ON SOME RATIONAL DIFFERENCE EQUATIONS

BRATISLAV IRIČANIN AND STEVO STEVIĆ

Abstract

We extend and give short proofs of some recent results regarding some classes of rational difference equations.

1. INTRODUCTION

Recently there has been a huge interest in studying rational difference equations, see, e.g., [1-32] and the references therein. In [1] and [2] were studied solutions of some particular cases of the following classes of difference equations

$$x_{n+1} = ax_n + \frac{bx_n x_{n-3}}{cx_{n-2} + dx_{n-3}}, \quad n \in \mathbb{N}_0, \tag{1}$$

where a, b, c and d are nonzero numbers, and

$$x_{n+1} = a + \frac{dx_{n-l}x_{n-k}}{b - cx_{n-s}}, \quad n \in \mathbb{N}_0, \tag{2}$$

where $k, l, s \in \mathbb{N}_0$ and $a, b, c, d > 0$. Explicit solution of some subclasses of equations (1) and (2) are given. Our aim here is to give short proofs of the main results in [1] and [2], as well as to extend some of them.

2. ON SOLUTIONS OF EQUATION (1)

In this section we prove that equation (1) can be solved explicitly, from which the main results in [1] easily follow.

Theorem 1. *All well-defined solutions of Eq.* (1) *can be solved explicitly.*

Proof. First note that if $x_{n_0} = 0$ for some $n_0 \in \mathbb{N}_0$, then $x_{n_0+1} = 0$, so that x_{n_0+4} is not defined. Hence, assume that $x_n \neq 0$, $n = -3, -2, -1, 0, 1, \dots$.
For such solutions of equation (1) we have that

$$\frac{x_{n+1}}{x_n} = \frac{ac\frac{x_{n-2}}{x_{n-3}} + ad + b}{c\frac{x_{n-2}}{x_{n-3}} + d}, \quad n \in \mathbb{N}_0.$$

If we use the following change $y_{n+1} = x_{n+1}/x_n$ we obtain

$$y_n = \frac{acy_{n-3} + ad + b}{cy_{n-3} + d}, \quad n \in \mathbb{N}. \tag{3}$$

By using the changes $z_m^{(l)} = y_{3m+l}$, $l = 1, 2, 3$, $m \in \mathbb{N}_0 \cup \{-1\}$, we obtain that equation (3) is equivalent with the following three equations

$$z_m^{(l)} = \frac{acz_{m-1}^{(l)} + ad + b}{cz_{m-1}^{(l)} + d}, \quad l = 1, 2, 3, \quad m \in \mathbb{N}_0. \tag{4}$$

Equations in (4) can be solved by using the change $z_m^{(l)} = v_{m+1}^{(l)}/v_m^{(l)} + f$, where f will be suitable chosen. Indeed, from (4) we have

$$c\frac{v_{m+1}^{(l)}}{v_{m-1}^{(l)}} + (cf + d)\frac{v_{m+1}^{(l)}}{v_m^{(l)}} + (f - a)c\frac{v_m^{(l)}}{v_{m-1}^{(l)}} + f^2c + fd - acf - ad - b = 0.$$

By choosing $f = -d/c$, we obtain

$$cv_{m+1}^{(l)} - (ac + d)v_m^{(l)} - bv_{m-1}^{(l)} = 0.$$

This equation is a homogeneous linear second-order difference equation with constant coefficients, whose solutions have the following form

$$v_m^{(l)} = c_1\lambda_1^m + c_2\lambda_2^m, \quad \text{where } c_1, c_2 \in \mathbb{R},$$

and

$$\lambda_{1,2} = \frac{ac + d \pm \sqrt{(ac + d)^2 + 4cb}}{2c},$$

or $v_m^{(l)} = (c_1 + mc_2)\lambda_1^m$, if $(ac + d)^2 + 4cb = 0$.

From all above mentioned the general solution of equation (1) can be easily found, finishing the proof of the theorem. □

Remark 1. Note that the method described above does not depend on the values of parameters a, b, c and d, so that it can be applied to, for example, the following four particular cases (see Theorems 4-7 in [1])

$$x_{n+1} = x_n \pm \frac{x_n x_{n-3}}{x_{n-2} \pm x_{n-3}}.$$

We leave the calculations to the reader.

Remark 2. Note that the following $m + k$-th order difference equation

$$x_n = ax_{n-k} + \frac{bx_{n-k}x_{n-m-k}}{cx_{n-m} + dx_{n-m-k}} \quad n \in \mathbb{N}_0,$$

where $k, m \in \mathbb{N}$, can be treated similarly as equation (1).

3. On the equation $x_{n+1} = a + \frac{x_n x_{n-1}}{a - x_n}$

In [2, Theorem 4] it was proved that every solution of the difference equation

$$x_{n+1} = a + \frac{x_n x_{n-1}}{a - x_n}, \quad n \in \mathbb{N}_0, \tag{5}$$

with $x_{-1}, x_0 \in \mathbb{R} \setminus \{0, a\}$, is periodic with period four. The proof was given by the method of induction. The next result gives a short proof of a slightly different result, as well as a natural explanation of it.

Theorem 2. *Every well-defined solution of Eq.* (5) *is periodic with period four.*

Proof. If $x_{n_1} = 0$, for some $n_1 \in \mathbb{N}_0$, then $x_{n_1+1} = a$ so that x_{n_1+2} is not defined. If $x_{-1} = 0$, then $x_1 = a$, so that x_2 is not defined. If $x_{n_2} = a$, for some $n_2 \in \mathbb{N}_0$, then x_{n_2+1} is not defined. Hence we may assume $x_n \neq 0$, $n \in \mathbb{N}_0 \cup \{-1\}$ and $x_n \neq a$, $n \in \mathbb{N}_0$.

Now note that for such solutions of equation (5) the following relation holds

$$\frac{x_{n+1} - a}{x_n} = -\left(\frac{x_n - a}{x_{n-1}}\right)^{-1}, \quad n \in \mathbb{N}_0,$$

and consequently the sequence $\frac{x_{n+1}-a}{x_n}$ is two periodic. Thus, we have

$$\frac{x_{2n+1} - a}{x_{2n}} = \frac{x_1 - a}{x_0} = \frac{x_{-1}}{a - x_0} := b \quad \text{and} \quad \frac{x_{2n} - a}{x_{2n-1}} = \frac{x_0 - a}{x_{-1}} = -\frac{1}{b}$$

or

$$x_{2n+1} = bx_{2n} + a \quad \text{and} \quad x_{2n} = -\frac{1}{b}x_{2n-1} + a. \tag{6}$$

Replacing the second equation in (6) into the first one we obtain

$$x_{2n+1} = b(-\frac{1}{b}x_{2n-1} + a) + a = -x_{2n-1} + a(1 + b). \tag{7}$$

By using (7) twice, we obtain

$$x_{2n+1} = -x_{2n-1} + a(1 + b) = -(-x_{2n-3} + a(1 + b)) + a(1 + b) = x_{2n-3}. \tag{8}$$

Similarly

$$x_{2n} = -\frac{1}{b}x_{2n-1} + a = -\frac{1}{b}(bx_{2n-2} + a) + a = -x_{2n-2} + a - \frac{a}{b},$$

from which it follows that

$$x_{2n} = -x_{2n-2} + a - \frac{a}{b} = -(-x_{2n-4} + a - \frac{a}{b}) + a - \frac{a}{b} = x_{2n-4}. \tag{9}$$

From (8) and (9) the result follows. □

Remark 3. The formulation of [2, Theorem 4] is not quite correct. Namely, if

$$x_0 = \frac{a^2}{a - x_{-1}},$$

where $x_{-1} \in \mathbb{R} \setminus \{0, a\}$, then $x_1 = 0$ and $x_2 = a$, so that x_3 is not defined. Hence, we add the phrase "well-defined" in our version of the result. The same remark can be given for the next two results which we are proved in the sequel.

4. On the equation $x_{n+1} = 1 + \frac{x_{n-1}x_{n-2}}{1-x_n}$

In [2, Theorem 5] was given a long inductive proof of the following result. Here we give a more natural short proof of Theorem 3.

Theorem 3. *Let $(x_n)_{n=-2}^{\infty}$ be a well-defined solution of the difference equation*

$$x_{n+1} = 1 + \frac{x_{n-1}x_{n-2}}{1 - x_n}, \quad n \in \mathbb{N}_0. \tag{10}$$

Then

$$x_{2n-1} = \sum_{j=0}^{n-1}\left(\frac{x_{-2}}{1-x_0}\right)^j + \left(\frac{x_{-2}}{1-x_0}\right)^n x_{-1}; \tag{11}$$

$$x_{2n} = \sum_{j=0}^{n-1}(-1)^j\left(\frac{1-x_0}{x_{-2}}\right)^j + (-1)^n\left(\frac{1-x_0}{x_{-2}}\right)^n x_0. \tag{12}$$

Proof. If $x_{n_3} = 0$ for some $n_3 \in \mathbb{N}_0 \cup \{-1\}$, then $x_{n_3+2} = 1$ so that x_{n_3+3} is not defined. If $x_{-2} = 0$, then $x_1 = 1$, so that x_2 is not defined. If $x_{n_4} = 1$, for some $n_4 \in \mathbb{N}_0$, then x_{n_4+1} is not defined. Hence we may assume $x_n \neq 0$, $n \in \mathbb{N}_0 \cup \{-2, -1\}$ and $x_n \neq 1$, $n \in \mathbb{N}_0$.

Now note that for such solutions of equation (10) the following equality holds $(x_{n+1} - 1)/x_{n-1} = -[(x_n - 1)/x_{n-2}]^{-1}$, from which it follows that the sequence $(x_{n+1} - 1)/x_{n-1}$ is two periodic. Thus

$$\frac{x_{2n+1} - 1}{x_{2n-1}} = \frac{x_1 - 1}{x_{-1}} = \frac{x_{-2}}{1 - x_0} \quad \Leftrightarrow \quad x_{2n+1} = \frac{x_{-2}}{1 - x_0}x_{2n-1} + 1$$

and

$$\frac{x_{2n} - 1}{x_{2n-2}} = \frac{x_0 - 1}{x_{-2}} \quad \Leftrightarrow \quad x_{2n} = \frac{x_0 - 1}{x_{-2}}x_{2n-2} + 1.$$

Now note that x_{2n} and x_{2n-1} are solutions of the following two linear first-order difference equations with constant coefficients

$$y_n = \frac{x_{-2}}{1 - x_0}y_{n-1} + 1 \quad \text{and} \quad z_n = \frac{x_0 - 1}{x_{-2}}z_{n-1} + 1.$$

The general solution of the linear first-order difference equation with constant coefficients can be found in any book on difference equation (see, for example, [12]), from which formulae (11) and (12) follow. □

5. On the equation $x_{n+1} = 1 + \frac{x_n^2}{1-2x_n}$

In [2, Theorem 6] the authors quoted the following interesting result without any proof. Here we give an elegant short proof of the result.

Theorem 4. *Let $(x_n)_{n=0}^{\infty}$ be a well-defined solution of the difference equation*

$$x_{n+1} = 1 + \frac{x_n^2}{1 - 2x_n}, \quad n \in \mathbb{N}_0. \tag{13}$$

Then

$$x_{2n-1} = \frac{(1 - x_0)^{2^{2n-1}}}{(1 - 2x_0)\prod_{i=1}^{2n-2}(-(1 - x_0)^{2^i} - x_0^{2^i})}; \tag{14}$$

$$x_{2n} = \frac{x_0^{4^n}}{(1-2x_0)\prod_{i=1}^{2n-1}(-(1-x_0)^{2^i} - x_0^{2^i})}. \tag{15}$$

Proof. Note that for every well-defined solution of equation (13) we have

$$x_{n+1} = \frac{(x_n-1)^2}{1-2x_n} = \left(\frac{(x_{n-1}-1)^2}{1-2x_{n-1}} - 1\right)^2 \left(1 - 2\frac{(x_{n-1}-1)^2}{1-2x_{n-1}}\right)^{-1} = \frac{x_{n-1}^4}{x_{n-1}^4 - (1-x_{n-1})^4}$$

and consequently

$$1 - \frac{1}{x_{n+1}} = \left(1 - \frac{1}{x_{n-1}}\right)^4. \tag{16}$$

From (16) we have

$$1 - \frac{1}{x_{2n}} = \left(1 - \frac{1}{x_{2n-2}}\right)^4 = \cdots = \left(1 - \frac{1}{x_0}\right)^{4^n} \quad \Rightarrow \quad x_{2n} = \frac{x_0^{4^n}}{x_0^{4^n} - (1-x_0)^{4^n}},$$

which is even better form for this part of the solution, than that given in (15).

Similarly, from (16) we obtain

$$1 - \frac{1}{x_{2n-1}} = \left(1 - \frac{1}{x_{2n-3}}\right)^4 = \cdots = \left(1 - \frac{1}{x_1}\right)^{4^{n-1}} = \left(\frac{x_0}{x_0 - 1}\right)^{2 \cdot 4^{n-1}}.$$

Thus

$$x_{2n-1} = \frac{(1-x_0)^{2^{2n-1}}}{(1-x_0)^{2^{2n-1}} - x_0^{2^{2n-1}}},$$

which is another form of the expression in (14), as desired. □

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Bratislav Iričanin, Faculty of Electrical Engineering, Bulevar Kralja Aleksandra 73, 11000 Beograd, Serbia, E-mail: iricanin@etf.bg.ac.rs

Stevo Stević, Mathematical Institute of the Serbian Academy of Science, Knez Mihailova 36/III, 11000 Beograd, Serbia, E-mail: sstevic@ptt.rs