The existence for large sets of disjoint pure directed triple systems with even orders *

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Abstract: A directed triple system of order v, denoted by $DTS(v)$, is a pair $(X, \mathcal{B})$ where X is a v-set and $\mathcal{B}$ is a collection of transitive triples on X such that every ordered pair of X belongs to exactly one triple of $\mathcal{B}$. A $DTS(v)$ is called pure and denoted by $PDTS(v)$ if $(x, y, z) \in \mathcal{B}$ implies $(z, y, x) \notin \mathcal{B}$. A large set of disjoint $PDTS(v)$ is denoted by $LPDTS(v)$. In this paper, we establish the existence of $LPDTS(v)$ for $v \equiv 0, 4 \pmod 6$, $v \geq 4$.

Keywords: large set, pure, directed triple system

1 Introduction

Let X be a finite set. In what follows, an *ordered pair* of X will always be an ordered pair (x, y) where $x \neq y \in X$. A *transitive triple* (x, y, z) from X is a set of three ordered pairs (x, y), (y, z) and (x, z) of X. A *directed triple system with holes* of order v (with index 1), denoted by $HDTS(v; \mathcal{G})$, is a pair $(X, \mathcal{B})$, where X is a v-set, $\mathcal{G} = \{Y_1, Y_2, \cdots, Y_m\}$ is a set of subsets of X with $| Y_i \cap Y_j | \leq 1$ for any $1 \leq i \neq j \leq m$, and $\mathcal{B}$ is a collection of transitive triples (called blocks) from X, such that every ordered pair (x, y) of X, $\{x, y\} \not\subseteq Y_i, 1 \leq i \leq m$, belongs to exactly one triple of $\mathcal{B}$.

Let $(X, \mathcal{B})$ be an $HDTS(v; \mathcal{G})$, $\mathcal{G} = \{Y_1, Y_2, \cdots, Y_m\}$. If $| Y_i | = t \geq 2$ for any $Y_i \in \mathcal{G}$, and $\mathcal{G}$ is a partition of X (i.e., $Y_i \cap Y_j = \phi$ for any $1 \leq i \neq j \leq m$, $X = \bigcup_{i=1}^{m} Y_i$ and $v = tm$), then the $HDTS(v; \mathcal{G})$ is denoted by $HDTS(t^m)$ and we write $(X, \mathcal{G}, \mathcal{B})$ instead of $(X, \mathcal{B})$. If $\mathcal{G} = \phi$, then the $HDTS(v; \phi)$

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is called *directed triple system* and denoted by $DTS(v)$. In other words, $(X, \mathcal{B})$ is a $DTS(v)$ if and only if every ordered pair (x, y) of X belongs to exactly one triple of $\mathcal{B}$.

An $HDTS(v;\mathcal{G}) = (X, \mathcal{B})$ is called *pure*, denoted by $PHDTS(v)$, if $(x, y, z) \in \mathcal{B}$ implies $(z, y, x) \notin \mathcal{B}$. Similarly, we can define pure $HDTS(t^m)$ and pure $DTS(v)$, which are denoted by $PHDTS(t^m)$ and $PDTS(v)$, respectively.

A *large set of directed triple system with holes* of order v, denoted by $LHDTS(v; \mathcal{G}_1, \mathcal{G}_2, \cdots, \mathcal{G}_{|R|})$, is a collection $\{(X, \mathcal{B}_r) : r \in R\}$ satisfying the following conditions:

(1) X is a v-set. $\bigcup_{r\in R} \mathcal{G}_r = \{Y_1, Y_2, \cdots, Y_m\}$ is a set of subsets of X, and $| Y_i \cap Y_j | \leq 2$ for any $1 \leq i \neq j \leq m$.

(2) Each $(X, \mathcal{B}_r)$ is an $HDTS(v; \mathcal{G}_r)$, $r \in R$.

(3) For any Y_i, $| \{r : Y_i \in \mathcal{G}_r, r \in R\} | = 3(| Y_i | -2), 1 \leq i \leq m$.

(4) Each transitive triple (x, y, z) from X, with $\{x, y, z\} \not\subseteq Y_i, 1 \leq i \leq m$, belongs to a unique $\mathcal{B}_r, r \in R$.

It is not difficult to see that an $LHDTS(v; \mathcal{G}_1, \mathcal{G}_2, \cdots, \mathcal{G}_{|R|})$ has $3(v-2)$ members, i.e., $|R| = 3(v-2)$. An $LHDTS(v; \mathcal{G}_1, \mathcal{G}_2, \cdots, \mathcal{G}_{3(v-2)})$ is called *pure* , denoted by $LPHDTS(v; \mathcal{G}_1, \mathcal{G}_2, \cdots, \mathcal{G}_{3(v-2)})$, if its members are pure.

Similarly, an $LPHDTS(t^m)$ will be a collection $\{(X, \mathcal{G}, \mathcal{B}_r) : r \in R\}$ of $PHDTS$ (t^m), such that each transitive triple (x, y, z) from X with $| \{x, y, z\} \cap G | \leq 1$ for any $G \in \mathcal{G}$ belongs to a unique $\mathcal{B}_r$. It is easy to see that an $LPHDTS(t^m)$ contains $3t(m-2)$ members.

Let $\{(X, \mathcal{B}_i) : 1 \leq i \leq 3(v-2)\}$ be an $LHDTS(v; \mathcal{G}_1, \mathcal{G}_2, \cdots, \mathcal{G}_{3(v-2)})$, if each $(X, \mathcal{B}_i)$ is a $DTS(v)$, then the $LHDTS(v; \mathcal{G}_1, \mathcal{G}_2, \cdots, \mathcal{G}_{3(v-2)})$ is called a *large set of directed triple system* and is denoted by $LDTS(v)$. Furthermore, if each $(X, \mathcal{B}_i)$ is a $PDTS(v)$, it is called pure $LDTS(v)$ and denoted by $LPDTS(v)$.

Example 1. $LPDTS(4) = \{(Z_4, \mathcal{B}_i^r) : i = 0, 1, r = 1, 2, 3\}$, where

$\mathcal{B}_0^1 = \{(0,2,3),(1,3,2),(2,0,1),(3,1,0)\}$;
$\mathcal{B}_0^2 = \{(2,3,0),(3,2,1),(0,1,2),(1,0,3)\}$;
$\mathcal{B}_0^3 = \{(3,0,2),(2,1,3),(1,2,0),(0,3,1)\}$;
$\mathcal{B}_1^1 = \{(1,2,3),(0,3,2),(2,1,0),(3,0,1)\}$;
$\mathcal{B}_1^2 = \{(2,3,1),(3,2,0),(1,0,2),(0,1,3)\}$;
$\mathcal{B}_1^3 = \{(3,1,2),(2,0,3),(0,2,1),(1,3,0)\}$.

Example 2. $LPDTS(6) = \{(Z_6, \mathcal{A}_i), (Z_6, \mathcal{B}_i) : i \in Z_6\}$, where

$$\begin{aligned}\mathcal{A}_0 = \{&(0,4,5), \ (1,2,0), \ (2,3,4), \ (4,0,2), \ (5,3,0), \\ &(0,3,1), \ (1,4,3), \ (2,1,5), \ (3,5,2), \ (5,4,1)\}.\end{aligned}$$

$$\begin{aligned}\mathcal{B}_0 = \{&(5,4,0), \ (0,2,1), \ (4,3,2), \ (2,0,4), \ (0,3,5), \\ &(3,1,0), \ (1,3,4), \ (1,2,5), \ (5,2,3), \ (4,5,1)\}.\end{aligned}$$

$\mathcal{A}_i = \mathcal{A}_0 + i, \quad \mathcal{B}_i = \mathcal{B}_0 + i, i \in Z_6.$

Example 3. $LPHDTS(2^3) = \{(X, \mathcal{G}, \mathcal{B}_i^r) : i \in Z_2, r = 1,2,3\}$, where

$$X = Z_3 \times Z_2, \qquad \mathcal{G} = \{\{k\} \times Z_2 : k \in Z_3\},$$

$$\begin{array}{lll}
\mathcal{B}_i^1: & ((0,i),(1,i),(2,i)), & ((2,1+i),(1,1+i),(0,1+i)), \\
& ((0,i),(1,1+i),(2,1+i)), & ((2,i),(1,i),(0,1+i)), \\
& ((0,1+i),(1,i),(2,1+i)), & ((2,i),(1,1+i),(0,i)), \\
& ((0,1+i),(1,1+i),(2,i)), & ((2,1+i),(1,i),(0,i)); \\
\mathcal{B}_i^2: & ((1,i),(2,i),(0,i)), & ((0,1+i),(2,1+i),(1,1+i)), \\
& ((1,i),(2,1+i),(0,1+i)), & ((0,i),(2,i),(1,1+i)), \\
& ((1,1+i),(2,i),(0,1+i)), & ((0,i),(2,1+i),(1,i)), \\
& ((1,1+i),(2,1+i),(0,i)), & ((0,1+i),(2,i),(1,i)); \\
\mathcal{B}_i^3: & ((2,i),(0,i),(1,i)), & ((1,1+i),(0,1+i),(2,1+i)), \\
& ((2,i),(0,1+i),(1,1+i)), & ((1,i),(0,i),(2,1+i)), \\
& ((2,1+i),(0,i),(1,1+i)), & ((1,i),(0,1+i),(2,i)), \\
& ((2,1+i),(0,1+i),(1,i)), & ((1,1+i),(0,i),(2,i)).
\end{array}$$

Example 4. $LPHDTS(2^4) = \{(X, \mathcal{G}, \mathcal{B}_{i,j}^r) : i,j \in Z_2, r = 1,2,3\}$, where

$$X = Z_4 \times Z_2, \qquad \mathcal{G} = \{\{k\} \times Z_2 : k \in Z_4\},$$

$$\begin{array}{lll}
\mathcal{B}_{i,j}^r: & ((x,j),(y,j),(z,j)), & ((x,j),(y,1+j),(z,1+j)), \\
& ((x,1+j),(y,j),(z,1+j)), & ((x,1+j),(y,1+j),(z,j)),
\end{array}$$

where $(x,y,z) \in \mathcal{B}_i^r$ in Example 1.

It is well known [1-5] that

(1) There exists a $DTS(v)$ if and only if $v \equiv 0,1 \pmod 3$ and $v \geq 3$.

(2) There exists a $PDTS(v)$ if and only if $v \equiv 0,1 \pmod 3$ and $v \geq 4$.

(3) There exists an $LDTS(v)$ if and only if $v \equiv 0,1 \pmod 3$ and $v \geq 3$.

Recently, in order to construct "Generalized Steiner Systems"— a type of new designs which are equivalent to maximum constant weight codes, Kevin Phelps and Carol Yin posed the open problem in [6] of finding large sets of disjoint pure $MTS(v)$. F.E.Bennett, Q.D.Kang, H. Zhang and J.G.Lei have given some preliminary results (see [7,8]). In this paper, we will study the analogous problem for large sets of disjoint pure $DTS(v)$ and give the existence of $LPDTS(v)$ for even v.

2 $LPDTS(2^n + 2)$

First, we introduce some definitions which come from Teirlinck's paper[9].

An $S(t,K,v)$, $t,v \in N$, $K \subseteq N \setminus \{1,2,\cdots,t-1\}$, is a pair $(X,\mathcal{B})$, where X is a v-set and $\mathcal{B}$ is a collection of subsets of X, called blocks, such that every t-subset of X is contained in exactly one block and such that $|B| \in K$ for any $B \in \mathcal{B}$. An $S(t,K,v)$ is often called a *t-wise balanced design.* An $S(2,K,v)$ is called a *pairwise balanced design* or PBD. If $(X,\mathcal{B})$ is an

$S(2, K, v)$ and if $a \neq b \notin X$, we denote the unique block through a and b by ab or by $a\mathcal{B}b$ if confusion is possible.

Let X be a v-set, $\infty_1 \neq \infty_2 \notin X$, $(X \bigcup \{\infty_1, \infty_2\}, \mathcal{B})$ be an $S(2, K, v+2)$, and $K_i = \{|B| : B \in \mathcal{B}, |B \bigcap \{\infty_1, \infty_2\}| = i\}$, $i = 0, 1, 2$, then we write $S(2, (K_0, K_1, K_2), v+2)$ instead of the $S(2, K, v+2)$. A *good* $S(2, (K_0, \{3\}, K_2), v+2)$ or $GS(2, (K_0, \{3\}, K_2), v+2)$ will be a 5-tuple $(X, \infty_1, \infty_2, \mathcal{B}, \mathcal{D})$, such that $(X \bigcup \{\infty_1, \infty_2\}, \mathcal{B})$ is an $S(2, (K_0, \{3\}, K_2), v+2)$, and such that $\mathcal{D}$ is a 1-regular digraph on X whose underlying undirected graph has edge set $\{\{x, y\} : x, y \in X, \{\infty_i, x, y\} \in \mathcal{B}, i \in \{1, 2\}\}$.

A $GLS(2, (3, K_0, \{3\}, K_2), v+2)$ will be a collection $\{(X, \infty_1, \infty_2, \mathcal{B}_r, \mathcal{D}_r) : r \in R\}$ of $GS(2, (K_0, \{3\}, K_2), v+2)$, such that:

(1) $(X \bigcup \{\infty_1, \infty_2\}, \bigcup_{r \in R} \mathcal{B}_r)$ is an $S(3, K_0 \bigcup \{3\} \bigcup K_2, v+2)$;

(2) For each $B \in \bigcup_{r \in R} \mathcal{B}_r$, there are exactly $|B| - 2$ elements of R such that $B \in \mathcal{B}_r$;

(3) Each ordered pair (x, y) of X not contained in some block $\infty_1 \mathcal{B}_r \infty_2$ occurs in a unique $\mathcal{D}_r$, $r \in R$.

Lemma 1.[9] (i) A $GLS(2, (3, K_0, \{3\}, K_2), v+2)$ must have v elements.

(ii) There exists a $GLS(2, (3, \{3, 4\}, \{3\}, \{2^{n-2}+2\}), 2^n+2)$ for $n \geq 5$.

Lemma 2. If there exist $GLS(2, (3, K_0, \{3\}, K_2), v+2)$ and $LPHDTS(2^k)$ for any $k \in K_0$, then there exists an $LPHDTS(2v+2 : \{H_{ijk} : i \in Z_v, j \in Z_2, k \in I_3\})$, where $|H_{ijk}| \in \{2l - 2 : l \in K_2\}$.

Proof. Let $\{(Z_v, \infty_1, \infty_2, \mathcal{B}_i, \mathcal{D}_i) : i \in Z_v\}$ be a $GLS(2, (3, K_0, \{3\}, K_2), v+2)$, where $\infty_1, \infty_2 \notin Z_v$ and $\infty_1 \neq \infty_2$. As an $S(2, (K_0, \{3\}, K_2), v+2)$ on the set $Z_v \bigcup \{\infty_1, \infty_2\}$, there is a unique block $B_\infty(i)$ in each $\mathcal{B}_i$, which contains ∞_1 and ∞_2. Furthermore, define

$B_{\infty_1}(i) = \{B : \infty_1 \in B \in \mathcal{B}_i \backslash \{B_\infty(i)\}\}$,

$B_{\infty_2}(i) = \{B : \infty_2 \in B \in \mathcal{B}_i \backslash \{B_\infty(i)\}\}$.

$\mathcal{D}_i^1 = \{(x, y) : (x, y) \in \mathcal{D}_i, \{\infty_1, x, y\} \in B_{\infty_1}(i)\}$,

$\mathcal{D}_i^2 = \{(x, y) : (x, y) \in \mathcal{D}_i, \{\infty_2, x, y\} \in B_{\infty_2}(i)\}$.

$H_{ijk} = ((B_\infty(i) \setminus \{\infty_1, \infty_2\}) \times Z_2) \bigcup \{\infty_1, \infty_2\}, \quad i \in Z_v, j \in Z_2, k \in I_3$.

Note that the block $B_\infty(i)$ is contained in $|B_\infty(i)| - 2$ block sets $\mathcal{B}_m$. Let $R_i(\infty) = \{m\text{: } B_\infty(i) \in \mathcal{B}_m\}$, then $i \in R_i(\infty)$ and $|R_i(\infty)| = |B_\infty(i)| - 2$. Thus $B_\infty(m) = B_\infty(i)$ for any $m \in R_i(\infty)$, and $H_{m0k} = H_{m1k} = H_{n0k} = H_{n1k}$ for any $m, n \in R_i(\infty), k \in \{1, 2, 3\}$. So, for any H_{ijk}, there exist $3(|H_{ijk}| - 2)$ $H_{i'j'k'}$ such that $H_{ijk} = H_{i'j'k'}$, where $i' \in Z_v$, $j' \in Z_2, k' \in \{1, 2, 3\}$.

Let $Y = (Z_v \times Z_2) \bigcup \{\infty_1, \infty_2\}$. The element $(x, i) \in Z_v \times Z_2$ can be denoted by x_i, briefly. Now, we construct transitive triple systems $\mathcal{A}_{ijk} (i \in Z_v, j \in Z_2, k \in I_3)$ on the set Y as follows:

(i) For any $B \in \bigcup_{i \in Z_v} \mathcal{B}_i$, $\{\infty_1, \infty_2\} \bigcap B = \phi$, let $R_B = \{i \in Z_v : B \in \mathcal{B}_i\}$, then $|R_B| = |B| - 2$ and $|B| \in K_0$. By the known condition, there exists an $LPHDTS(2^{|B|}) = \{(B \times Z_2, \mathcal{C}_{ijk}(B)) : i \in R_B, j \in Z_2, k \in I_3\}$.

(ii) For any $B \in \bigcup_{i \in Z_v} \mathcal{B}_i$, $|B \bigcap \{\infty_1, \infty_2\}| = 1$, i.e., $|B| = 3$, there exists a unique $i \in Z_v$ such that $B \in \mathcal{B}_i$.

For any ordered pair $(x, y) \in \mathcal{D}_i^1$, i.e., $\{\infty_1, x, y\} \in \mathcal{B}_{\infty_1}(i)$, let

$$\begin{aligned} \mathcal{C}_{ij1}(B) = \{ & (\infty_1, x_t, y_{t+j}), (y_{t+j}, x_{1+t}, \infty_2), (x_{1+t}, y_{t+j}, x_t) : & t \in Z_2\}, \\ \mathcal{C}_{ij2}(B) = \{ & (y_{t+j}, \infty_1, x_t), (\infty_2, y_{t+j}, x_{1+t}), (x_t, x_{1+t}, y_{t+j}) : & t \in Z_2\}, \\ \mathcal{C}_{ij3}(B) = \{ & (x_t, y_{t+j}, \infty_1), (x_{1+t}, \infty_2, y_{t+j}), (y_{t+j}, x_t, x_{1+t}) : & t \in Z_2\}. \end{aligned}$$

For any ordered pair $(x, y) \in \mathcal{D}_i^2$, i.e., $\{\infty_2, x, y\} \in \mathcal{B}_{\infty_2}(i)$, let

$$\begin{aligned} \mathcal{C}_{ij1}(B) &= \{(x_t, y_{t+j+1}, \infty_1), (\infty_2, y_{t+j+1}, x_{1+t}), (x_{1+t}, y_{t+j+1}, x_t) : t \in Z_2\}, \\ \mathcal{C}_{ij2}(B) &= \{(y_{t+j+1}, \infty_1, x_t), (y_{t+j+1}, x_{1+t}, \infty_2), (x_t, x_{1+t}, y_{t+j+1}) : t \in Z_2\}, \\ \mathcal{C}_{ij3}(B) &= \{(\infty_1, x_t, y_{t+j+1}), (x_{1+t}, \infty_2, y_{t+j+1}), (y_{t+j+1}, x_t, x_{1+t}) : t \in Z_2\}. \end{aligned}$$

Define

$$\mathcal{A}_{ijk} = \bigcup_{B \in \mathcal{B}_i \setminus B_\infty(i)} \mathcal{C}_{ijk}(B), i \in Z_v, j \in Z_2, k \in I_3.$$

Then $\{(Y, \mathcal{A}_{ijk}) : i \in Z_v, j \in Z_2, k \in I_3\}$ is an $LPHDTS(2v+2;\ \{H_{ijk} : i \in Z_v, j \in Z_2, k \in I_3\})$. □

Lemma 3. If there exist a $PHDTS(v; \mathcal{G})$, $\mathcal{G} = \{Y_1, Y_2, \cdots, Y_m\}$, and a $PHDTS\ (|Y_i|; \mathcal{G}_i)$ for any i, $1 \le i \le m$, then there exists a $PHDTS(v; \mathcal{G}')$, where $\mathcal{G}' = \bigcup_{i=1}^{m} \mathcal{G}_i$.

Proof. Let X be a v-set and $(X, \mathcal{B})$ be a $PHDTS(v; \mathcal{G})$. For each Y_i, $1 \le i \le m$, let $(Y_i, \mathcal{B}_i)$ be a $PHDTS(|Y_i|; \mathcal{G}_i)$, and let $\mathcal{G}' = \bigcup_{i=1}^{m} \mathcal{G}_i = \{W_1, W_2, \cdots, W_h\}$. Define

$$\mathcal{A} = \mathcal{B} \bigcup (\bigcup_{i=1}^{m} \mathcal{B}_i),$$

then $(X, \mathcal{A})$ is a $PHDTS(v; \mathcal{G}')$. In fact, for any ordered pair (x, y), $\{x, y\} \not\subseteq W_j$, $1 \le j \le h$, if $\{x, y\} \not\subseteq Y_i$, $1 \le i \le m$, then there is exactly one transitive triple of $\mathcal{B}$ containing (x, y); if $\{x, y\} \subseteq Y_i$ for some i, $1 \le i \le m$, by the definition of $PHDTS$, i is unique (since $|Y_j \bigcap Y_l| \le 1$ for any $1 \le j \ne l \le m$), then (x, y) is contained in a unique transitive triple of $\mathcal{B}_i$. Thus $(X, \mathcal{A})$ is a $PHDTS(v; \mathcal{G}')$. □

Corollary 1. If there exist a $PHDTS(v; \mathcal{G})$ and a $PDTS(|Y|)$ for each $Y \in \mathcal{G}$, then there exists a $PDTS(v)$.

Lemma 4. If there exists an $LPHDTS(v;\ \mathcal{G}_1,\ \mathcal{G}_2,\ \cdots,\ \mathcal{G}_{3(v-2)})$, $\bigcup_{i=1}^{3(v-2)} \mathcal{G}_i = \{Y_1, \cdots, Y_m\}$, and an $LPHDTS(|Y_i|; T_i(j_1), T_i(j_2), \cdots, T_i(j_{3(|Y_i|-2)}))$, $1 \le i \le m$, where $\{j_1, j_2, \cdots, j_{3(|Y_i|-2)}\} \subseteq \{h : Y_i \in \mathcal{G}_h, 1 \le h \le 3(v-2)\}$, then there exists an $LPHDTS(v;\mathcal{G}'_1,\mathcal{G}'_2,\cdots,\mathcal{G}'_{3(v-2)})$, where $\mathcal{G}'_k=\bigcup_{Y_i\in\mathcal{G}_k}T_i(k)$.

Proof. Let X be a v-set, $\{(X,\mathcal{B}_k):1\le k\le 3(v-2)\}$ be an $LPHDTS(v;\mathcal{G}_1,\mathcal{G}_2,\cdots,\mathcal{G}_{3(v-2)})$, $\bigcup_{i=1}^{3(v-2)}\mathcal{G}_i=\{Y_1,Y_2,\cdots,Y_m\}$. Define $R_i=\{h:Y_i\in\mathcal{G}_h,1\le h\le 3(v-2)\}$ for $1\le i\le m$, then $|R_i|=3(|Y_i|-2)$. Let $LPHDTS(|Y_i|;T_i(j_1),T_i(j_2),\cdots,T_i(j_{3(|Y_i|-2)}))=\{(Y_i,\mathcal{C}_k(i)):k\in R_i\}$, where $j_l\in R_i$, $1\le l\le |R_i|$. Let $\mathcal{G}'_k=\bigcup_{Y_i\in\mathcal{G}_k}T_i(k)$ and $\bigcup_{k=1}^{m}\mathcal{G}'_k=\{W_1,W_2,\cdots,W_s\}$. Define

$$\mathcal{A}_k=\mathcal{B}_k\bigcup(\bigcup_{Y_i\in\mathcal{G}_k}\mathcal{C}_k(i)).$$

By Lemma 3, each $(X,\mathcal{A}_k)$ is a $PHDTS(v;\mathcal{G}'_k),1\le k\le 3(v-2)$.

Let (x,y,z) be a transitive triple, $\{x,y,z\}\not\subseteq W_n$, $1\le n\le s$. If $\{x,y,z\}$ is not contained in any Y_i, then by the definition of $LPHDTS$, there exists a unique k, $1\le k\le 3(v-2)$, such that $(x,y,z)\in\mathcal{B}_k\subseteq\mathcal{A}_k$. If $\{x,y,z\}$ is contained in some Y_i, then Y_i is unique. Since $\{(Y_i,\mathcal{C}_k(i)):k\in R_i\}$ is an $LPHDTS(|Y_i|;T_i(j_1),T_i(j_2),\cdots,T_i(j_{3(|Y_i|-2)}))$, there exists a unique k such that $(x,y,z)\in\mathcal{C}_k(i)\subseteq\mathcal{A}_k$. Therefore, $\{(X,\mathcal{A}_k):1\le k\le 3(v-2)\}$ is an $LPHDTS(v;\mathcal{G}'_1,\mathcal{G}'_2,\cdots,\mathcal{G}'_{3(v-2)})$. □

Corollary 2. If there exists an $LPHDTS(v;\mathcal{G}_1,\mathcal{G}_2,\cdots,\mathcal{G}_{3(v-2)})$, $\bigcup_{i=1}^{3(v-2)}\mathcal{G}_i=\{Y_1,\cdots,Y_m\}$, and there exists an $LPDTS(|Y_i|)$ for any i, $1\le i\le m$, then there exists an $LPDTS(v)$.

Theorem 1. There exists an $LPDTS(v)$ for $v=10,12,18$.

$LPDTS(10)=\{(\{a,b\}\cup Z_8,\mathcal{B}^r_x):x\in Z_8,r=1,2,3\}$, where

$\mathcal{B}^1_0$:	0 1 2	2 7 0	3 4 7	6 2 5	7 4 6	7 1 3	6 0 3	
	4 3 2	2 1 4	5 0 7	5 1 6	0 6 4	4 1 5	3 0 5	
	1 0 a	2 6 a	a 3 1	5 a 4	7 a 5	3 a 6	a 7 2	b 4 a
	4 b 0	1 7 b	2 3 b	5 b 3	b 5 2	b 6 1	6 b 7	a 0 b
$\mathcal{B}^2_0$:	2 0 1	7 0 2	7 3 4	2 5 6	4 6 7	1 3 7	0 3 6	
	2 4 3	1 4 2	0 7 5	1 6 5	6 4 0	5 4 1	5 3 0	
	a 1 0	6 a 2	3 1 a	a 4 5	5 7 a	a 6 3	2 a 7	4 a b
	b 0 4	7 b 1	3 b 2	b 3 5	5 2 b	6 1 b	b 7 6	0 b a
$\mathcal{B}^3_0$:	1 2 0	0 2 7	4 7 3	5 6 2	6 7 4	3 7 1	3 6 0	
	3 2 4	4 2 1	7 5 0	6 5 1	4 0 6	1 5 4	0 5 3	
	0 a 1	a 2 6	1 a 3	4 5 a	a 5 7	6 3 a	7 2 a	a b 4
	0 4 b	b 1 7	b 2 3	3 5 b	2 b 5	1 b 6	7 6 b	b a 0

$\mathcal{B}^r_x=\mathcal{B}^r_0+x,x\in Z_8,r=1,2,3.$

$LPDTS(12) = \{(\{a,b\} \cup Z_{10}, \mathcal{B}_x^r) : x \in Z_{10}, r = 1,2,3\}$, where

$\mathcal{B}_0^1$:	9 0 1	1 5 9	4 9 5	5 2 4	3 2 5	5 8 3	8 5 0	0 2 8	4 2 0
	0 7 4	7 0 3	3 *a* 7	*a* 3 8	8 9 *a*	2 *a* 9	9 *b* 2	7 2 *b*	*b* 9 7
	6 7 9	9 8 6	7 6 8	8 4 7	*b* 4 8	8 1 *b*	1 8 2	2 7 1	5 1 7
	7 *a* 5	5 *a b*	*b* 6 5	0 5 6	6 *b* 0	0 *b a*	*a* 1 0	1 *a* 4	4 6 1
	6 4 *a*	*a* 2 6	6 2 3	3 1 6	*b* 1 3	3 4 *b*	9 4 3	3 0 9	

$\mathcal{B}_0^2$:	0 1 9	9 1 5	5 4 9	2 4 5	5 3 2	8 3 5	5 0 8	2 8 0	0 4 2
	7 4 0	0 3 7	*a* 7 3	3 8 *a*	*a* 8 9	9 2 *a*	2 9 *b*	*b* 7 2	7 *b* 9
	9 6 7	6 9 8	8 7 6	4 7 8	8 *b* 4	1 *b* 8	8 2 1	1 2 7	7 5 1
	5 7 *a*	*a b* 5	5 *b* 6	6 0 5	0 6 *b*	*b a* 0	1 0 *a*	*a* 4 1	6 1 4
	4 *a* 6	6 *a* 2	2 3 6	1 6 3	3 *b* 1	4 *b* 3	3 9 4	9 3 0	

$\mathcal{B}_0^3$:	1 9 0	5 9 1	9 5 4	4 5 2	2 5 3	3 5 8	0 8 5	8 0 2	2 0 4
	4 0 7	3 7 0	7 3 *a*	8 *a* 3	9 *a* 8	*a* 9 2	*b* 2 9	2 *b* 7	9 7 *b*
	7 9 6	8 6 9	6 8 7	7 8 4	4 8 *b*	*b* 8 1	2 1 8	7 1 2	1 7 5
	a 5 7	*b* 5 *a*	6 5 *b*	5 6 0	*b* 0 6	*a* 0 *b*	0 *a* 1	4 1 *a*	1 4 6
	a 6 4	2 6 *a*	3 6 2	6 3 1	1 3 *b*	*b* 3 4	4 3 9	0 9 3	

$\mathcal{B}_x^r = \mathcal{B}_0^r + x, x \in Z_{10}, r = 1,2,3.$

$LPDTS(18) = \{(X, \mathcal{B}_x^r) : x \in Z_{14}, r = 1,2,3\} \bigcup \{(X, \mathcal{A}_i^r) : i \in Z_2, r = 1,2,3\}$, where $X = Z_{14} \cup \{a,b,c,d\}$, $\mathcal{B}_x^r = \mathcal{B}_0^r + x$ for $x \in Z_{14}, r = 1,2,3$ and

$\mathcal{B}_0^1$:	4 13 10	2 13 4	1 2 0	2 1 11	10 11 4	4 0 2	0 4 1
	4 11 9	12 11 10	3 10 6	5 1 4	11 1 6	10 3 12	5 6 10
	3 1 5	6 1 3	1 7 13	1 12 9	9 12 4	13 6 5	6 0 8
	12 8 2	4 7 3	5 9 13	5 7 12	10 7 5	3 7 0	5 2 3
	6 9 11	6 12 7	8 0 13	7 2 6	11 3 13	13 12 1	10 9 0
	8 11 5	9 8 1	8 9 3	2 9 10	9 2 7	5 11 0	8 7 4
	0 9 5	3 11 8	13 7 8	8 12 6	*d* 10 13	11 *d* 2	*d* 7 1
	d 0 6	2 *d* 12	12 *d* 5	13 9 *d*	3 *d* 4	0 *d* 3	6 4 *d*
	8 10 *d*	12 3 *c*	9 6 *c*	2 5 *c*	*c* 4 12	*c* 13 0	*c* 6 2
	13 *c* 11	0 7 *c*	*c* 1 10	11 *c* 7	10 8 *c*	13 2 *b*	4 *b* 5
	b 12 13	0 *b* 10	5 *b* 8	1 8 *b*	3 9 *b*	12 *b* 0	*b* 4 6
	10 *b* 2	7 *b* 11	11 12 *a*	*a* 3 2	2 8 *a*	7 *a* 9	*a* 7 10
	a 6 13	*a* 0 11	13 3 *a*	4 *a* 8	10 1 *a*	0 *a* 12	*d c* 8
	1 *c d*	*d* 11 *b*	*b* 7 *d*	*d* 9 *a*	*a* 5 *d*	*a* 4 *c*	*c* 5 *a*
	c b 9	*b c* 3	*b a* 1	6 *a b*			

$\mathcal{B}_0^2$:	10 4 13	13 4 2	0 1 2	11 2 1	11 4 10	2 4 0	1 0 4
	9 4 11	10 12 11	6 3 10	4 5 1	1 6 11	12 10 3	10 5 6
	1 5 3	3 6 1	13 1 7	12 9 1	4 9 12	6 5 13	0 8 6
	2 12 8	7 3 4	9 13 5	7 12 5	5 10 7	0 3 7	3 5 2
	11 6 9	12 7 6	13 8 0	2 6 7	13 11 3	1 13 12	9 0 10
	11 5 8	1 9 8	3 8 9	10 2 9	7 9 2	0 5 11	4 8 7
	5 0 9	8 3 11	7 8 13	6 8 12	13 d 10	2 11 d	7 1 d
	6 d 0	d 12 2	5 12 d	d 13 9	d 4 3	3 0 d	4 d 6
	10 d 8	3 c 12	c 9 6	c 2 5	12 c 4	0 c 13	6 2 c
	11 13 c	7 c 0	10 c 1	c 7 11	8 c 10	2 b 13	5 4 b
	12 13 b	10 0 b	b 8 5	8 b 1	9 b 3	b 0 12	6 b 4
	b 2 10	11 7 b	a 11 12	2 a 3	8 a 2	a 9 7	7 10 a
	13 a 6	11 a 0	3 a 13	a 8 4	1 a 10	12 0 a	c 8 d
	d 1 c	b d 11	d b 7	9 a d	d a 5	4 c a	5 a c
	b 9 c	c 3 b	a 1 b	b 6 a			

$\mathcal{B}_0^3$:	13 10 4	4 2 13	2 0 1	1 11 2	4 10 11	0 2 4	4 1 0
	11 9 4	11 10 12	10 6 3	1 4 5	6 11 1	3 12 10	6 10 5
	5 3 1	1 3 6	7 13 1	9 1 12	12 4 9	5 13 6	8 6 0
	8 2 12	3 4 7	13 5 9	12 5 7	7 5 10	7 0 3	2 3 5
	9 11 6	7 6 12	0 13 8	6 7 2	3 13 11	12 1 13	0 10 9
	5 8 11	8 1 9	9 3 8	9 10 2	2 7 9	11 0 5	7 4 8
	9 5 0	11 8 3	8 13 7	12 6 8	10 13 d	d 2 11	1 d 7
	0 6 d	12 2 d	d 5 12	9 d 13	4 3 d	d 3 0	d 6 4
	d 8 10	c 12 3	6 c 9	5 c 2	4 12 c	13 0 c	2 c 6
	c 11 13	c 0 7	1 10 c	7 11 c	c 10 8	b 13 2	b 5 4
	13 b 12	b 10 0	8 5 b	b 1 8	b 3 9	0 12 b	4 6 b
	2 10 b	b 11 7	12 a 11	3 2 a	a 2 8	9 7 a	10 a 7
	6 13 a	0 11 a	a 13 3	8 4 a	a 10 1	a 12 0	8 d c
	c d 1	11 b d	7 d b	a d 9	5 d a	c a 4	a c 5
	9 c b	3 b c	1 b a	a b 6			

$\mathcal{A}_0^1$:	0 10 8	11 9 1	10 2 12	11 3 13	12 4 0	13 5 1	6 2 0
	7 3 1	8 4 2	5 3 9	6 4 10	5 11 7	6 12 8	7 13 9
	10 0 3	3 7 10	1 4 11	11 4 8	5 12 2	9 12 5	6 13 3
	10 13 6	0 4 7	7 11 0	1 5 8	8 12 1	9 13 2	2 6 9
	13 0 11	1 2 13	4 1 3	3 5 6	7 8 5	9 10 7	12 9 11
	0 1 12	3 0 2	2 4 5	7 4 6	8 9 6	8 10 11	12 13 10
	8 a 0	a 1 9	a 2 10	a 3 11	a 4 12	a 5 13	0 a 6
	1 a 7	2 a 8	9 3 a	10 4 a	11 5 a	12 6 a	13 7 a
	b 0 9	9 4 b	5 0 b	b 10 5	1 10 b	b 6 1	11 6 b
	b 2 11	7 2 b	b 12 7	3 12 b	b 8 3	13 8 b	b 4 13
	c 0 13	c 2 1	3 c 4	c 6 5	8 7 c	10 9 c	11 c 12
	1 0 c	2 c 3	5 4 c	6 c 7	c 9 8	c 11 10	13 12 c

	d 9 0	4 9 *d*	0 5 *d*	*d* 5 10	10 1 *d*	*d* 1 6	6 11 *d*
	d 11 2	2 7 *d*	*d* 7 12	12 3 *d*	*d* 3 8	8 13 *d*	*d* 13 4
	a b c	*b a d*	*c d a*	*d c b*			

$\mathcal{A}_0^2$:	8 0 10	1 11 9	12 10 2	3 13 11	0 12 4	5 1 13	2 0 6
	3 1 7	4 2 8	9 5 3	4 10 6	11 7 5	12 8 6	9 7 13
	3 10 0	7 10 3	4 11 1	8 11 4	2 5 12	12 5 9	13 3 6
	6 10 13	4 7 0	0 7 11	8 1 5	1 8 12	13 2 9	6 9 2
	11 13 0	2 13 1	1 3 4	6 3 5	5 7 8	10 7 9	9 11 12
	12 0 1	0 2 3	5 2 4	6 7 4	9 6 8	10 11 8	13 10 12
	0 8 *a*	9 *a* 1	2 10 *a*	11 *a* 3	4 12 *a*	13 *a* 5	*a* 6 0
	7 1 *a*	*a* 8 2	3 *a* 9	*a* 10 4	5 *a* 11	6 *a* 12	*a* 13 7
	0 9 *b*	*b* 9 4	*b* 5 0	10 5 *b*	*b* 1 10	6 1 *b*	*b* 11 6
	2 11 *b*	*b* 7 2	12 7 *b*	*b* 3 12	8 3 *b*	*b* 13 8	4 13 *b*
	0 13 *c*	1 *c* 2	*c* 4 3	5 *c* 6	*c* 8 7	9 *c* 10	*c* 12 11
	c 1 0	3 2 *c*	4 *c* 5	7 6 *c*	8 *c* 9	11 10 *c*	12 *c* 13
	9 0 *d*	*d* 4 9	*d* 0 5	5 10 *d*	*d* 10 1	1 6 *d*	*d* 6 11
	11 2 *d*	*d* 2 7	7 12 *d*	*d* 12 3	3 8 *d*	*d* 8 13	13 4 *d*
	c a b	*d b a*	*a c d*	*b d c*			

$\mathcal{A}_0^3$:	10 8 0	9 1 11	2 12 10	13 11 3	4 0 12	1 13 5	0 6 2
	1 7 3	2 8 4	3 9 5	10 6 4	7 5 11	8 6 12	13 9 7
	0 3 10	10 3 7	11 1 4	4 8 11	12 2 5	5 9 12	3 6 13
	13 6 10	7 0 4	11 0 7	5 8 1	12 1 8	2 9 13	9 2 6
	0 11 13	13 1 2	3 4 1	5 6 3	8 5 7	7 9 10	11 12 9
	1 12 0	2 3 0	4 5 2	4 6 7	6 8 9	11 8 10	10 12 13
	a 0 8	1 9 *a*	10 *a* 2	3 11 *a*	12 *a* 4	5 13 *a*	6 0 *a*
	a 7 1	8 2 *a*	*a* 9 3	4 *a* 10	*a* 11 5	*a* 12 6	7 *a* 13
	9 *b* 0	4 *b* 9	0 *b* 5	5 *b* 10	10 *b* 1	1 *b* 6	6 *b* 11
	11 *b* 2	2 *b* 7	7 *b* 12	12 *b* 3	3 *b* 8	8 *b* 13	13 *b* 4
	13 *c* 0	2 1 *c*	4 3 *c*	6 5 *c*	7 *c* 8	*c* 10 9	12 11 *c*
	0 *c* 1	*c* 3 2	*c* 5 4	*c* 7 6	9 8 *c*	10 *c* 11	*c* 13 12
	0 *d* 9	9 *d* 4	5 *d* 0	10 *d* 5	1 *d* 10	6 *d* 1	11 *d* 6
	2 *d* 11	7 *d* 2	12 *d* 7	3 *d* 12	8 *d* 3	13 *d* 8	4 *d* 13
	b c a	*a d b*	*d a c*	*c b d*			

$\mathcal{A}_1^1$:	6 0 4	3 5 13	2 4 12	3 11 1	2 10 0	13 1 9	0 8 12
	7 11 13	6 10 12	11 5 9	8 10 4	9 3 7	6 8 2	7 1 5
	1 0 3	1 13 12	13 11 10	9 8 11	7 6 9	5 4 7	2 5 3
	0 13 2	12 11 0	12 10 9	7 10 8	8 6 5	4 3 6	4 2 1
	7 4 0	0 11 7	13 6 3	10 6 13	1 11 4	4 11 8	3 0 10
	10 7 3	12 8 1	5 1 8	9 5 12	12 5 2	2 9 6	9 2 13
	a 0 9	9 4 *a*	5 0 *a*	*a* 10 5	1 10 *a*	*a* 6 1	11 6 *a*
	a 2 11	7 2 *a*	*a* 12 7	3 12 *a*	*a* 8 3	13 8 *a*	*a* 4 13

b 9 0	4 9 b	0 5 b	b 5 10	10 1 b	b 1 6	6 11 b
b 11 2	2 7 b	b 7 12	12 3 b	b 3 8	8 13 b	b 13 4
0 6 c	c 13 5	c 12 4	c 11 3	c 10 2	c 9 1	c 8 0
13 c 7	12 c 6	5 11 c	4 10 c	3 9 c	2 8 c	1 7 c
d 0 1	d 12 13	d 10 11	d 8 9	6 7 d	d 5 6	4 5 d
3 d 2	13 0 d	11 12 d	9 10 d	8 d 7	d 3 4	1 2 d
b a c	a b d	d c a	c d b			

$\mathcal{A}_1^2$:

4 6 0	5 13 3	12 2 4	11 1 3	10 0 2	9 13 1	8 12 0
11 13 7	10 12 6	5 9 11	4 8 10	3 7 9	2 6 8	1 5 7
3 1 0	12 1 13	10 13 11	8 11 9	9 7 6	4 7 5	3 2 5
2 0 13	11 0 12	9 12 10	10 8 7	5 8 6	6 4 3	1 4 2
0 7 4	7 0 11	3 13 6	6 13 10	4 1 11	11 8 4	0 10 3
7 3 10	1 12 8	8 5 1	12 9 5	5 2 12	6 2 9	13 9 2
0 9 a	a 9 4	a 5 0	10 5 a	a 1 10	6 1 a	a 11 6
2 11 a	a 7 2	12 7 a	a 3 12	8 3 a	a 13 8	4 13 a
9 0 b	b 4 9	b 0 5	5 10 b	b 10 1	1 6 b	b 6 11
11 2 b	b 2 7	7 12 b	b 12 3	3 8 b	b 8 13	13 4 b
c 0 6	13 5 c	4 c 12	3 c 11	2 c 10	1 c 9	0 c 8
c 7 13	6 12 c	11 c 5	10 c 4	9 c 3	8 c 2	7 c 1
0 1 d	13 d 12	11 d 10	9 d 8	d 6 7	6 d 5	5 d 4
d 2 3	d 13 0	12 d 11	10 d 9	7 8 d	3 4 d	2 d 1
c b a	d a b	a d c	b c d			

$\mathcal{A}_1^3$:

0 4 6	13 3 5	4 12 2	1 3 11	0 2 10	1 9 13	12 0 8
13 7 11	12 6 10	9 11 5	10 4 8	7 9 3	8 2 6	5 7 1
0 3 1	13 12 1	11 10 13	11 9 8	6 9 7	7 5 4	5 3 2
13 2 0	0 12 11	10 9 12	8 7 10	6 5 8	3 6 4	2 1 4
4 0 7	11 7 0	6 3 13	13 10 6	11 4 1	8 4 11	10 3 0
3 10 7	8 1 12	1 8 5	5 12 9	2 12 5	9 6 2	2 13 9
9 a 0	4 a 9	0 a 5	5 a 10	10 a 1	1 a 6	6 a 11
11 a 2	2 a 7	7 a 12	12 a 3	3 a 8	8 a 13	13 a 4
0 b 9	9 b 4	5 b 0	10 b 5	1 b 10	6 b 1	11 b 6
2 b 11	7 b 2	12 b 7	3 b 12	8 b 3	13 b 6	4 b 13
6 c 0	5 c 13	12 4 c	11 3 c	10 2 c	9 1 c	8 0 c
7 13 c	c 6 12	c 5 11	c 4 10	c 3 9	c 2 8	c 1 7
1 d 0	12 13 d	10 11 d	8 9 d	7 d 6	5 6 d	d 4 5
2 3 d	0 d 13	d 11 12	d 9 10	d 7 8	4 d 3	d 1 2
a c b	b d a	c a d	d b c			

□

Let $Z_v = \{0, 1, \cdots, v-1\}$ and $Z_u = \{0, 1, \cdots, u-1\}$. Let $(Z_v, \circ)$ be an idempotent quasigroup. For ordered pair (p, q), $p \neq q \in Z_u$, define transitive triple set $\mathcal{D}(p,q) = \bigcup_{x \neq y \in Z_v} d_{x,y}(p,q)$ on $Z_v \times \{p, q\}$, where

$$d_{x,y}(p,q) = \{((x,p),(y,p),(x\circ y,q)),((x,p),(x\circ y,q),(y,p)),$$
$$((x\circ y,q),(x,p),(y,p))\}.$$

An idempotent quasigroup $(Z_v,\circ)$ is said to be transitive provided that $\mathcal{D}(p,q)$ can be partitioned into three sets $\mathcal{D}^1(p,q),\mathcal{D}^2(p,q)$ and $\mathcal{D}^3(p,q)$ such that

(1) three transitive triples in each $d_{x,y}(p,q)$ belong to different $\mathcal{D}^s(p,q)$, $s=1,2,3$,

(2) if $x\neq y\in Z_v$, each of ordered pairs $((x,p),(y,q))$ and $((y,q),(x,p))$ belongs to exactly one transitive triple in each $\mathcal{D}^s(p,q), s=1,2,3$.

In what follows, we call $\mathcal{D}^1(p,q),\mathcal{D}^2(p,q),\mathcal{D}^3(p,q)$ a separation of $\mathcal{D}(p,q)$ determined by $(Z_v,\circ)$. For a permutation π on Z_v, $\mathcal{D}^s(p,q)\pi$ represents the transitive triples obtained from $\mathcal{D}^s(p,q)$ by replacing (z,q) with $(z\pi,q)$ for $z\in Z_v$. It is well known that every idempotent quasigroup is transitive [10].

Lemma 5. For any $v>3$ and $v\neq 6$, if there exists an $LPDTS(v+1)$, then there exists an $LPDTS(3v+1)$.

Construction. Let $(Z_v,\circ)$ be an idempotent quasigroup with the property $x\circ y\neq y\circ x$ for any $x\neq y\in Z_v$. (The quasigrup exists for $v>3$ from [7].) Then $(Z_v,\circ)$ is transitive. For any $p\neq q\in Z_3$, let $\mathcal{D}^s(p,q)(s\in\{1,2,3\})$ be a separation of $\mathcal{D}(p,q)$ determined by $(Z_v,\circ)$. Furthermore, let L be an orthogonal array $OA(v,4)$ over Z_v, (It is well known that there is an $OA(v,4)$ for $v>3$ and $v\neq 6$ from [7]). Define

$$L_k=\{(x,y,z):(x,y,z,k)\in L\}, k\in Z_v,$$

then $|L_k| = v$ and $\{x : (x,\star,*,k)\in L\} = \{y : (\star,y,*,k)\in L\} = \{z : (\star,*,z,k)\in L\} = Z_v$. Furthermore, define three permutations as follows:

$$\alpha_k(x)=y \text{ if } (x,y,*,k)\in L;$$
$$\beta_k(x)=y \text{ if } (*,x,y,k)\in L;$$
$$\gamma_k(x)=y \text{ if } (y,*,x,k)\in L.$$

Let $\pi=(0,1,\cdots,v-1)$ be a cycle permutation on Z_v. For any $x\in Z_v$, there exists a unique pair of elements $\{y,z\} \subseteq Z_v$, such that $(x,y,z) \in L_k$. Define $LPDTS(4)$ on the set $\{\infty,(x,0),(y,1),(z,2)\}$ as $\{\mathcal{A}^s_{x,k} : 1 \le s \le 6\}$.

Let $\{(\{\infty\} \cup Z_v, \mathcal{C}^r_i) : 1 \le i \le v-1, r = 1,2,3\}$ be an $LPDTS(v+1)$, where $\infty \notin Z_v$. Set $X = \{\infty\} \cup (Z_v \times Z_3)$. Define $6v + 3(v-1) = 3(3v-1)$ transitive triple systems on X as follows:

Part 1. $6v$ transitive triple systems $(X, \mathcal{A}^s_k), 0 \le k \le v-1, 1 \le s \le 6$.

$\mathcal{A}^s_k (s = 1,2,3)$:

(1) $\bigcup_{x \in Z_v} \mathcal{A}^s_{x,k}$

(2) $\mathcal{D}^s(0,1)\alpha_k \cup \mathcal{D}^s(1,2)\beta_k \cup \mathcal{D}^s(2,0)\gamma_k$

$\mathcal{A}^{s+3}_k (s = 1,2,3)$:

(1) $\bigcup_{x \in Z_v} \mathcal{A}^{s+3}_{x,k}$

(2) $\mathcal{D}^s(1,0)\alpha_k^{-1} \cup \mathcal{D}^s(2,1)\beta_k^{-1} \cup \mathcal{D}^s(0,2)\gamma_k^{-1}$

Part 2. $3(v-1)$ transitive triple systems $(X, \mathcal{B}^r_i \cup \mathcal{E}^r_i)$, $1 \le i \le v-1, 1 \le r \le 3$.

$\mathcal{B}^r_i = \{((x,t),(y,t),(z,t)) : (x,y,z) \in \mathcal{C}^r_i, t \in Z_3\}$, if ∞ appears then t is omitted .

$\mathcal{E}^r_i = \{S^r((x,0),(y,1),(z+\xi(i),2)), S^{-r}((z+\eta(i),2),(y,1),(x,0)) : (x,y,z,*) \in L\}$. where ξ, η are permutations on Z_v (0 is the unique fixed point)

$$\xi = (0)(1,2,3,\cdots,v-1), \qquad \eta = (0)(1,v-1,v-2,\cdots,2),$$

which satisfy $\xi(i) \ne \eta(i)$ for any $\forall i \in Z_v^*$. For an ordered triple (u,v,w), we use the symbols $S^r(u,v,w)$ and $S^{-r}(u,v,w)$ to represent the cyclic shifts:

$$S^1(u,v,w) = (u,v,w), \quad S^2(u,v,w) = (v,w,u), \quad S^3(u,v,w) = (w,u,v);$$
$$S^{-1}(u,v,w) = (u,v,w), \quad S^{-2}(u,v,w) = (w,u,v), \quad S^{-3}(u,v,w) = (v,w,u).$$

Then $\{(X, \mathcal{A}^s_k) : 0 \le k \le v-1, 1 \le s \le 6\} \bigcup \{(X, \mathcal{B}^r_i \cup \mathcal{E}^r_i) : 1 \le i \le v-1, r = 1,2,3\}$ is an $LPDTS(3v+1)$.

Proof. (1) Each $(X, \mathcal{A}^s_k)$ is a $PDTS(3v+1), 0 \le k \le v-1, 1 \le s \le 6$.

First, $|\mathcal{A}^s_k| = 4v + 3v(v-1) = \frac{(3v+1)3v}{3}$, just as expected.

Second, each ordered pair P with distinct elements appears one time.

(a) $P = (\infty, (x,t)), x \in Z_v, t \in Z_3$. For given k and $x \in Z_v$, there exists $(x,y,z) \in L_k$ for $t = 0$ ($(y,x,z) \in L_k$ for $t = 1$, $(y,z,x) \in L_k$ for $t = 2$) and $\mathcal{A}^s_{x,k}$ is a $PDTS(4)$ on $\{\infty, (x,0), (y,1), (z,2)\}$. So, P appears in $\mathcal{A}^s_{x,k} \subseteq \mathcal{A}^s_k$.

(b) $P = ((x,t),(y,t)), x \neq y \in Z_v, t \in Z_3$. Since $\mathcal{D}^s(p,q)(s = 1,2,3)$ is a separation of $\mathcal{D}(p,q)$, $P \in \mathcal{D}^s(p,q)$. If $t = 0, P \in \mathcal{D}^s(0,1)\alpha_k$ (s=1,2,3) or $P \in \mathcal{D}^{s-3}(0,2)\gamma_k^{-1}$ (s=4,5,6). If $t = 1, P \in \mathcal{D}^s(1,2)\beta_k$ (s=1,2,3) or $P \in \mathcal{D}^{s-3}(1,0)\alpha_k^{-1}$ (s=4,5,6). If $t = 2, P \in \mathcal{D}^s(2,0)\gamma_k$ (s=1,2,3) or $P \in \mathcal{D}^{s-3}(2,1)\beta_k^{-1}$ (s=4,5,6).

(c) $P = ((x,m),(y,n)), x \neq y \in Z_v, m \neq n \in Z_3$. We only prove the case $m = 0, n = 1$, the other cases are similar.

1) When $s \in \{1,2,3\}$, for the given $k \in \{0,1,\cdots,v-1\}$, there is $u \in Z_v$ such that $(u,y,*) \in L_k$. Let $u = h \circ x, h \in Z_v$. If $h = x$, i.e., $h = x = u$, then $(x,y,*) \in L_k$, P appears in Part 1. If $h \neq x$, then $(h \circ x, y, *) \in L_k$, P appears in $\mathcal{D}^s(0,1)\alpha_k$ (which is defined on $\{(h,0),(x,0),((h\circ x)\alpha_k = y,1)\}$).

2) When $s \in \{4,5,6\}$, for given $k \in \{0,1,\cdots,v-1\}$, there is $u \in Z_v$ such that $(x,u,*) \in L_k$. Let $u = y \circ h, h \in Z_v$. If $y = h$, i.e., $h = y = u$, then $(x,y,*) \in L_k$, P appears in Part 1. If $h \neq y$, then $(x, y \circ h, *) \in L_k$, P appears in $\mathcal{D}^s(0,1)\alpha_k^{-1}$ (which is defined on $\{(y,1),(h,1),((y\circ h)\alpha_k^{-1} = x,0)\}$).

At last, $\mathcal{A}^s_k$ is pure. Since $\mathcal{A}^s_{x,k}$ is a $PDTS(4)$ and $(Z_v, \circ)$ is an idempotent quasigroup with the property $x \circ y \neq y \circ x$ for any $x \neq y \in Z_v$.

(2) Each $(X, \mathcal{B}^r_i \cup \mathcal{E}^r_i)$ is a $PDTS(3v+1), 1 \leq i \leq v-1, r \in \{1,2,3\}$.

$|\mathcal{B}^r_i \cup \mathcal{E}^r_i| = 3\frac{v(v+1)}{3} + 2v^2 = \frac{(3v+1)3v}{3}$, just as expected.

Each ordered pair P with distinct elements appears in one block of $\mathcal{B}^r_i \cup \mathcal{E}^r_i$.

(a) $P = (\infty, (x,t)), ((x,t),\infty)$ or $((x,t),(y,t))$, $t \in Z_3, x \neq y \in Z_v$. It is easy to see that P is contained in $\mathcal{B}^r_i$.

(*b*) $P = ((x,p),(y,q)), x,y \in Z_v, p \neq q \in Z_3$. Here, we only consider the case $p = 0, q = 2$, the other cases are similar.

For given i, there exist $y' \in Z_v$ and $z \in Z_v$ such that $y = y' + \xi(i)$ and $(x,z,y',*) \in L$, so P appears in $\mathcal{C}_i^r$.

Finally, since $\mathcal{E}_i^r$ is pure and $\xi(i) \neq \eta(i)(1 \leq i \leq v-1)$, $\mathcal{B}_i^r \cup \mathcal{E}_i^r$ is pure.

(3) $\mathcal{A}_k^s$ ($0 \leq k \leq v-1, 1 \leq s \leq 6$) and $(X, \mathcal{B}_i^r \cup \mathcal{E}_i^r)$ $(1 \leq i \leq v-1, 1 \leq r \leq 3)$ form the large set. Here, we only indicate that every transitive triple T from X belongs to $\mathcal{A}_k^s$ or $(X, \mathcal{B}_i^r \cup \mathcal{E}_i^r)$.

(*a*) $T = (\infty,(x,t),(y,t)), x \neq y \in Z_v, t \in Z_3$. Since $\mathcal{C}_i^r(1 \leq i \leq v-1, r = 1,2,3)$ is an $LPDTS(v+1)$ on $\{\infty\} \cup Z_v$, there are i and r such that $(\infty,x,y) \in \mathcal{C}_i^r$. Similarly, for $T = ((x,t),\infty,(y,t))$, $((x,t),(y,t),\infty)$.

(*b*) $T = (\infty,(x,p),(y,q)), x,y \in Z_v, p \neq q \in Z_3$. Since L is an $OA(v,4)$ on Z_v, there is $(u_0,u_1,u_2,k) \in L$, i.e., $(u_0,u_1,u_2) \in L_k$, such that $u_p = x, u_q = y$. Further, $\mathcal{A}_{z,k}^s(1 \leq s \leq 6)$ form an $LDTS(4)$ on $\{\infty,(u_0,0),(u_1,1),(u_2,2)\}$, there is an s such that $T \in \mathcal{A}_{z,k}^s \subseteq \mathcal{A}_k^s$. Similarly, for $T = ((x,p),\infty,(y,q))$, $((x,p),(y,q),\infty)$.

(*c*) $T = ((x,t),(y,t),(z,t))$, $x,y,z \in Z_v$ and $x \neq y \neq z, t \in Z_3$. Similar to the case (*a*), there are i and r such that $(x,y,z) \in \mathcal{C}_i^r$.

(*d*) $T = ((x,p),(y,p),(z,q)), x \neq y, z \in Z_v, p \neq q \in Z_3$. Let $x \circ y = z'$, then $((x,p),(y,p),(z',q)) \in d_{x,y}(p,q) \subseteq \mathcal{D}^s(p,q), 1 \leq s \leq 3$. For example, when $p = 0, q = 1$, there is a k such that $\alpha_k(z') = z$ and $T \in \mathcal{D}^s(0,1)\alpha_k \subseteq \mathcal{A}_k^s$. The other cases are similar to prove. Similarly, for $T = ((x,p),(z,q),(y,p)), ((z,q),(x,p),(y,p))$.

(*e*) $T = ((x,p),(y,q),(z,l)), x,y,z \in Z_v, \{p,q,l\} = Z_3$. We consider the case $p = 0, q = 1, l = 2$, the other cases are similar. Let $(x,y,\overline{z},k) \in L$. If $\overline{z} = z$ then T arrears in $LPDTS(4)$ on $\{\infty,(x,0),(y,1),(z,2)\}$, $T \in \bigcup_{1 \leq s \leq 6} \mathcal{A}_{x,k}^s \subseteq \mathcal{A}_k^s$. If $\overline{z} \neq z$, then there is an $i \in Z_v^*$ such that $z = \overline{z} + \xi(i)$, $T \in \mathcal{C}_i^1$. □

Theorem 2. There exists an $LPDTS(2^n + 2)$ for $n \geq 1$.

Proof. We use induction on n. There exists an $LPDTS(2^n+2)$ for $n \in \{1,2,3,4,5\}$($n=1,2$, see Example 1,2; $n=3,4$, see Theorem 1; $n=5$, by Theorem 1 and Lemma 5). Suppose that it is true for $n \le n_0$, where $n_0 \ge 5$ is an integer. By Lemma 1 (ii), there exists a $GLS(2,(3,\{3,4\},\{3\},\{2^{n_0-2}+2\}),2^{n_0}+2)$. Example 3 and Example 4 present the constructions of an $LPHDTS(2^3)$ and an $LPHDTS(2^4)$ respectively. Hence an $LPHDTS$ $(2^{n_0+1}+2;\{H_{ik}: 1 \le i \le 2^{n_0+1}, k \in \{1,2,3\}\})$ exists by Theorem 4, where $|H_{ik}| = 2^{n_0-1}+2$, $1 \le i \le 2^{n_0+1}, k \in \{1,2,3\}$. It follows that an $LPDTS(2^{n_0+1}+2)$ exists by Corollary 2. Thus the theorem is true. □

3 The $nm+2$ construction

In what follows, we discuss the recursive construction: $n+2 \longrightarrow nm+2$, where $m>3$.

Let $n \equiv 1,5 \pmod 6$, $Z_n = \{0,1,\cdots,n-1\}$, and $Z_m = \{0,1,\cdots,m-1\}$. Given $i \in Z_n$, we define a binary operation $\circ_i$ on Z_n : $p \circ_i q = z$ if and only if $p+q+r \equiv 3i \pmod n$. Then $(Z_n, \circ_i)$ is a quasigroup with the following properties:

(1) It has exactly one idempotent element, i.e., i;

(2) $\{x \circ_i x : \text{all } x \ne i\} = Z_n \backslash \{i\}$.

It is not difficult to find that property (2) induces a permutation: $p \mapsto f_i(p) \equiv 3i-2p \pmod n$ on $Z_n\backslash\{i\}$, which partitions $Z_n\backslash\{i\}$ into pairwise disjoint cycles $(p, f_i(p), f_i^2(p), \cdots)$s. From property (1) and $p \ne i$, we know $p \ne f_i(p), p \ne f_i^2(p) \equiv 4p-3i \pmod n), p \ne f_i^3(p) \equiv 9i-8p \pmod n$. So the length of each cycle is at least 4.

For any $a \ne b \notin Z_m \times Z_n, x \in Z_m$ and $p \in Z_n\backslash\{i\}$, we define some transitive triple sets on $(\{x\} \times (Z_n\backslash\{i\})) \cup \{a,b\}$ as follows:

$$A(x,p) = \{(a,(x,p),(x,f_i(p))),(b,(x,f_i(p)),(x,p))\},$$
$$B(x,p) = \{((x,f_i(p)),a,(x,p)),((x,p),b,(x,f_i(p)))\},$$
$$E(x,p) = \{((x,p),(x,f_i(p)),a),((x,f_i(p)),(x,p),b)\}.$$

where $f_i(p) = p \circ_i p$, i.e., $p + p + f_i(p) \equiv 3i \pmod{n}$.

Now, we consider all the cycles. Let $C = (p, f_i(p), f_i^2(p), \cdots, f_i^{t-1}(p))$ be a cycle of length t. Consequently, we define some transitive triple sets $\mathcal{A}_i^r(x, C)(r = 1, 2, 3, x \in Z_m)$:

(1) If t is even, let $t = 2s, s \geq 1$. Define

$$\mathcal{A}_i^1(x, C) = (\bigcup_{k=0}^{s-1} A(x, f_i^{2k}(p))) \bigcup (\bigcup_{k=1}^{s} E(x, f_i^{2k-1}(p)));$$

$$\mathcal{A}_i^2(x, C) = \bigcup_{k=1}^{2s} B(x, f_i^{2k-1}(p));$$

$$\mathcal{A}_i^3(x, C) = (\bigcup_{k=1}^{s} A(x, f_i^{2k-1}(p))) \bigcup (\bigcup_{k=0}^{s-1} E(x, f_i^{2k}(p))).$$

(2) If t is odd, we consider two cases:

(a) $t = 5$, define

$$\mathcal{A}_i^1(x, C) = A(x, p) \cup E(x, f_i(p)) \cup B(x, f_i^2(p)) \cup A(x, f_i^3(p)) \cup E(x, f_i^4(p));$$

$$\mathcal{A}_i^2(x, C) = B(x, p) \cup B(x, f_i(p)) \cup A(x, f_i^2(p)) \cup E(x, f_i^3(p)) \cup B(x, f_i^4(p));$$

$$\mathcal{A}_i^3(x, C) = E(x, p) \cup A(x, f_i(p)) \cup E(x, f_i^2(p)) \cup B(x, f_i^3(p)) \cup A(x, f_i^4(p)).$$

For convenience, we denote the above sets as U, V, T respectively in the following case $t \geq 7$.

(b) $t = 2s + 5, s \geq 1$, define

$$\mathcal{A}_i^1(x, C) = U \bigcup (\bigcup_{k=3}^{s+2} A(x, f_i^{2k-1}(p))) \bigcup (\bigcup_{k=3}^{s+2} E(x, f_i^{2k}(p)));$$

$$\mathcal{A}_i^2(x, C) = V \bigcup (\bigcup_{k=6}^{2s+5} B(x, f_i^{k-1}(p)));$$

$$\mathcal{A}_i^3(x, C) = T \bigcup (\bigcup_{k=3}^{s+2} A(x, f_i^{2k}(p))) \bigcup (\bigcup_{k=3}^{s+2} E(x, f_i^{2k-1}(p))).$$

Obviously, $\mathcal{A}_i^1(x, C), \mathcal{A}_i^2(x, C), \mathcal{A}_i^3(x, C)$ are pairwise disjoint. For each element α of cycle C, each of the ordered pairs $(a, (x, \alpha))$, $(b, (x, \alpha))$, $((x, \alpha), a)$, $((x, \alpha), b)$, $((x, \alpha), (x, f_i(\alpha)))$ and $((x, f_i(\alpha)), (x, \alpha))$ is contained in exactly one transitive triple of $\mathcal{A}_i^r(x, C)(r = 1, 2, 3)$, but also

$$\bigcup_{r=1}^{3} \mathcal{A}_i^r(x, C) = \bigcup_{k=0}^{t-1} \{A(x, f_i^k(p)) \cup B(x, f_i^k(p)) \cup E(x, f_i^k(p))\}.$$

For convenience, we call $\mathcal{A}_i^r(x, C)$ $(r = 1, 2, 3)$ a separation of cycle $C = (p, f_i(p), f_i^2(p), \cdots, f_i^{t-1}(p))$.

Let $\Gamma = \{C(1), C(2), \cdots, C(h)\}$ be the cycle partition determined by f_i

from $Z_n\backslash\{i\}$, and $\mathcal{A}_i^r(x, C(l))$ $(r = 1,2,3)$ be a separation of $C(l)$. Define

$$\mathcal{B}_i^r(x) = \bigcup_{l=1}^{h} \mathcal{A}_i^r(x, C(l)),\ r = 1,2,3.$$

Then $\mathcal{B}_i^1(x), \mathcal{B}_i^2(x), \mathcal{B}_i^3(x)$ are pairwise disjoint. Moreover, for each element $p \in Z_n\backslash\{i\}$, the ordered pairs $(a,(x,p)), (b,(x,p)), ((x,p),a), ((x,p),b), ((x, p),(x,f_i(p))), ((x,f_i(p)),(x,p))$ are contained in exactly one transitive triple of $\mathcal{B}_i^r(x)$ respectively. In what follows, we call $\mathcal{B}_i^1(x), \mathcal{B}_i^2(x), \mathcal{B}_i^3(x)$ a separation of the set $(\{x\} \times (Z_n\backslash\{i\})) \cup \{a,b\}$, $i \in Z_n, x \in Z_m$. It is worthy noting that every transitive triple T of $\mathcal{B}_i^r(x)$ contains three elements a (or b), (x,p) and $(x,f_i(p))$.

Let $\pi = (0,1,\cdots,m-1)$ be a cyclic permutation on Z_m. For $j \in Z_m$, $T\pi^j$ represents the transitive triple obtained from transitive triple $T \in \mathcal{B}_i^r(x)$ by replacing $(x,f_i(p))$ with $(\pi^j(x), f_i(p))$. Define

$$\mathcal{B}_i^r(x)\pi^j = \{T\pi^j : T \in \mathcal{B}_i^r(x)\},$$
$$\mathcal{B}_i^r\pi^j = \bigcup_{x\in Z_m} \mathcal{B}_i^r(x)\pi^j.$$

On the other side, the transitive triple R of $D(p,q)$ or $D^r(p,q)(r = 1,2,3)$ (which are defined as section two), contains three elements (x,p), (y,p) and $(x \circ y, q)$, where $q = f_i(p)$. Similar to the variation $T \longrightarrow T\pi^j$, we can define $R\pi^j$ and

$$D^r(p,f_i(p))\pi^j = \{R\pi^j : R \in D^r(p,f_i(p))\},$$
$$\mathcal{D}_i^r\pi^j = \bigcup_{p\in Z_n\backslash\{i\}} \mathcal{D}^r(p,f_i(p))\pi^j.$$

Lemma 6. For any $m > 3$ and $n \equiv 1,5 \pmod 6$, if there exists an $LPDTS(m+2)$, then there exists an $LPDTS(nm+2)$.

Construction. Let $(Z_m, \circ)$ be an idempotent quasigroup with the property $x \circ y \neq y \circ x$ for any $x \neq y \in Z_m$. $\pi = (0,1,\cdots,m-1)$ be an m-cyclic permutation on Z_m. Define a quasigroup $(Z_n, \circ_i)$ as above for $i \in Z_n$. Let $\{(\{a,b\} \cup Z_m, \mathcal{A}_j^r); j \in Z_m, r = 1,2,3\}$ be an $LPDTS(m+2)$. Let $X = \{a,b\} \cup (Z_m \times Z_n)$, define $3nm$ transitive triple systems on X as follows:

$$\Omega^r_{i,j} = \mathcal{A}^r_{ij} \cup \mathcal{B}^r_i\pi^j \cup \mathcal{D}^r_i\pi^j \cup \mathcal{C}^r_{ij}, \quad i \in Z_n, j \in Z_m, r = 1,2,3.$$

Where

$\mathcal{A}^r_{ij} = \{((x,i),(y,i),(z,i)) : (x,y,z) \in \mathcal{A}^r_j\}$, if a, b appear then i is omitted .

$\mathcal{B}^r_i\pi^j, \mathcal{D}^r_i\pi^j$ are as above.

$$\mathcal{C}^r_{ij} = \{S^r((x,u),(y,v),(\pi^{j+1}(xoy),w)), S^{-r}((\pi^j(xoy),w),(y,v),(x,u)) : \\ 0 \le u < v < w \le n-1, u+v+w \equiv 3i \pmod n),\ x,y \in Z_m\}.$$

where the symbols S^r, S^{-r} see also Lemma 5. Then

$$\{(X, \Omega^r_{i,j}) : i \in Z_n, j \in Z_m, r = 1,2,3\}$$

is an $LPDTS(nm+2)$.

Proof. (1) $\Omega^r_{i,j}$ is a $PDTS(nm+2), i \in Z_n, j \in Z_m, r = 1,2,3$.

First, $|\Omega^r_{i,j}| = \frac{(m+2)(m+1)}{3} + (n-1)\cdot m(m-1) + (n-1)\cdot 2m + \frac{(n-1)(n-2)}{3}\cdot m^2$
$= \frac{(nm+2)(nm+1)}{3}$, just as expected.

Second, each ordered pair P of X is contained in $\Omega^r_{i,j}$.

(a) $P = (a,b),(b,a)$ are contained in $\mathcal{A}^r_{ij}$.

(b) $P = (a,(x,p)), x \in Z_m, p \in Z_n$. If $p = i$, then P appears in $\mathcal{A}^r_{ij}$; if $p \ne i$, then there exists a block $T \in \mathcal{B}^r_i\pi^j$ defined on $\{a,(x,p),(\pi^j(x), f_i(p))\}$ containing P. Similarly, for $P = ((x,p),a)$, $(b,(x,p))$ and $((x,p),b)$.

(c) $P = ((x,p),(y,p)), x \ne y \in Z_m, p \in Z_n$. If $p = i$ then P appears in $\mathcal{A}^r_{ij}$; if $p \ne i$, then there exists a block $T \in \mathcal{D}^r_i\pi^j$ defined on $\{(x,p),(y,p),(\pi^j(x \circ y), f_i(p))\}$ which contains P.

(d) $P = ((x,p),(y,q)), x,y \in Z_m, p \ne q \in Z_n$. When $q = f_i(p)$, if $y = \pi^j(x)$ then P appears in $\mathcal{B}^r_i\pi^j$; if there exists $z \ne x \in Z_m$ such that $y = \pi^j(z \circ x)$, then P appears in $\mathcal{D}^r_i\pi^j$. When $p = f_i(q)$, if $x = \pi^j(y)$, then P appears in $\mathcal{B}^r_i\pi^j$; if there exists $z \ne y \in Z_m$ such that $x = \pi^j(z \circ y)$, then P appears in $\mathcal{D}^r_i\pi^j$. If there exists $l \in Z_n \backslash \{p,q\}$ such that $p+q+l \equiv 3i \pmod n$, then P appears in $\mathcal{C}^r_{ij}$.

At last, by the construction, $\mathcal{B}^r_i\pi^j$ is a pure transitive triple set. From $\mathcal{A}^r_j$ is pure, $(Z_m, \circ)$ has property $xoy \ne yox$ for any $x \ne y \in Z_m$ and $\pi^{j+1} \ne$

π^j, we know $\mathcal{A}^r_{ij}, \mathcal{D}^r_i\pi^j$ $\mathcal{C}^r_{ij}$ are pure. On the other hand, the structures of $\mathcal{B}^r_i\pi^j$, $\mathcal{A}^r_{ij}, \mathcal{D}^r_i\pi^j$ and $\mathcal{C}^r_{ij}$ are different. So, $\Omega^r_{i,j}$ is a $PDTS(nm+2)$.

(3)All transitive triples T from X are partitioned into $\Omega^r_{i,j}, i \in Z_n, j \in Z_m, r = 1,2,3$.

(a) $T = (a, b, (x, i)), x \in Z_m, i \in Z_n$. Since $\{(\{a, b\} \cup Z_m, \mathcal{A}^r_j); j \in Z_m, r = 1,2,3\}$ is an $LPDTS(m+2)$, there exist $j \in Z_m$ and $r \in \{1,2,3\}$ such that $(a, b, x) \in \mathcal{A}^r_j$. So $T \in \mathcal{A}^r_{ij} \subseteq \Omega^r_{i,j}$. Similarly, for $T = (a, (x, i), b), ((x, i), a, b), (b, a, (x, i)), (b, (x, i), a), ((x, i), b, a)$.

(b) $T = (a, (x, i), (y, i)), x \neq y \in Z_m, i \in Z_n$. Similar to (a), there exist $j \in Z_m$ and $r \in \{1,2,3\}$ such that $(a, x, y) \in \mathcal{A}^r_j$, $T \in \Omega^r_{i,j}$. Similarly, for $T = ((x, i), a, (y, i)), ((x, i), (y, i), a), (b, (x, i), (y, i)), ((x, i), b, (y, i)), ((x, i), (y, i), b)$.

(c) $T = (a, (x, p), (y, q)), x, y \in Z_m, p \neq q \in Z_n$. For given p, q, there exists a unique element $i \in Z_n$ such that $q = f_i(p)$, there exists $j \in Z_m$ such that $y = \pi^j(x)$. By the definition of $\mathcal{B}^r_i(x)\pi^j \subseteq \mathcal{B}^r_i\pi^j$, there exists a unique $r \in \{1,2,3\}$ such that $T \in \mathcal{B}^r_i(x)\pi^j \subseteq \mathcal{B}^r_i\pi^j \subseteq \Omega^r_{i,j}$. Similarly, for $T = (b, (x, p), (y, q)), ((x, p), a, (y, q)), ((x, p), b, (y, q)), ((x, p), (y, q), a), ((x, p), (y, q), b)$.

(d) $T = ((x, p), (y, p), (z, p)), x \neq y \neq z \in Z_m, p \in Z_n$. There exist $j \in Z_m$ and $r \in \{1,2,3\}$ such that $(x, y, z) \in \mathcal{A}^r_j$, $T \in \mathcal{A}^r_{p,j} \subseteq \Omega^r_{p,j}$.

(e) $T = ((x, p), (y, p), (z, q)), p \neq q \in Z_n, x \neq y, z \in Z_m$. There exists a unique element $i \in Z_n$ such that $q = f_i(p)$, there exists a unique element $j \in Z_m$ such that $z = \pi^j(x \circ y)$. By the definition of $\mathcal{D}^r_i\pi^j$, there exists a unique $r \in \{1,2,3\}$ such that $T \in \mathcal{D}^r_i\pi^j \subseteq \Omega^r_{i,j}$. Similarly, for $T = ((x, p), (z, q), (y, p)), ((z, q), (x, p), (y, p))$.

(f) $T = ((x, p), (y, q), (z, l)), p \neq q \neq l \in Z_n, x, y, z \in Z_m$. There exists a unique element $i \in Z_n$ such that $p + q + l \equiv 3i \pmod n$,

i) if $p < q < l$, then there exists $j \in Z_m$ such that $z = \pi^{j+1}(x \circ y), T \in \mathcal{C}^1_{ij} \subseteq \Omega^1_{i,j}$;

if $p > q > l$, then there exists $j \in Z_m$, such that $x = \pi^j(z \circ y), T \in C^1_{ij} \subseteq \Omega^1_{i,j}$;

ii) if $q > p > l$, then there exists $j \in Z_m$ such that $y = \pi^{j+1}(z \circ x), T \in C^2_{ij} \subseteq \Omega^2_{i,j}$;

if $q > l > p$, then there exists $j \in Z_m$ such that $y = \pi^j(x \circ z), T \in C^2_{ij} \subseteq \Omega^2_{i,j}$;

iii) if $p > l > q$, there exists $j \in Z_m$ such that $x = \pi^{j+1}(y \circ z), T \in C^3_{ij} \subseteq \Omega^3_{i,j}$;

if $l > p > q$, there exists $j \in Z_m$ such that $z = \pi^j(y \circ x), T \in C^3_{ij} \subseteq \Omega^3_{i,j}$. □

4 The $2n + 2$ construction

Similar to the $nm + 2$ construction, let $n \equiv 1, 5 \pmod 6$, define a binary operation $\circ_i (\forall i \in Z_n)$ on Z_n, give a quasigroup $(Z_n, \circ_i)$, a permutation f_i on $Z_n \backslash \{i\}$ and the corresponding cycle partition. For any $a \neq b \notin Z_2 \times Z_n$, $p \in Z_n \backslash \{i\}$, define some transitive triple sets as follows($j \in Z_2$):

$$A_j(p) = \{(a, (t,p), (t+j, f_i(p))),\ (b, (t+1,p), (t+j, f_i(p))),\\ ((t+j, f_i(p)), (t,p), (t+1,p)) :\ t \in Z_2\};$$

$$B_j(p) = \{((t+j, f_i(p)), a, (t,p)),\ ((t+j, f_i(p)), b, (t+1,p)),\\ ((t,p), (t+1,p), (t+j, f_i(p))) :\ t \in Z_2\};$$

$$E_j(p) = \{((t,p), (t+j, f_i(p)), a)),\ ((t+j, f_i(p)), (t+1,p), b),\\ ((t+1,p), (t+j, f_i(p)), (t,p)) :\ t \in Z_2\}.$$

Let $C = (p, f_i(p), f_i^2(p), \cdots, f_i^{s-1}(p))$ be a cycle satisfying the conditions above with length s. Define three collections of transitive triples as follows:

(1) if t is even

$$\mathcal{A}^1_j(C) = A_j(p) \cup E_j(f_i(p)) \cup A_j(f_i^2(p)) \cup E_j(f_i^3(p)) \cup \cdots \cup E_j(f_i^{s-1}(p));$$

(alternate between $A_j(f_i(p))$ and $E_j(f_i(p))$)

$$\mathcal{A}^2_j(C) = \bigcup_{k=0}^{s-1} B_j(f_i^k(p));$$

$$\mathcal{A}^3_j(C) = E_j(p) \cup A_j(f_i(p)) \cup E_j(f_i^2(p)) \cup A_j(f_i^3(p)) \cup \cdots \cup A_j(f_i^{s-1}(p));$$

(alternate between $E_j(f_i(p))$ and $A_j(f_i(p))$)

(2) if t is odd. We consider two cases:

(a) $t = 5$.

$\mathcal{A}_j^1(C) = A_j(p) \cup E_j(f_i(p)) \cup B_j(f_i^2(p)) \cup A_j(f_i^3(p)) \cup E_j(f_i^4(p))$;

$\mathcal{A}_j^2(C) = B_j(p) \cup B_j(f_i(p)) \cup A_j(f_i^2(p)) \cup E_j(f_i^3(p)) \cup B_j(f_i^4(p))$;

$\mathcal{A}_j^3(C) = E_j(p) \cup A_j(f_i(p)) \cup E_j(f_i^2(p)) \cup B_j(f_i^3(p)) \cup A_j(f_i^4(p))$.

For convenience, we denote the above sets as U, V, T respectively in the following case $t \geq 7$.

(b) $t \geq 7$,

$\mathcal{A}_j^1(C) = U \cup A_j(f_i^5(p)) \cup E_j(f_i^6(p)) \cup \cdots \cup A_j(f_i^{s-2}(p)) \cup E_j(f_i^{s-1}(p))$;

$\mathcal{A}_j^2(C) = V \bigcup (\bigcup_{k=5}^{s-1} B_j(f_i^k(p))$;

$\mathcal{A}_j^3(C) = T \cup E_j(f_i^5(p)) \cup A_j(f_i^6(p)) \cup \cdots \cup E_j(f_i^{s-2}(p)) \cup A_j(f_i^{s-1}(p))$.

Obviously, $\mathcal{A}_j^1(C)$, $\mathcal{A}_j^2(C), \mathcal{A}_j^3(C)$ are pairwise disjoint. We call $\mathcal{A}_j^1(C)$, $\mathcal{A}_j^2(C), \mathcal{A}_j^3(C)$ a separation of the cycle C. Let $\Gamma = \{C(1), C(2), \cdots, C(h)\}$ be the cycle partition determined by f_i from $Z_n \backslash \{i\}$, and $\mathcal{A}_j^r(C(l))(r = 1, 2, 3)$ be a separation of the cycle $C(l)$. Define

$$\mathcal{B}_{ij}^r = \bigcup_{l=1}^{h} \mathcal{A}_j^r(C(l)), \quad i \in Z_n, j \in Z_2, r = 1, 2, 3.$$

Theorem 3. There exists an $LPDTS(2n+2)$ for $n \equiv 1, 5 \pmod 6$.

Construction: For $i \in Z_n$, let $\{(\{a, b\} \cup (Z_2 \times \{i\}), \mathcal{A}_{ij}^r) : j \in Z_2, r = 1, 2, 3\}$ be an $LPDTS(4)$. For $j \in Z_2, r = 1, 2, 3$, define the quasigroup $(Z_n, \circ_i)$ and the transitive triple collections $\mathcal{B}_{ij}^r$ as above. Furthermore, define transitive triple collection $\mathcal{C}_{ij}^r$ as follows:

$$\mathcal{C}_{ij}^r = \{S^r((x, u), (y, v), (z, w)), S^{-r}((z', w), (y', v), (x', u)) : 0 \leq u < v < w \leq n-1, u+v+w \equiv 3i \pmod n), (x, y, z) \in T_0, (x', y', z') \in T_1\}.$$

where the symbols S^r, S^{-r} see also Lemma 5, and

$T_0 = \{(0,0,0), (0,1,1), (1,0,1), (1,1,0)\}$;

$T_1 = \{(1,1,1), (0,0,1), (0,1,0), (1,0,0)\}$.

Let $X = \{a, b\} \cup (Z_2 \times Z_n)$, define

$$\Omega_{i,j}^r = \mathcal{A}_{ij}^r \cup \mathcal{B}_{ij}^r \cup \mathcal{C}_{ij}^r, \quad i \in Z_n, j \in Z_2, r = 1, 2, 3.$$

Then $\{(X, \Omega^r_{i,j}) : i \in Z_n, j \in Z_2, r = 1,2,3\}$ is an $LPDTS(2n+2)$.

Proof: (1) $\Omega^r_{i,j}$ is a $PDTS(2n+2), i \in Z_n, j \in Z_2, r = 1,2,3$.

First, $|\Omega^r_{i,j}| = 4 + 3 \cdot 2 \cdot (n-1) + 4 \cdot \frac{(n-1)(n-2)}{3} = \frac{(2n+2)(2n+1)}{3}$, just as expected.

Second, each ordered pair P of X is contained in one triple of $\Omega^r_{i,j}$.

(a) If $P = (a,b), (b,a), ((x,i),a), ((x,i),b), (a,(x,i))$ or $(b,(x,i))$, where $x \in Z_2$, then it appears in $\mathcal{A}^r_{ij}$.

(b) $P = (a,(x,p)), x \in Z_2, p \in Z_n \backslash \{i\}$. Since Γ is the cycle partition of $Z_n \backslash \{i\}$, there exists a unique cycle $C(l)$ containing p, so P appears in $\mathcal{A}^r_j(C(l)) \subseteq \mathcal{B}^r_{ij} \subseteq \Omega^r_{i,j}$. Similarly, for $P = ((x,p),a), (b,(x,p)), ((x,p),b)$, where $p \in Z_n \backslash \{i\}$.

(c) $P = ((x,p),(x+1,p)), x \in Z_2,\ p \in Z_n$. If $p = i$, then P appears in $\mathcal{A}^r_{ij}$. If $p \neq i$, P appears in $\mathcal{B}^r_{ij} \subseteq \Omega^r_{i,j}$.

(d) $P = ((x,p),(y,q)), x,y \in Z_2, p \neq q \in Z_n$. If $p = f_i(q)$ or $q = f_i(p)$, then there exists a unique cycle $C(l) \in \Gamma$ containing p and q. So, P appears in $\mathcal{A}^r_j(C(l)) \subseteq \mathcal{B}^r_{ij} \subseteq \Omega^r_{i,j}$. If there exists $s \in Z_n$ such that $p + q + s \equiv 3i \pmod{n}$, then P appears in $\mathcal{C}^r_{ij}$.

Finally, it is easy to see that $\mathcal{A}^r_{ij}, \mathcal{B}^r_{ij}$ are pure. Moreover, $(x,y,z) \in T_0, (x',y',z') \in T_1$, $\mathcal{C}^r_{ij}$ is pure. On the other side, the structures of $\mathcal{A}^r_{ij}, \mathcal{B}^r_{ij}, \mathcal{C}^r_{ij}$ are different, so $\Omega^r_{i,j}$ is a pure $DTS(2n+2)$.

(2) All transitive triples T from X are partitioned into $\Omega^r_{i,j}, i \in Z_n, j \in Z_2, r = 1,2,3$.

(a) Foe any $i \in Z_n$, since $\{\mathcal{A}^r_{ij}; j \in Z_2, r = 1,2,3\}$ is an $LPDTS(4)$ on $W_i = \{a,b\} \cup (Z_2 \times \{i\})$, the transitive triples of W_i appear in $\mathcal{A}^r_{ij} \subseteq \Omega^r_{i,j}$.

(b) $T = (a,(x,p),(y,q)), p \neq q \in Z_n, x,y \in Z_2$. From $3 \dagger n$, there exists a unique $i \in Z_n$ such that $q = f_i(p)$. By the definition of Γ, there exists a unique cycle $C(l) \in \Gamma$ containing p and q. So, there is $j \in Z_2$ such that $T \in \mathcal{A}^r_j(C(l)) \subseteq \mathcal{B}^r_{ij} \subseteq \Omega^r_{i,j}$. Similarly, for $T = ((x,p),a,(y,q)), ((x,p),(y,q),a), (b,(x,p),(y,q)), ((x,p),b,(y,q)), ((x,p),(y,q),b)$.

(c) $T = ((x,p),(x+1,p),(y,q)), p \neq q \in Z_n, x,y \in Z_2$. There exists $i \in Z_n$ such that $q = f_i(p)$. In addition, there exists a unique cycle $C(l) \in \Gamma$ containing p and q. Hence, by the definition of $\mathcal{B}^r_{ij}$, there exist r and j such that $T \in \mathcal{B}^r_{ij} \subseteq \Omega^r_{i,j}$. Similarly, for $T = ((x,p),(y,q),(x+1,p)), ((y,q),(x,p),(x+1,p))$.

(d) $T = ((x,p),(y,q),(z,s)), p \neq q \neq s \in Z_n, x,y,z \in Z_2$. There exists a unique $i \in Z_n$ such that $p + q + s \equiv 3i \pmod{n}$. In fact, $\mathcal{C}^r_{ij}, r = 1,2,3$, are a partition of the whole transitive triples on the set $\{(0,p),(1,p),(0,q),(1,q),(0,s),\ (1,s)\}$(the transitive triples don't contain the ordered pairs $((t,p),(t+1,p)), ((t,q),(t+1,q)), ((t,s),(t+1,s)), t \in Z_2$). So, $T \in \mathcal{C}^r_{ij} \subseteq \Omega^r_{i,j}$. □

5 Conclusion

Theorem 4. There exists an $LPDTS(v)$ for $v \equiv 0,4 \pmod 6$ and $v \geq 4$.
Proof. When $v \equiv 0,4 \pmod 6$ and $v \geq 0$, we can denote it as $v = 2^n \cdot u + 2$, where $u \equiv 1,5 \pmod 6$ and $n \geq 1$. By Theorem 2, there exists an $LPDTS(2^n + 2)$, $n \geq 1$. Furthermore, if $n = 1$, i.e., $v = 2u + 2 \equiv 0,4 \pmod{12}$, there exists an $LPDTS(v)$ by theorem 3. If $n > 1$, then an $LPDTS(v)$ exists by Lemma 6. So there exists an $LPDTS(v)$ for $v \equiv 0,4 \pmod 6$ and $v \geq 4$. □

Note: By using overlarge sets of disjoint pure $DTS(v)$ as auxiliary design, we have obtained some results on $LPDTS(v)$ for odd v. Here we don't narrate any longer.

References:

[1]S. H. Y. Hung and N. S. Mendelsohn, *Directed triple systems,* J. Combin. Theory A, 14(1973), 310–318.

[2]H. Shen, *Embeddings of Pure Mendelsohn Triple system and Pure Directed Triple Systems,* J.C.D., 1995,3(1):41-50.

[3]C.C.Lindner and Anne Street, *Construction of large sets of pairwise disjoint transitive triple systems*, European J.Combin. 4(1983), 335-346.

[4]C.C.Lindner, *Construction of large sets of pairwise disjoint transitive triple systems II*, Discrete Mathematics 65(1987), 65-74.

[5]Q. Kang and Y. Chang, *A completion of the spectrum for large set of disjoint transitive triple systems*, Journal of Combinatorial Theory, Ser.A, Vol.60, No,2(1992),287-294.

[6]Kevin Phelps and Carol Yin, *Generalized Steiner systems with block size three and group size $g \equiv 3$ (mod 6)*, Journal of Combinatorial Designs, 5(1997), 417-432.

[7]F. E. Bennett, Q. Kang, J. Lei and H. Zhang, *Large sets of disjoint pure Mendelsohn triple systems,* JSPI, 95(2001), 89-115.

[8]J. Lei, *Further results on Large Set of Disjoint Pure Mendelsohn Triple Systems,* Journal of Combinatorial Designs, 8(2000), 274-290.

[9]Luc Teirlinck, *Large sets with holes*, Journal of Combin. Designs, 1(1993), 69–94.

[10]C.C.Lindner and Anne Street, *Ordered triple systems and transitive triple systems*, ARS Combin., 17A(1984), 297–306.