ON THE SECOND ORDER LINEAR RECURRENCES BY GENERALIZED DOUBLY STOCHASTIC MATRICES

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ABSTRACT. In this paper, we consider the relationships between the second order linear recurrences, and the generalized doubly stochastic permanents and determinants.

1. INTRODUCTION

The Fibonacci sequence, $\{F_n\}$, is defined by the recurrence relation, for $n \geq 1$

$$F_{n+1} = F_n + F_{n-1} \tag{1.1}$$

where $F_0 = 0$, $F_1 = 1$. The Lucas Sequence, $\{L_n\}$, is defined by the recurrence relation, for $n \geq 1$

$$L_{n+1} = L_n + L_{n-1} \tag{1.2}$$

where $L_0 = 2$, $L_1 = 1$.

The well-known Fibonacci, Lucas and Pell numbers can be generalized as follows: Let A and B be nonzero, relatively prime integers such that $D = A^2 - 4B \neq 0$. Define sequences $\{u_n\}$ and $\{v_n\}$ by, for all $n \geq 2$ (see [14]),

$$u_n = Au_{n-1} - Bu_{n-2} \tag{1.3}$$

$$v_n = Av_{n-1} - Bv_{n-2} \tag{1.4}$$

where $u_0 = 0$, $u_1 = 1$ and $v_0 = 2$, $v_1 = A$. If $A = 1$ and $B = -1$, then $u_n = F_n$ (the nth Fibonacci number) and $v_n = L_n$ (the nth Lucas number). If $A = 2$ and $B = -1$, then $u_n = P_n$ (the nth Pell number).

An alternative is to let the roots of the equation $t^2 - At + B = 0$ be, for $n \geq 0$

$$u_n = \frac{\sigma^n - \gamma^n}{\sigma - \gamma} \quad \text{and} \quad v_n = \sigma^n + \gamma^n. \tag{1.5}$$

There are many connections between permanents or determinants of tridiagonal matrices and the Fibonacci and Lucas numbers. For example,

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Minc [12] define a $n \times n$ super diagonal $(0,1)$ −matrix $F(n,k)$ for $n > k \geq 2$, and show that the permanent of $F(n,k)$ equals to the generalized order-k Fibonacci numbers. Also he give some relations involving the permanents of some $(0,1)$ −Circulant matrices and the usual Fibonacci numbers.

In [8], the authors present a nice result involving the permanent of an $(-1,0,1)$-matrix and the Fibonacci Number F_{n+1}. The authors then explore similar directions involving the positive subscripted Fibonacci and Lucas Numbers as well as their uncommon negatively subscripted counterparts. Finally the authors explore the generalized order-k Lucas numbers, (see [19] and [7] for more detail the generalized Fibonacci and Lucas numbers), and their permanents.

In [9] and [10], the authors gave the relations involving the generalized Fibonacci and Lucas numbers and the permanent of the $(0,1)$ −matrices. The results of Minc, [12], and the result of Lee, [9], on the generalized Fibonacci numbers are the same because they use the same matrix. However, Lee proved the same result by a different method, contraction method for the permanent (for more detail of the contraction method see [1]).

In [11], Lehmer proves a very general result on permanents of tridiagonal matrices whose main diagonal and super-diagonal elements are ones and whose subdiagonal entries are somewhat arbitrary.

Also in [16] and [17], the authors define a family of tridiagonal matrices $M(n)$ and show that the determinants of $M(n)$ are the Fibonacci numbers F_{2n+2}. In [4] and[3], the family of tridiagonal matrices $H(n)$ and the authors show that the determinants of $H(n)$ are the Fibonacci numbers F_n. In a similar family of matrices, the $(1,1)$ element of $H(n)$ is replaced with a 3. The determinants, [2], now generate the Lucas sequence L_n.

In [5], the authors find the families of $(0,1)$ −matrices such that permanents of the matrices, equal to the sums of Fibonacci and Lucas numbers.

Recently, in [6], the authors define two tridiagonal matrices and then give the relationships the permanents and determinants of these matrices and the second order linear recurrences.

The *permanent* of an n-square matrix $A = (a_{ij})$ is defined by

$$\text{per}A = \sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i\sigma(i)}$$

where the summation extends over all permutations σ of the symmetric group S_n. Also one can find more applications of permanents in [13].

Let $A = [a_{ij}]$ be an $m \times n$ real matrix row vectors $\alpha_1, \alpha_2, \ldots, \alpha_m$. We say A is *contractible on column* (resp. *row*.) k *if column* (resp. *row*.) k contains exactly two nonzero entries. Suppose A is contractible on column k with $a_{ik} \neq 0 \neq a_{jk}$ and $i \neq j$. Then the $(m-1) \times (n-1)$ matrix $A_{ij:k}$ obtained from A by replacing row i with $a_{jk}\alpha_i + a_{ik}\alpha_j$ and deleting row

j and column k is called *the contraction of A on column k relative to rows i and j*. If A is contractible on row k with $a_{ki} \neq 0 \neq a_{kj}$ and $i \neq j$, then the matrix $A_{k:ij} = \left[A^T_{ij:k}\right]^T$ is called *the contraction of A on row k relative to columns i and j*. Every contraction used in this paper will be on the first column using the first and second rows. We say that A can be contracted to a matrix B if either $B = A$ or exist matrices $A_0, A_1, \ldots A_t$ $(t \geq 1)$ such that $A_0 = A$, $A_t = B$, and A_r is a contraction of A_{r-1} for $r = 1, 2, \ldots, t$. One can find the following fact in [1]: let A be a nonnegative integral matrix of order $n > 1$ and let B be a contraction of A. Then

$$per A = per B. \tag{1.6}$$

We also recall the following definitions :

Definition 1. *A matrix $A = (a_{ij})$ of order n is said to be nonnegative if $a_{ij} \geq 0$, $i, j = 1, 2, \ldots, n$.*

Definition 2. *A nonnegative $n \times n$ matrix A is called row stochastic, or simply stochastic, if all its rows sum 1.*

Definition 3. *A nonnegative $n \times n$ matrix A is called row stochastic, if all its rows and colums sum 1.*

We give the following definitions (see [15] and [18], respectively).

Definition 4. *A matrix $A = (a_{kj})$ of order n is said to be generalized stochastic if*

$$\sum_{j=1}^{n} a_{kj} = s, \qquad k = 1, 2, \ldots, n$$

where s is a complex number.

Definition 5. *If $A = (a_{kj})$ is such that*

$$\sum_{j=1}^{n} a_{kj} = s, \qquad k = 1, 2, \ldots, n \qquad \text{and} \qquad \sum_{k=1}^{n} a_{kj} = s, \qquad j = 1, 2, \ldots, n$$

then A is said to be generalized doubly stochastic matrix.

Note that a generalized stochastic or generalized doubly stochastic matrix need to be nonnegative.

In this paper, we give the relationships between the permanents of some generalized symmetric doubly stochastic matrices and the second order linear recurrences.

2. The Main Results

In this section, we define a $n \times n$ generalized symmetric doubly stochastic matrix D_n and then show that its permanent equals to the nth term of the sequence $\{v_n\}$.

We define a $n \times n$ generalized symmetric doubly stochastic matrix D_n with $d_{11} = \frac{\alpha}{\alpha+\beta}$, $d_{ii} = 0$ for $2 \leq i \leq n-1$, let n be an even number, $d_{2k,2k+1} = \frac{\alpha}{\alpha+\beta}$ for $1 \leq k \leq \frac{n-2}{2}$, $d_{2k-1,2k} = \frac{\beta}{\alpha+\beta}$ for $1 \leq k \leq \frac{n}{2}$ and $d_{nn} = \frac{\alpha}{\alpha+\beta}$, and, let n be an odd number, $d_{2k,2k+1} = \frac{\alpha}{\alpha+\beta}$ and $d_{2k-1,2k} = \frac{\beta}{\alpha+\beta}$ for $1 \leq k \leq \frac{n-1}{2}$, and $d_{nn} = \frac{\beta}{\alpha+\beta}$. Clearly, if n is an even , then

$$D_n = \begin{bmatrix} \frac{\alpha}{\alpha+\beta} & \frac{\beta}{\alpha+\beta} & & & & 0 \\ \frac{\beta}{\alpha+\beta} & 0 & \frac{\alpha}{\alpha+\beta} & & & \\ & \frac{\alpha}{\alpha+\beta} & 0 & \frac{\beta}{\alpha+\beta} & & \\ & & \frac{\beta}{\alpha+\beta} & \ddots & \ddots & \\ & & & \ddots & 0 & \frac{\beta}{\alpha+\beta} \\ 0 & & & & \frac{\beta}{\alpha+\beta} & \frac{\alpha}{\alpha+\beta} \end{bmatrix}.$$

We note that the rows and colums sums of the matrix D_n equal to 1. However, in general, since the entries of the matrix D_n, $\frac{\alpha}{\alpha+\beta}$ and $\frac{\beta}{\alpha+\beta}$ are not nonnegative, the matrix D_n is a generalized doubly stochastic matrix.

Then we have the following Theorem.

Theorem 1. *Let the matrix D_n be as before. Then, for $n \geq 2$*

$$per D_n = \frac{\alpha^n + \beta^n}{(\alpha+\beta)^n}.$$

Proof. If $n = 2$, then we have

$$per D_2 = per \begin{bmatrix} \frac{\alpha}{\alpha+\beta} & \frac{\beta}{\alpha+\beta} \\ \frac{\beta}{\alpha+\beta} & \frac{\alpha}{\alpha+\beta} \end{bmatrix} = \frac{\alpha^2 + \beta^2}{(\alpha+\beta)^2}.$$

If $n = 3$, then we have

$$per D_3 = per \begin{bmatrix} \frac{\alpha}{\alpha+\beta} & \frac{\beta}{\alpha+\beta} & \\ \frac{\beta}{\alpha+\beta} & 0 & \frac{\alpha}{\alpha+\beta} \\ & \frac{\alpha}{\alpha+\beta} & \frac{\beta}{\alpha+\beta} \end{bmatrix} = \frac{\alpha^3 + \beta^3}{(\alpha+\beta)^3}.$$

We suppose that n is an even number and let D_n^k be the kth contraction of D_n, $1 \leq k \leq n-2$. Since the definition of the matrix D_n, the matrix D_n can be contracted on column 1 so that

$$D_n^1 = \begin{bmatrix} \frac{\beta^2}{(\alpha+\beta)^2} & \frac{\alpha^2}{(\alpha+\beta)^2} & & & & 0 \\ \frac{\alpha}{\alpha+\beta} & 0 & \frac{\beta}{\alpha+\beta} & & & \\ & \frac{\beta}{\alpha+\beta} & 0 & \frac{\alpha}{\alpha+\beta} & & \\ & & \frac{\alpha}{\alpha+\beta} & \ddots & \ddots & \\ & & & \ddots & 0 & \frac{\beta}{\alpha+\beta} \\ 0 & & & & \frac{\beta}{\alpha+\beta} & \frac{\alpha}{\alpha+\beta} \end{bmatrix}.$$

Since the matrix D_n^1 can be contracted on column 1,

$$D_n^2 = \begin{bmatrix} \frac{\alpha^3}{(\alpha+\beta)^3} & \frac{\beta^3}{(\alpha+\beta)^3} & & & & 0 \\ \frac{\beta}{\alpha+\beta} & 0 & \frac{\alpha}{\alpha+\beta} & & & \\ & \frac{\alpha}{\alpha+\beta} & 0 & \frac{\beta}{\alpha+\beta} & & \\ & & \frac{\beta}{\alpha+\beta} & \ddots & \ddots & \\ & & & \ddots & 0 & \frac{\beta}{\alpha+\beta} \\ 0 & & & & \frac{\beta}{\alpha+\beta} & \frac{\alpha}{\alpha+\beta} \end{bmatrix}.$$

Continuing this process, we have, for even number k,

$$D_n^k = \begin{bmatrix} \frac{\alpha^{k+1}}{(\alpha+\beta)^{k+1}} & \frac{\beta^{k+1}}{(\alpha+\beta)^{k+1}} & & & & 0 \\ \frac{\beta}{\alpha+\beta} & 0 & \frac{\alpha}{\alpha+\beta} & & & \\ & \frac{\alpha}{\alpha+\beta} & 0 & \frac{\beta}{\alpha+\beta} & & \\ & & \frac{\beta}{\alpha+\beta} & \ddots & \ddots & \\ & & & \ddots & 0 & \frac{\beta}{\alpha+\beta} \\ 0 & & & & \frac{\beta}{\alpha+\beta} & \frac{\alpha}{\alpha+\beta} \end{bmatrix}$$

for $3 \leq k \leq n-4$. Hence,

$$D_n^{(n-3)} = \begin{bmatrix} \frac{\beta^{n-2}}{(\alpha+\beta)^{n-2}} & \frac{\alpha^{n-2}}{(\alpha+\beta)^{n-2}} & 0 \\ \frac{\alpha}{\alpha+\beta} & 0 & \frac{\beta}{\alpha+\beta} \\ 0 & \frac{\beta}{\alpha+\beta} & \frac{\alpha}{\alpha+\beta} \end{bmatrix}$$

which, by contraction of $D_n^{(n-4)}$ on column 1, gives

$$D_n^{(n-2)} = \begin{bmatrix} \frac{\alpha^{n-1}}{(\alpha+\beta)^{n-1}} & \frac{\beta^{n-1}}{(\alpha+\beta)^{n-1}} \\ \frac{\beta}{\alpha+\beta} & \frac{\alpha}{\alpha+\beta} \end{bmatrix}.$$

By applying (1.6), we have $perD_n = perD_n^{(n-2)} = \frac{\alpha^n+\beta^n}{(\alpha+\beta)^n}$.

When the case n is odd number, the proof easily follows from the above case n is even number.

So the proof is complete. □

Now we give a relationship between the result of Theorem 1 and second order recurrence $\{v_n\}$.

Corollary 1. *Let α and β be the roots of the equation $t^2 - At + B = 0$. Then, for $n \geq 2$*

$$perD_n = \frac{v_n}{A^n}$$

where v_n is the nth term of the sequence $\{v_n\}$ and $A = \alpha + \beta$.

Proof. For the sequence $\{v_n\}$, the Binet formula is given by $v_n = \alpha^n + \beta^n$ and since Theorem 1, the proof is easily seen. □

For example, when $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$, by Corollary 1, we have

$$per \begin{bmatrix} \frac{\alpha}{\alpha+\beta} & \frac{\beta}{\alpha+\beta} & & & & 0 \\ \frac{\beta}{\alpha+\beta} & 0 & \frac{\alpha}{\alpha+\beta} & & & \\ & \frac{\alpha}{\alpha+\beta} & 0 & \frac{\beta}{\alpha+\beta} & & \\ & & \frac{\beta}{\alpha+\beta} & \ddots & \ddots & \\ & & & \ddots & 0 & \frac{\alpha}{\alpha+\beta} \\ 0 & & & & \frac{\alpha}{\alpha+\beta} & \frac{\beta}{\alpha+\beta} \end{bmatrix}_{n\times n} = L_n$$

where L_n is the nth Lucas number.

Second, we define a $n \times n$ generalized symmetric doubly stochastic matrix H_n with $h_{11} = \frac{\alpha}{\alpha-\beta}$, $h_{ii} = 0$ for $2 \le i \le n-1$, let n be an even number, $h_{2k,2k+1} = \frac{\alpha}{\alpha-\beta}$ for $1 \le k \le \frac{n-2}{2}$, $h_{2k-1,2k} = \frac{-\beta}{\alpha-\beta}$ for $1 \le k \le \frac{n}{2}$ and $h_{nn} = \frac{\alpha}{\alpha-\beta}$, and, let n be an odd number, $h_{2k,2k+1} = \frac{\alpha}{\alpha-\beta}$ and $h_{2k-1,2k} = \frac{-\beta}{\alpha-\beta}$ for $1 \le k \le \frac{n-1}{2}$, and $h_{nn} = \frac{-\beta}{\alpha-\beta}$. Clearly, for even number n, the matrix

$$H_n = \begin{bmatrix} \frac{\alpha}{\alpha-\beta} & \frac{-\beta}{\alpha-\beta} & & & & 0 \\ \frac{-\beta}{\alpha-\beta} & 0 & \frac{\alpha}{\alpha-\beta} & & & \\ & \frac{\alpha}{\alpha-\beta} & 0 & \frac{-\beta}{\alpha-\beta} & & \\ & & \frac{-\beta}{\alpha-\beta} & \ddots & \ddots & \\ & & & \ddots & 0 & \frac{-\beta}{\alpha-\beta} \\ 0 & & & & \frac{-\beta}{\alpha-\beta} & \frac{\alpha}{\alpha-\beta} \end{bmatrix}.$$

We note that the rows and colums sums of the matrix H_n equal to 1. However, in general, since the entries of the matrix H_n, $\frac{\alpha}{\alpha-\beta}$ and $\frac{-\beta}{\alpha-\beta}$ are not nonnegative, the matrix H_n is a generalized doubly stochastic matrix.

Then we have the following Theorem.

Theorem 2. *Let the generalized doubly stochastic matrix H_n be as before. Then, for $n \geq 2$*

$$per H_n = \begin{cases} \frac{\alpha^n - \beta^n}{(\alpha-\beta)^n} & \text{if } n \text{ is odd number,} \\ \frac{\alpha^n + \beta^n}{(\alpha-\beta)^n} & \text{if } n \text{ is even number.} \end{cases}$$

Proof. We consider the first case n is odd number. If $n = 3$, then we have

$$per H_3 = \begin{bmatrix} \frac{\alpha}{\alpha-\beta} & \frac{-\beta}{\alpha-\beta} & 0 \\ \frac{-\beta}{\alpha-\beta} & 0 & \frac{\alpha}{\alpha-\beta} \\ 0 & \frac{\alpha}{\alpha-\beta} & -\beta \end{bmatrix} = \frac{\alpha^3 - \beta^3}{(\alpha-\beta)^3}.$$

If $n = 5$, then we have

$$per H_5 = \begin{bmatrix} \frac{\alpha}{\alpha-\beta} & \frac{-\beta}{\alpha-\beta} & & & \\ \frac{-\beta}{\alpha-\beta} & 0 & \frac{\alpha}{\alpha-\beta} & & \\ & \frac{\alpha}{\alpha-\beta} & 0 & \frac{-\beta}{\alpha-\beta} & \\ & & \frac{-\beta}{\alpha-\beta} & 0 & \frac{\alpha}{\alpha-\beta} \\ & & & \frac{\alpha}{\alpha-\beta} & \frac{-\beta}{\alpha-\beta} \end{bmatrix} = \frac{\alpha^5 - \beta^5}{(\alpha-\beta)^5}.$$

Let H_n^k be the kth contraction of H_n, $1 \leq k \leq n-2$. Since the definition of the matrix H_n, the matrix H_n can be contracted on column 1 so that

$$H_n^1 = \begin{bmatrix} \frac{\beta^2}{(\alpha-\beta)^2} & \frac{\alpha^2}{(\alpha-\beta)^2} & & & & \\ \frac{\alpha}{\alpha-\beta} & 0 & \frac{-\beta}{\alpha-\beta} & & & \\ & \frac{-\beta}{\alpha-\beta} & 0 & \frac{\alpha}{\alpha-\beta} & & \\ & & \frac{\alpha}{\alpha-\beta} & \ddots & \ddots & \\ & & & \ddots & 0 & \frac{\alpha}{\alpha-\beta} \\ & & & & \frac{\alpha}{\alpha-\beta} & \frac{-\beta}{\alpha-\beta} \end{bmatrix}.$$

Since the matrix H_n^1 can be contracted on column 1,

$$H_n^2 = \begin{bmatrix} \frac{\alpha^3}{(\alpha-\beta)^3} & \frac{-\beta^3}{(\alpha-\beta)^3} & & & & \\ \frac{-\beta}{\alpha-\beta} & 0 & \frac{\alpha}{\alpha-\beta} & & & \\ & \frac{\alpha}{\alpha-\beta} & 0 & \frac{-\beta}{\alpha-\beta} & & \\ & & \frac{-\beta}{\alpha-\beta} & \ddots & \ddots & \\ & & & \ddots & 0 & \frac{\alpha}{\alpha-\beta} \\ & & & & \frac{\alpha}{\alpha-\beta} & \frac{-\beta}{\alpha-\beta} \end{bmatrix}.$$

Continuing this process, we have, for odd number k,

$$H_n^k = \begin{bmatrix} \frac{\beta^{k+1}}{(\alpha-\beta)^{k+1}} & \frac{\alpha^{k+1}}{(\alpha-\beta)^{k+1}} & & & & \\ \frac{\alpha}{\alpha-\beta} & 0 & \frac{-\beta}{\alpha-\beta} & & & \\ & \frac{-\beta}{\alpha-\beta} & 0 & \frac{\alpha}{\alpha-\beta} & & \\ & & \frac{\alpha}{\alpha-\beta} & \ddots & \ddots & \\ & & & \ddots & 0 & \frac{\alpha}{\alpha-\beta} \\ & & & & \frac{\alpha}{\alpha-\beta} & \frac{-\beta}{\alpha-\beta} \end{bmatrix}$$

for $3 \leq k \leq n-4$. Hence,

$$H_n^{(n-3)} = \begin{bmatrix} \frac{\alpha^{n-2}}{(\alpha-\beta)^{n-2}} & \frac{-\beta^{n-2}}{(\alpha-\beta)^{n-2}} & 0 \\ \frac{-\beta}{\alpha-\beta} & 0 & \frac{\alpha}{\alpha-\beta} \\ 0 & \frac{\alpha}{\alpha-\beta} & \frac{-\beta}{\alpha-\beta} \end{bmatrix}$$

which, by contraction of $H_n^{(n-4)}$ on column 1, gives

$$H_n^{(n-2)} = \begin{bmatrix} \frac{\beta^{n-1}}{(\alpha-\beta)^{n-1}} & \frac{\alpha^{n-1}}{(\alpha-\beta)^{n-1}} \\ \frac{\alpha}{\alpha-\beta} & \frac{-\beta}{\alpha-\beta} \end{bmatrix}.$$

By applying (1.6), we have $PerH_n = perH_n^{(n-2)} = \frac{\alpha^n-\beta^n}{(\alpha-\beta)^n}$.

Now we consider the second case n is even number. If $n = 2$, then we have

$$perH_2 = \begin{bmatrix} \frac{\alpha}{\alpha-\beta} & \frac{-\beta}{\alpha-\beta} \\ \frac{-\beta}{\alpha-\beta} & \frac{\alpha}{\alpha-\beta} \end{bmatrix} = \frac{\alpha^2+\beta^2}{(\alpha-\beta)^2}.$$

Taking succsessive contrations of the matrix H_n, similarly the above, we have, for even number k,

$$H_n^k = \begin{bmatrix} \frac{\alpha^{k+1}}{(\alpha-\beta)^{k+1}} & \frac{-\beta^{k+1}}{(\alpha-\beta)^{k+1}} & & & & \\ \frac{-\beta}{\alpha-\beta} & 0 & \frac{\alpha}{\alpha-\beta} & & & \\ & \frac{\alpha}{\alpha-\beta} & 0 & \frac{-\beta}{\alpha-\beta} & & \\ & & \frac{-\beta}{\alpha-\beta} & \ddots & \ddots & \\ & & & \ddots & 0 & \frac{-\beta}{\alpha-\beta} \\ & & & & \frac{-\beta}{\alpha-\beta} & \frac{\alpha}{\alpha-\beta} \end{bmatrix}$$

for $3 \le k \le n-4$. Hence,

$$H_n^{(n-3)} = \begin{bmatrix} \frac{\beta^{n-2}}{(\alpha-\beta)^{n-2}} & \frac{\alpha^{n-2}}{(\alpha-\beta)^{n-2}} & 0 \\ \frac{\alpha}{\alpha-\beta} & 0 & \frac{-\beta}{\alpha-\beta} \\ 0 & \frac{-\beta}{\alpha-\beta} & \frac{\alpha}{\alpha-\beta} \end{bmatrix}$$

which, by contraction of $H_n^{(n-4)}$ on column 1, gives

$$H_n^{(n-2)} = \begin{bmatrix} \frac{\alpha^{n-1}}{(\alpha-\beta)^{n-1}} & \frac{-\beta^{n-1}}{(\alpha-\beta)^{n-1}} \\ \frac{-\beta}{\alpha-\beta} & \frac{\alpha}{\alpha-\beta} \end{bmatrix}..$$

By applying (1.6), we have $PerH_n = perH_n^{(n-2)} = \frac{\alpha^n+\beta^n}{(\alpha-\beta)^n}$ for even number n.

So the proof is complete. □

Now we give a relationship between the result of Theorem 2 and sequence $\{u_n\}$.

Corollary 2. *Let α and β be the roots of the equation $t^2 - At + B = 0$. Then, for $n > 1$*

$$perH_n = \begin{cases} \frac{u_n}{(\alpha-\beta)^{n-1}} & \text{if } n \text{ is odd number,} \\ \frac{v_n}{(\alpha-\beta)^n} & \text{if } n \text{ is even number.} \end{cases}$$

where u_n and v_n are the nth terms of the sequences $\{u_n\}$ and $\{v_n\}$.

Proof. Considering the Binet formulas for the sequences $\{u_n\}$ and $\{v_n\}$, and the result of Theorem 2, the proof is easily seen. □

For example, when $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$, by Corollary 2, we have

$$per \begin{bmatrix} \frac{\alpha}{\alpha-\beta} & \frac{-\beta}{\alpha-\beta} & & & & 0 \\ \frac{-\beta}{\alpha-\beta} & 0 & \frac{\alpha}{\alpha-\beta} & & & \\ & \frac{\alpha}{\alpha-\beta} & 0 & \frac{-\beta}{\alpha-\beta} & & \\ & & \frac{-\beta}{\alpha-\beta} & \ddots & \ddots & \\ & & & \ddots & 0 & \frac{\alpha}{\alpha-\beta} \\ 0 & & & & \frac{\alpha}{\alpha-\beta} & \frac{-\beta}{\alpha-\beta} \end{bmatrix}_{n\times n} = \frac{F_n}{(\sqrt{5})^{n-1}}$$

for even number n, where F_n is the nth Fibonacci number, and

$$per \begin{bmatrix} \frac{\alpha}{\alpha-\beta} & \frac{-\beta}{\alpha-\beta} & & & & 0 \\ \frac{-\beta}{\alpha-\beta} & 0 & \frac{\alpha}{\alpha-\beta} & & & \\ & \frac{\alpha}{\alpha-\beta} & 0 & \frac{-\beta}{\alpha-\beta} & & \\ & & \frac{-\beta}{\alpha-\beta} & \ddots & \ddots & \\ & & & \ddots & 0 & \frac{-\beta}{\alpha-\beta} \\ 0 & & & & \frac{-\beta}{\alpha-\beta} & \frac{\alpha}{\alpha-\beta} \end{bmatrix} = \frac{L_n}{(\sqrt{5})^{n}}$$

for odd number n, where L_n is the nth Lucas number.

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