-1)+1}$ defines S as a subgraph of G'. □

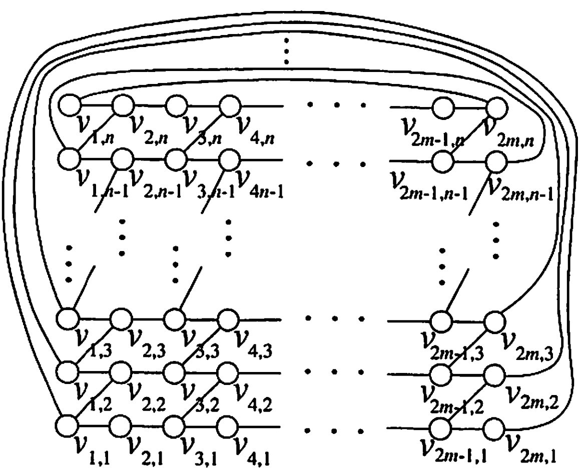

Figure 8: Planar embedding of a graph obtained from $\mathcal{H}_{C_m \times C_n}$ by the removal of a set with m edges. Hence, $sk(\mathcal{H}_{C_m \times C_n}) \leq m$.

4 Results on $\mathcal{H}_{C_m \times C_n}$

We prove that the values of skewness, splitting number and vertex deletion of $\mathcal{H}_{C_m \times C_n}$ are all $\min\{m,n\}$ as follows. In Lemma 10 we show the upper bound $\min\{m,n\}$ for $sk(\mathcal{H}_{C_m \times C_n})$. In Lemma 13 we prove that $\min\{m,n\}$ is a lower bound for $vd(\mathcal{H}_{C_m \times C_n})$. Fact 1 gives the claimed equality.

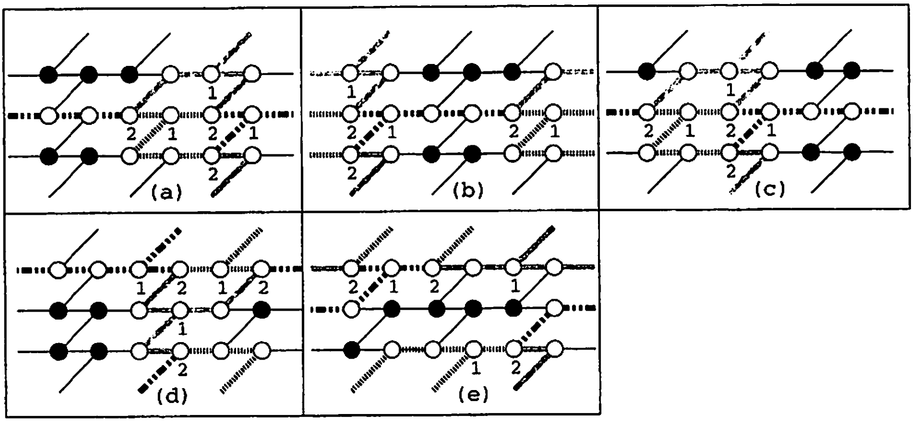

Figure 9: Five subdivisions of $K_{3,3}$ depicted as a subgraph of $\mathcal{H}_{C_3 \times C_3}$.

Lemma 10 *$sk(\mathcal{H}_{C_m \times C_n})$ is at most* $\min\{m, n\}$.

Proof. Figure 8 displays a planar drawing of the graph obtained from $\mathcal{H}_{C_m \times C_n}$ by the removing the set $\{v_{2,1}v_{1,n}, v_{4,1}v_{3,n}, v_{6,1}v_{5,n}, \ldots, v_{2m,1}v_{2m-1,n}\}$ with m edges. Hence, $sk(\mathcal{H}_{C_m \times C_n}) \leq \min\{m, n\}$. □

Lemma 11 *$vd(\mathcal{H}_{C_3 \times C_3})$ is at least* 3.

Proof. We establish $vd(\mathcal{H}_{C_3 \times C_3}) \geq 3$ by considering that the graph defined by the removal a subset R with 2 vertices from $\mathcal{H}_{C_3 \times C_3}$ yields a nonplanar graph. Figure 9 is used in the proof of Lemma 11. In Figure 9 are shown 5 subdivisions of $K_{3,3}$, each as a subgraph of $\mathcal{H}_{C_3 \times C_3}$. Each subdivision is defined in white vertices and coloured edges, such that vertices of each partition of vertices of the bipartite $K_{3,3}$ are labelled with 1 and 2, and the other vertices of the subdivision are not labelled. Vertices not belonging to the subdivision are depicted black. We show three different colours used to depict the paths emanating from the three vertices of partition 1.

From Figures 9(a,b,c,d,e) we note that for each vertex $v \in \mathcal{H}_{C_3 \times C_3}$ there is a subdivision S of $K_{3,3}$ as a subgraph of $\mathcal{H}_{C_3 \times C_3}$, such that $v \notin S$. Note also that vertex $v_{1,1}$ does not belong to any of the 5 subdivisions.

Let u and v be the vertices of a subset R of $V(\mathcal{H}_{C_3 \times C_3})$. As graph $\mathcal{H}_{C_3 \times C_3}$ is vertex transitive, we assume $u = v_{1,1}$. Hence, for every possible set R with 2 vertices of $\mathcal{H}_{C_3 \times C_3}$, there is a subdivision of $K_{3,3}$ as a subgraph of the resulting graph obtained by removing R from $\mathcal{H}_{C_3 \times C_3}$. □

Lemma 12 *$vd(\mathcal{H}_{C_m \times C_n})$ is at least m.*

Proof. We prove this assertion by induction on m. The induction basis with the case $m = 3$ is established in Lemma 11. It is enough to prove that

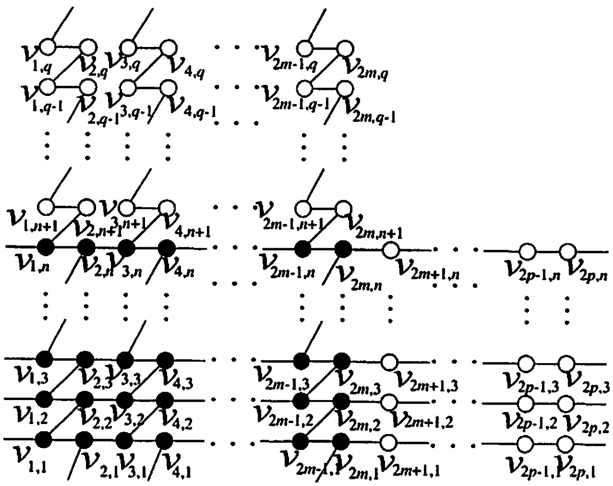

Figure 10: Subdivision of $\mathcal{H}_{C_m \times C_n}$ as a subgraph of $\mathcal{H}_{C_p \times C_q}$, $p > m$, $q > n$.

for every vertex v of $\mathcal{H}_{C_p \times C_q}$ with $p > m$ and $q > n$, there is a subdivision of $\mathcal{H}_{C_m \times C_n}$ as a subgraph of $\mathcal{H}_{C_p \times C_q} - v$. Hence $\mathcal{H}_{C_{m+1} \times C_{m+1}}$ less a vertex has a subdivision of $\mathcal{H}_{C_m \times C_m}$ as a subgraph. As graph $\mathcal{H}_{C_p \times C_q}$ is vertex transitive, we set v to be vertex $v = v_{2p,2q}$. Figure 10 is used in the proof of Lemma 12. Figure 10 shows a drawing for a subdivision of $\mathcal{H}_{C_m \times C_n}$ in bold edges and black vertices, as a subgraph of $\mathcal{H}_{C_p \times C_q} - v$. We observe that white vertices adjacent to bold edges define paths to be contracted in order to define the subdivision of $\mathcal{H}_{C_m \times C_n}$. □

Lemma 13 $vd(\mathcal{H}_{C_m \times C_n}) = min\{m, n\}$.

Proof. It follows from the observation that $\mathcal{H}_{C_m \times C_n}$ has a subdivision of $\mathcal{H}_{C_m \times C_m}$ which by Lemma 12 has vertex deletion at least m. □

Theorem 14 $vd(\mathcal{H}_{C_m \times C_n}) = sp(\mathcal{H}_{C_m \times C_n}) = sk(\mathcal{H}_{C_m \times C_n}) = \min\{m, n\}$.

Proof. It follows from Fact 1, Lemma 10, and Lemma 13. □

5 Final remarks

A natural interesting question which arises from observing the values derived to the vertex deletion, splitting number and skewness of the dual graphs $\mathcal{T}(C_m \times C_n)$ and $\mathcal{H}(C_m \times C_n)$ is whether the fact that G and H are dual graphs in the torus implies that $sp(G) = vd(H)$, or $sp(G) = sp(H)$, or $vd(G) = vd(H)$. However, none of these results is true. The graph $C_3 \times C_5$ is auto-dual in the torus and yet satisfies $sp(C_3 \times C_5) = 3 \neq 2 = vd(C_3 \times C_5)$, see [1, 2]. Now consider $G = K_5$, to see an example where G admits an embedding in the torus whose dual H is a graph with 5 vertices which is not the K_5, and therefore planar.

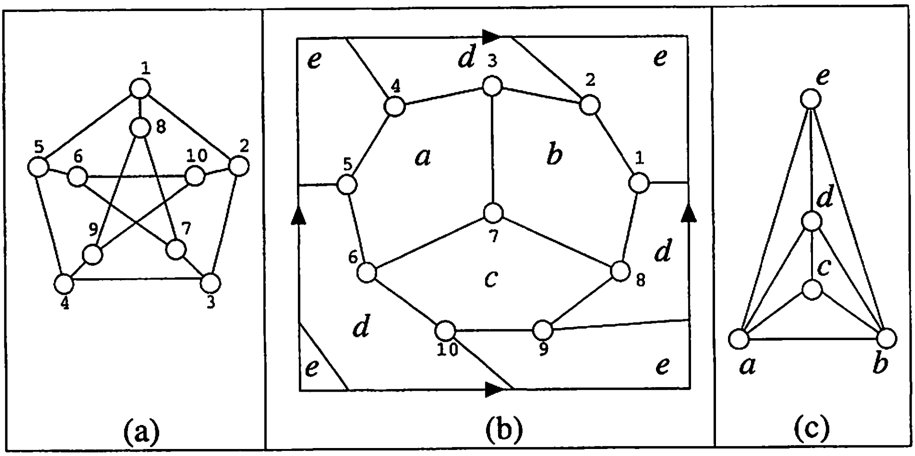

Figure 11: (a) the Petersen graph, (b) an embedding in the torus of the Petersen graph, and (c) the plane graph corresponding to the dual in the torus of the Petersen graph.

In Figure 11(a) we show a drawing of the Petersen graph, in Figure 11(b) we show an embedding in the torus of the Petersen graph, and in Figure 11(c) we show a plane drawing of the dual of the Petersen graph. We observe that the Petersen graph is vertex-transitive, using this property we can note that the removal of any vertex leaves a subdivision of $K_{3,3}$, proving that the vertex deletion number of the Petersen graph is 2.

Given G^* the dual graph in the torus of graph G, a question that could be posed open here is whether the difference between $vd(G)$ and $vd(G^*)$ can be arbitrarily large.

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