A family of tetravalent Frobenius graphs *

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Abstract

As a part of the author's work of enumerating the edge-forwarding indices of Frobenius graphs, I give a class of valency four Frobenius graphs derived from the Frobenius groups $\mathbb{Z}_{4n^2+1} \rtimes \mathbb{Z}_4$. Following the method of Fang, Li and Praeger, some properties including the diameter and the type of this class of graphs are given(Theorem 3.2).

Key words: Cayley graph; Frobenius graph; Edge-forwarding index
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1 Introduction

In [2], Chung et al introduced the concept of forwarding index of communication networks. In general, we use a graph to model an interconnection network which consists of hardware and/or software entities that are interconnected to facilitate efficient computation and communications [5]. Then in [6], Heydemann et al defined the edge-forwarding index $\pi(\Gamma)$ of a finite graph Γ as a measure of the maximal load carried by an edge of Γ. One may also refer to [13] for more details.

A *Frobenius group* is a transitive permutation group on a set V which is not regular on V, but has the property that the only element of G which fixes more than one point of V is the identity element of G. It was shown by Thompson [7, 8] that a finite Frobenius group G has a nilpotent normal subgroup K, called the *Frobenius kernel*, which acts regularly on V. Thus, K is the direct product of its Sylow subgroups and G is the semidirect product $K \rtimes H$, where H is the stabilizer of a point of V. Each such subgroup H is called a *Frobenius complement* of K in G. Gorenstein [4, pp. 38 and 339] showed that every element of $H \setminus \{1\}$ induces an automorphism of K by conjugation which fixes only the identity element of K. For a group-theoretic terminology not defined in this paper, we refer the reader to [4, 9].

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Given a finite group G and a generating set S of G such that $S = S^{-1}$ and $1 \notin S$, the *Cayley graph* $\Gamma = C(G, S)$ on G *relative to* S has vertex set G and edge set $\{\{g, gs\} \mid g \in G,\ s \in S\}$. For a graph-theoretic terminology not defined in this paper, we refer the reader to [1, 10]. Fang et al in [3] introduced the G-Frobenius graph Γ as a connected orbital graph of a Frobenius group $G = K \rtimes H$ with Frobenius kernel K and Frobenius complement H. They showed that almost all finite orbital regular graphs are Frobenius graphs and identified a G-Frobenius graph Γ, $G = K \rtimes H$, as a Cayley graph $\Gamma = \mathrm{Cay}(K, S)$ for K and for some Cayley subset S. For a G-Frobenius graph Γ, where $G = K \rtimes H$, we say that Γ has *type*-$(n_1, \ldots, n_d)$ if d is the diameter of Γ and, for each $i = 1, \ldots, d$, n_i is the number of H-orbits of vertices at distance i from the identity element in Γ.

In this paper, I give a class of Frobenius graphs of valency 4 arising from the Frobenius group G, $G = K \rtimes H$ where $K = \mathbb{Z}_{4n^2+1}$ and $H = \mathbb{Z}_4$. As a result, the diameter and the edge-forwarding index of each graph are given (Theorem 3.2).

2 Some known results of Frobenius graphs

Given a permutation group G on a set V, the G-action on V induces a natural action on $V \times V$ by $(x, y)^g = (x^g, y^g)$ for $(x, y) \in V \times V$ and $g \in G$. The orbits of G in the action on $V \times V$ are called *orbitals*. Note that the set $\Delta = \{(x, x) \mid x \in V\}$ is G-invariant as well as the set $\Delta^c = \{(x, y) \mid x, y \in V, x \neq y\}$. A G-orbit in Δ is called a *trivial orbital* and that in Δ^c is called a *nontrivial orbital*. Let Γ be a connected graph with vertex set V, and let $G \leq \mathrm{Aut}(\Gamma)$. Then Γ is said to be a *G-orbital regular graph* if G is regular on each of its orbitals in Δ^c, and there is a nontrivial G-orbital O such that the edge set is $E(\Gamma) = \{\{x, y\} \mid (x, y) \in O\}$. A graph Γ is *orbital regular* if it is G-orbital regular for some $G \leq \mathrm{Aut}(\Gamma)$. Fang et al. [3] introduced a Frobenius graph as follows:

Definition 2.1 Let G be a Frobenius group on a set V. A *G-Frobenius graph* is defined to be a connected graph Γ with vertex set $V(\Gamma) = V$ and edge set $E(\Gamma) = \{\{x, y\} \mid (x, y) \in O\}$ for some nontrivial G-orbital O in Δ^c.

Let $G = K \rtimes H$ be a Frobenius group on a set V and let Γ be a G-Frobenius graph. Since K is regular on the vertex set V of Γ, we may identify V with K in such a way that K acts by left multiplication.

Example 2.1 For any prime number p, the group $G = \mathbb{Z}_p \rtimes \mathbb{Z}_{p-1}$ is a Frobenius group, where $K = \mathbb{Z}_p$ and $H = \mathbb{Z}_{p-1}$. Here, the group G acts on K in such a way that K acts on itself by translation and H acts on K

by multiplication. Thus G acts regularly on each nontrivial orbital and the G-Frobenius graph is isomorphic to the complete graph K_p.

Lemma 2.1 ([3, Theorem 1.4]) *Let $G = K \rtimes H$ be a Frobenius group with Frobenius kernel K and Frobenius complement H. Then a G-Frobenius graph is a Cayley graph $C(K, S)$ for K and for some generating subset S of the form*

$$S = \begin{cases} x^H & \text{if } |H| \text{ is even or } |x| = 2, \\ x^H \cup (x^{-1})^H & \text{if } |H| \text{ is odd and } |x| \neq 2, \end{cases} \quad (1)$$

where $x \in K$ such that $\langle x^H \rangle = K$. Conversely, if $x \in K$ satisfies $\langle x^H \rangle = K$, then $C(K, S)$ is G-Frobenius with S defined in the equation (1).

Now we turn to the problem of computing the edge-forwarding indices of Frobenius graphs. The load of an edge e in a given routing R of a graph Γ is the number of paths of R going through e. We use $\pi(\Gamma, R, e)$ to denote the load of an edge e in a given routing R of a graph Γ. Heydemann et al defined *the edge-forwarding index of* (Γ, R) as $\pi(\Gamma, R) = \max_{e \in E(\Gamma)} \pi(\Gamma, R, e)$ and the edge-forwarding index $\pi(\Gamma)$ of Γ as $\pi(\Gamma) = \min_R \pi(\Gamma, R)$. As for G-Frobenius graphs, Fang et al gave the following expression for $\pi(\Gamma)$ in terms of the type of Γ.

Lemma 2.2 ([3, Theorem 1.6]) *Let $G = K \rtimes H$ be a Frobenius group and let Γ be a G-Frobenius graph of type-$(\delta_1, \delta_2, ..., \delta_d)$, then*

$$\pi(\Gamma) = \begin{cases} 2\sum_{i=1}^{d} i\delta_i & \text{if } |H| \text{ is even or } |x| = 2, \\ \sum_{i=1}^{d} i\delta_i & \text{if } |H| \text{ is odd and } |x| > 2. \end{cases} \quad (2)$$

Moreover, $|H|$ is odd and $|x| > 2$ if and only if $\delta_1 = 2$.

3 A Class of Valency-4 Frobenius Graphs

Lemma 3.1 *Let $G = K \rtimes H$ be a Frobenius group, where $K \cong \mathbb{Z}_{4n^2+1}$ and $H \cong \mathbb{Z}_4$. View K as an additive abelian group, then $S = \{\pm 1, \pm 2n\}$ is a Cayley subset of K satisfying Lemma 2.1.*

Proof: Assume $H \cong \langle \sigma \rangle$. If $1^\sigma = i$ for some integer $1 < i < 4n^2 + 1$, then $1^{\sigma^4} = i^4 \equiv 1 \pmod{4n^2 + 1}$. An easy calculation shows that $i = 2n$ fits for. Thus $S = \{\pm 1, \pm 2n\}$. □

As an example of such Frobenius graphs, we refer the reader to Figure 1, where some edges are omitted from the graph. One may refer to [11, 12] for more information on Frobenius graphs. In a graph Γ, we use $N_i(u)$ to denote the set of vertices in Γ with distance i from a vertex u and $d(u,v)$ the distance between u and v.

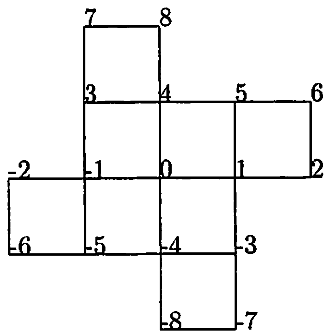

Figure 1: A tetravalent Frobenius graph for $n = 2, K = \mathbb{Z}_{17}$ and $H = \mathbb{Z}_4$

Theorem 3.2 *Let $G = K \rtimes H$ be a Frobenius group with $K \cong \mathbb{Z}_{4n^2+1}$ and $H \cong \mathbb{Z}_4$. If we choose $S = \{\pm 1, \pm(2n)\}$, then the following statements hold.*

(1) *Any vertex $k \in N_i(0)$ of the Frobenius graph $\Gamma = \mathrm{Cay}(K,S)$ can be written uniquely as $k = x + 2ny$ for some integers x, y satisfying $|x| + |y| = i$;*
(2) *the diameter of Γ is $2n-1$;*
(3) *the type of Γ is $(1, 2, ..., n-1, n, n-1, ..., 2, 1)$ and its edge-forwarding index is $2n^3$.*

Proof: Because $K \cong \mathbb{Z}_{4n^2+1}$, one can write K as $K = \{0, \pm 1, \pm 2, ..., \pm(2n^2)\}$. For any $k \in \Gamma$, to find a shortest way in Γ from 0 to k is to express k by the least number of elements in S. Notice that if we choose 1 we don't need to choose -1, and vice versa, in order to acquire the most brief expression. The case is the same for $(2n+1)$ and $-(2n+1)$. So if $k \in N_i(0)$, k can be uniquely expressed as $k = x + (2n+1)y$ for some integers x, y with $|x| + |y| = i$. On the contrary, a vertex k having an expression $k = x + (2n+1)y$ clearly satisfies $d(k,0) \leq |x| + |y|$.

One can see that $N_1(0) = S$. Let $\Lambda_i = \{(x,y) \mid |x| + |y| = i\}$ and $\Delta_i = \{x + 2ny| \ (x,y) \in \Lambda_i\}$, then $N_i(0) \subseteq \Delta_i$. So in order to count the vertices in $N_i(0)$ we need only consider the integer pairs in Λ_i. Define a function $f : \mathbb{Z} \times \mathbb{Z} \mapsto \mathbb{Z}$ which maps the integer pair (a,b) to $(a + 2nb)$. A direct calculation shows that the image set of f on Λ_n is $f(\Lambda_n) = \{\pm 2n^2, \pm(2n^2 - 2n+1), \pm(2n^2-2n-1), \ldots, \pm(3n-1), \pm(n+1), \pm n\}$. Because $\pm 2n^2 \in N_n(0)$ and when $|x|+|y| \leq n$ and $(x,y) \neq (0, \pm n)$, $|x+2ny \pmod{4n^2+1}| < 2n^2$. So, the elements we acquire by now are different from each other and there are $4i$ elements in $N_i(0)$ for $1 \leq i \leq n$.

When $i \geq 1$, $f(\Lambda_{n+i}) = \{\pm(2n^2-2ni+1), \pm(2n^2-2ni+2n), \ldots, \pm(2n^2-2n+i), \pm(2n^2-i+1), \pm(2n^2-i), \ldots, \pm(2ni+n), \pm(2ni-n), \pm(2ni-n+1), \ldots, \pm(n-i+1), \pm(n+i)\}$. But $\pm(2n^2-2ni+1) \in N_{n-i+1}(0), \pm(2n^2-2ni+2n) \in N_{n-i+1}(0), \ldots, \pm(2n^2-2n+i) \in N_{n+i-1}(0), \pm(2n^2-i+1) \in N_{n+i-1}(0), \pm(2n^2-i) \in N_{n+i}(0), \ldots, \pm(2ni+n) \in N_{n+i}(0), \pm(2ni-n) \in N_{n+i-1}(0), \pm(2ni-n+1) \in N_{n+i-1}(0), \ldots, \pm(n-i+1) \in N_{n-i+1}(0), \pm(n+i) \in N_{n-i+1}(0)$. Therefore, the number of elements in $N_{n+i}(0)$ is $4(n-i)$.

Following the preceding discussion, there are $4i$ elements in $N_i(0)$ for $1 \leq i \leq n$ and $4(n-j)$ elements in $N_{n+j}(0)$ when $j \geq 1$. However, $1 + \Sigma_{i=1}^{n} 4i + \Sigma_{j=1}^{n-1} 4(n-j) = 4n^2+1$ which shows that the diameter of Γ is $d = 2n-1$. By Lemma 2.2, the type of Γ is $(1, 2, ..., n-1, n, n-1, ..., 2, 1)$ and $\pi(\Gamma) = 2n^3$. □

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