Minus Edge Domination in Graphs*

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ABSTRACT. Let γ_E^- and γ_S^- be the minus edge domination and minus star domination numbers of a graph, respectively, and γ_E, β_1, α_1 be the edge domination, matching and edge covering numbers of a graph. In this paper, we present some bounds on γ_E^- and γ_S^- and characterize the extremal graphs of even order n attaining the upper bound $\frac{n}{2}$ on γ_E^-. We also investigate the relationships between the above parameters.

Keywords: edge domination, minus edge domination, minus star domination, matching, edge cover

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1 Introduction

We consider only finite undirected graphs without loops or multiple edges. For a graph $G = (V, E)$ with *vertex set* V and *edge set* E, the *open neighborhood* of $v \in V$ is $N_G(v) = \{u \in V | uv \in E\}$ and the *closed neighborhood* of v is $N_G[v] = N_G(v) \cup \{v\}$. The *degree* of v in G is denoted by $d_G(v)$, and $\delta(G)$ and $\Delta(G)$ denote the *minimum degree* and *maximum degree* of G respectively. For a subgraph G_1 of G, we let $d_{G_1}(v)$ denote the number of vertices in G_1 that are adjacent to v. For $e = uv \in E(G)$, $N_G(e) = \{e' \in E(G) |\ e'$ is adjacent to $e\}$ is called the *open edge-neighborhood* of e in G, and $N_G[e] = N_G(e) \cup \{e\}$ is called the closed one. If $v \in V$, then $E_G(v) = \{uv \in E |\ u \in V\}$ is called the *edge-neighborhood*

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of v in G. If confusion is unlikely, the above notations are denoted by $N(v)$, $N[v]$, $d(v)$, $N(e)$, $N[e]$ and $E(v)$, respectively. For $S \subseteq V$, $G[S]$ denotes the subgraph of G induced by S. The *matching number* $\beta_1(G)$ is the maximum cardinality among the independent sets of edges of a graph G. A graph G of order n is said to be have a *perfect matching* if n is even and $\beta_1(G) = \frac{n}{2}$, and have a *near-perfect matching* if n is odd and $\beta_1(G) = \frac{n-1}{2}$. For terminology and notation not given here, the reader is referred to [5].

For a real-valued function $f : E \rightarrow R$, the weight of f is $w(f) = \sum_{e \in E} f(e)$. For $E' \subseteq E$, we define $f(E') = \sum_{e \in E'} f(e)$, so that $w(f) = f(E)$. For a vertex $v \in V$, we write $d^*(v)$ for $\sum_{e \in E(v)} f(e)$.

A function $f : E \rightarrow \{0, 1\}$ is called the *edge dominating function* (EDF) of G if $\sum_{e' \in N[e]} f(e') \geq 1$ for every $e \in E$. The *edge domination number* of G is defined as $\gamma_E(G) = \min\{w(f) \mid f \text{ is an EDF of } G\}$. For a graph G without isolated vertices, a function $f : E \rightarrow \{0, 1\}$ is called the *edge covering function* (ECF) of G if $d^*(v) \geq 1$ for every $v \in V$. Clearly, the set of edges assigned 1 under f forms an *edge cover* of G. The *edge covering number* of G is defined as $\alpha_1(G) = \min\{w(f) \mid f \text{ is an ECF of } G\}$.

Now we generalize the above concepts by changing the weight $\{0, 1\}$ into $\{-1, 0, +1\}$.

A function $f : E \rightarrow \{-1, 0, +1\}$ is called the *minus edge dominating function* (MEDF) of G if $\sum_{e' \in N[e]} f(e') \geq 1$ for every $e \in E$. The *minus edge domination number* of G is defined as $\gamma_E^-(G) = \min\{w(f) \mid f \text{ is an MEDF of } G\}$. Let $G = (V, E)$ be a graph without isolated vertices. A function $f : E \rightarrow \{-1, 0, +1\}$ is called the *minus star dominating function* (MSDF) of G if $d^*(v) \geq 1$ for every $v \in V$. The *minus star domination number* of G is defined as $\gamma_S^-(G) = \min\{w(f) \mid f \text{ is an MSDF of } G\}$. In particular, we define $\gamma_E^-(G) = 0$ and $\gamma_S^-(G) = 0$ if G is a totally disconnected graph.

Similar concepts are signed edge domination and signed star domination where only labels $+1$ and -1 are allowed [9]. Other dominating functions in graphs have been studied in [1-3, 5-9] and elsewhere.

In this paper, we first present some bounds on γ_E^- and γ_S^- for a graph, which generalize some previous results on signed edge domination due to Xu [9]. We also characterize the extremal graphs of even order n attaining the upper bound $\frac{n}{2}$ on γ_E^-. Finally, we investigate the relationships between these two parameters and other graph parameters such as γ_E, β_1 and α_1.

2 Main results

Firstly, we present bounds on the minus edge domination number γ_E^- for graphs.

Theorem 1 *Let G be a connected graph of order n and size $m \geq 1$. Then*

$$\gamma_E^-(G) \geq \tfrac{m-(\Delta-\delta)(\Delta-2)(n-\delta)}{2\Delta-1}.$$

Proof. If $\Delta = 1$, then $G = K_2$, and the assertion obviously holds. We therefore assume that $\Delta \geq 2$. Let f be an MEDF of G such that $w(f) = \gamma_E^-(G)$. We partition E into three subsets as follows.

$$\begin{aligned} E_0 &= \{e \in E(G) \mid f(e) = 0\}, \\ E_1 &= \{e \in E(G) \mid f(e) = +1\}, \\ E_2 &= \{e \in E(G) \mid f(e) = -1\}. \end{aligned}$$

Let G_i be the subgraph of G induced by E_i $(i = 1, 2)$. For each vertex $u \in V(G)$, we have $d^*(u) = d_{G_1}(u) - d_{G_2}(u)$. Furthermore, we partition V into two subsets $V_0 = \{u \in V(G) \mid d^*(u) \leq 0\}$ and $V_1 = \{u \in V(G) \mid d^*(u) \geq 1\}$.

For any edge $e = uv \in E(G)$, we have $\sum_{e' \in N_G[e]} f(e') \geq 1$ by the definition of MEDF, that is, $d^*(u) + d^*(v) - f(uv) \geq 1$. Then we have

$$\begin{aligned} \sum_{u \in V(G)} d_G(u) d^*(u) &= \sum_{uv \in E(G)} (d^*(u) + d^*(v)) \\ &\geq \sum_{uv \in E(G)} (f(uv) + 1) \\ &= \gamma_E^-(G) + m. \end{aligned}$$

and

$$\begin{aligned} \sum_{u \in V(G)} d_G(u) d^*(u) &= \sum_{u \in V_0} d_G(u) d^*(u) + \sum_{u \in V_1} d_G(u) d^*(u) \\ &\leq \delta \sum_{u \in V_0} d^*(u) + \Delta \sum_{u \in V_1} d^*(u) \\ &= \Delta \sum_{u \in V(G)} d^*(u) + (\delta - \Delta) \sum_{u \in V_0} d^*(u). \end{aligned}$$

Hence,

$$\Delta \sum_{u \in V(G)} d^*(u) + (\delta - \Delta) \sum_{u \in V_0} d^*(u) \geq m + \gamma_E^-(G).$$

Note that $\sum_{u\in V(G)} d^*(u) = 2\gamma_E^-(G)$. Therefore,

$$(2\Delta - 1)\gamma_E^-(G) \geq m + (\Delta - \delta) \sum_{u\in V_0} d^*(u). \tag{1}$$

To complete the proof of the theorem, it is sufficient to prove that $\sum_{u\in V_0} d^*(u) \geq -(\Delta - 2)(n - \delta)$ by Eq. (1).

If $d^*(u) = 0$ for each vertex $u \in V_0$, then the desired inequality clearly holds.

If there is a vertex $u_0 \in V_0$ such that $d^*(u_0) \neq 0$, then it must be the case that $N_G(u_0) \subseteq V_1$. Otherwise, there is a vertex $v_0 \in V_0$ such that $u_0v_0 \in E(G)$. Note that $\sum_{e\in N_G[u_0v_0]} f(e) = d^*(u_0) + d^*(v_0) - f(u_0v_0) \geq 1$, we have $f(u_0v_0) \leq -2$ since $d^*(u_0) \leq -1$ and $d^*(v_0) \leq 0$, a contradiction. Then $|V_1| \geq |N_G(u_0)| \geq \delta$, and thus $|V_0| \leq n - \delta$. Furthermore, we show that $d^*(u) \geq -(\Delta - 2)$ for each vertex $u \in V_0$. Suppose to the contrary that there exists a vertex $u_0 \in V_0$ such that $d^*(u_0) \leq -(\Delta - 1)$. Then $d^*(u_0) = -\Delta$ or $d^*(u_0) = -\Delta + 1$, hence there is an edge $e = u_0v_0 \in E(G)$ such that $f(e) = -1$. This implies that $d^*(v_0) = d_{G_1}(v_0) - d_{G_2}(v_0) \leq d_G(v_0) - 2 \leq \Delta - 2$. Since $\sum_{e'\in N_G[e]} f(e') = d^*(u_0) + d^*(v_0) - f(e) \geq 1$, we have $d^*(u_0) \geq -(\Delta - 2)$, a contradiction. Therefore, $\sum_{u\in V_0} d^*(u) \geq -(\Delta - 2)|V_0| \geq -(\Delta - 2)(n - \delta)$ and the desired result follows. ■

When we apply some little changes to Xu's proofs of Theorem 2.1 in [9], then we immediately obtain another sharp lower bound on γ_E^- of a graph in terms of its size and order.

Theorem 2 *Let G be a graph of order n, size m and $\delta(G) \geq 1$. Then $\gamma_E^-(G) \geq n - m$ and this bound is sharp.*

Let G be a connected graph. Obviously, an EDF of G is an MEDF of G, and each maximal matching of G is also an edge dominating set of G. So we immediately have

Theorem 3 *For a connected graph G of order n, $\gamma_E^-(G) \leq \gamma_E(G) \leq \beta_1(G) \leq \frac{n}{2}$.*

Next, we characterize the extremal graphs of even order attaining the upper bound. First, we recall a lemma that will be useful in what follows.

Lemma 4 **([1])** *For any connected graph G of even order n, $\gamma_E(G) = \frac{n}{2}$ if and only if G is isomorphic to K_n or $K_{\frac{n}{2},\frac{n}{2}}$, where K_n and $K_{\frac{n}{2},\frac{n}{2}}$ are a complete graph and a complete bipartite graph respectively.*

In following, we compute the numbers $\gamma_{\bar{E}}(K_n)$ and $\gamma_{\bar{E}}(K_{\frac{n}{2},\frac{n}{2}})$.

Theorem 5 *For any complete graph K_n, $\gamma_{\bar{E}}(K_n) = \lfloor \frac{n}{2} \rfloor$.*

Proof. By Theorem 3, it suffices to prove that $\gamma_{\bar{E}}(K_n) \geq \lfloor \frac{n}{2} \rfloor$. Let f be any MEDF of K_n, we show that $w(f) \geq \lfloor \frac{n}{2} \rfloor$.

By the definition of MEDF, for each edge $e = uv \in E(K_n)$, we have $d^*(u)+d^*(v)-f(uv) \geq 1$, i.e., $d^*(v) \geq 1+f(uv)-d^*(u)$. We now distinguish several cases.

Case 1. If there exists a vertex $u_0 \in V(K_n)$ such that $d^*(u_0) \leq -2$. Then we have $d^*(v) \geq 3 + f(u_0v) \geq 2$ for each vertex v of $V(K_n) - \{u_0\}$. Hence

$$w(f) = \sum_{e\in E(K_n)} f(e) = \frac{1}{2} \sum_{v\in V(K_n)} d^*(v) \geq n-2 \geq \lfloor \frac{n}{2} \rfloor.$$

Case 2. If there exists a vertex $u_0 \in V(K_n)$ such that $d^*(u_0) = -1$. Then there is a vertex $v_0 \in V(K_n) - \{u_0\}$ such that $f(u_0v_0) = 0$ or $f(u_0v_0) = 1$, hence $d^*(v_0) \geq 1 - d^*(u_0)$. Since $d^*(v) \geq 2 + f(u_0v) \geq 1$ for each vertex $v \in V(K_n) - \{u_0\}$, it follows that

$$w(f) = \sum_{e\in E(K_n)} f(e) = \frac{1}{2} \sum_{v\in V(K_n)} d^*(v) \geq \frac{n-1}{2}.$$

Case 3. If there exists a vertex $u_0 \in V(K_n)$ such that $d^*(u_0) = 0$. Let $V_1 = \{v \in N(u_0) \mid f(u_0v) = -1\}$ and $V_2 = \{v \in N(u_0) \mid f(u_0v) = +1\}$. Then $|V_2| \geq |V_1|$. For each $v \in V_1$, we have $d^*(v) \geq 1 + f(u_0v) = 0$; for each $v \in V_2$, we have $d^*(v) \geq 1 + f(u_0v) = 2$; for each $v \in N(u_0) - V_1 - V_2$, we have $d^*(v) \geq 1$ as $f(u_0v) = 0$. Then

$$w(f) = \frac{1}{2} \sum_{v\in V(K_n)} d^*(v) \geq \frac{1}{2}(2|V_2| + n - 1 - |V_1| - |V_2|) \geq \frac{n-1}{2}.$$

Case 4. If $d^*(v) \geq 1$ for each vertex $v \in V(K_n)$. Then $\sum_{e\in E(K_n)} f(e) = \frac{1}{2} \sum_{v\in V(K_n)} d^*(v) \geq \frac{n}{2}$.

In either case, we have $\gamma_{\bar{E}}(K_n) \geq \lfloor \frac{n}{2} \rfloor$, and the desired result follows. ■

Theorem 6 *For any complete bipartite graph $K_{p,q}$, $\gamma_{\bar{E}}(K_{p,q}) = min\{p, q\}$.*

Proof. First, we show that $\gamma_E^-(K_{p,q}) \geq min\{p,q\}$. Let X and Y be a bipartition of $K_{p,q}$ and let $X = \{u_1, u_2, \ldots, u_p\}$ and $Y = \{v_1, v_2, \ldots, v_q\}$. Without loss of generality, we may assume $p \geq q$. Let f be an MEDF of $K_{p,q}$. As before, we have $d^*(v) \geq 1 + f(uv) - d^*(u)$ for each edge $e = uv \in E(K_{p,q})$. We consider the following three cases.

Case 1. If there exists a vertex $u_1 \in X$ such that $d^*(u_1) \leq -1$. Then $d^*(v) \geq 2 + f(u_1v) \geq 1$ for each vertex $v \in Y$. Note that $\sum_{u\in X} d^*(u) = \sum_{v\in Y} d^*(v)$. Hence,

$$w(f) = \sum_{e\in E(K_{p,q})} f(e) = \frac{1}{2} \sum_{v\in V(K_{p,q})} d^*(v) = \sum_{v\in Y} d^*(v) \geq |Y| = q.$$

Case 2. If there exists a vertex $u_1 \in X$ such that $d^*(u_1) = 0$. Then $d^*(v) \geq 1 + f(u_1v)$ for any $v \in Y$. Let $Y_1 = \{v \in Y \mid f(u_1v) = -1\}$ and $Y_2 = \{v \in Y \mid f(u_1v) = +1\}$. Then $|Y_1| \leq \frac{q}{2}$ and $|Y_2| \geq |Y_1|$. For each vertex $v \in Y_1$, we have $d^*(v) \geq 0$; for each vertex $v \in Y_2$, we have $d^*(v) \geq 2$; for each vertex $v \in Y - Y_1 - Y_2$, we have $d^*(v) \geq 1$ as $f(u_1v) = 0$. Hence,

$$w(f) = \sum_{e\in E(K_{p,q})} f(e) = \sum_{v\in Y} d^*(v) \geq 2|Y_2| + q - |Y_1| - |Y_2| \geq q.$$

Case 3. If $d^*(u) \geq 1$ for each vertex $u \in X$. Then clearly

$$w(f) = \sum_{e\in E(K_{p,q})} f(e) = \frac{1}{2} \sum_{v\in V(K_{p,q})} d^*(v) = \sum_{u\in X} d^*(u) \geq p \geq q.$$

On the other hand, we show that $\gamma_E^-(K_{p,q}) \leq q$. In order to prove it, we define an MEDF $f : E(K_{p,q}) \to \{-1, 0, 1\}$ as follows:

$$f(u_iv_j) = \begin{cases} 1 & i = j \text{ and } 1 \leq i \leq q, \\ 0 & \text{otherwise.} \end{cases}$$

Then it is easy to see that f is an MEDF of $K_{p,q}$. Thus $\gamma_E^-(K_{p,q}) \leq \sum_{e\in E(K_{p,q})} f(e) = q$. Consequently, $\gamma_E^-(K_{p,q}) = q = min\{p,q\}$. ■

For a connected graph G of order n, $\gamma_E^-(G) = \lfloor \frac{n}{2} \rfloor$ implies $\gamma_E(G) = \lfloor \frac{n}{2} \rfloor$ by Theorem 3. Combining Lemma 4, Theorem 5 and 6, we obtain the following result.

Theorem 7 *For any connected graph G of even order n, $\gamma_E^-(G) = \frac{n}{2}$ if and only if G is isomorphic to K_n or $K_{\frac{n}{2},\frac{n}{2}}$.*

Theorem 8 *For any positive integer k, there is a connected graph G such that $\gamma_E(G) - \gamma_E^-(G) \geq k$ and $\beta_1(G) - \gamma_E^-(G) \geq k$.*

Proof. Let H be a complete graph of order $n \geq 2k$ and $V(G) = \{v_1, v_2, \ldots, v_n\}$. Let F be the union of n copies of $K_{1,2}$. We construct a connected graph G from H by adding n disjoint copies of F, say $F_1, F_2, \ldots, F_n$, and joining each center of each copy $K_{1,2}$ of F_i with v_i, for $i = 1, 2, \ldots, n$. Let P be the set of all pendant edges of G. Define a function $f_1 : E(G) \to \{-1, 0, 1\}$ of G as follows:

$$f_1(e) = \begin{cases} -1 & e \in E(H), \\ 0 & e \in P, \\ 1 & \text{otherwise.} \end{cases}$$

Then f_1 is an MEDF of G and $\gamma_E^-(G) \leq \sum_{e \in E(G)} f_1(e) = n^2 - |E(H)|$. In order to edge-dominate each pendant edge of G, every edge dominating set of G contains at least n^2 edges. Hence, $\gamma_E(G) \geq n^2$. Define a function $f_2 : E(G) \to \{0, 1\}$ of G as follows:

$$f_2(e) = \begin{cases} 1 & e \in E(G) - E(H) - P, \\ 0 & \text{otherwise.} \end{cases}$$

Then f_2 is an EDF of G and $\gamma_E(G) \leq \sum_{e \in E(G)} f_2(e) = n^2$. So $\gamma_E(G) = n^2$. Since H is a complete graph, $\beta_1(H) = \lfloor \frac{n}{2} \rfloor$. So $\beta_1(G) \geq \beta_1(H) + \beta_1(G - H) \geq \lfloor \frac{n}{2} \rfloor + n^2$. Therefore, $\gamma_E(G) - \gamma_E^-(G) \geq |E(H)| = \frac{n(n-1)}{2} \geq k$ and $\beta_1(G) - \gamma_E^-(G) \geq \lfloor \frac{n}{2} \rfloor + |E(H)| \geq k$. ■

Now we begin to give bounds on γ_S^-. First, we recall the well-known equality due to Gallai [4].

Lemma 9 ([4]) *For any connected graph G of order n, $\alpha_1(G) + \beta_1(G) = n$.*

By the previous definitions and the Gallai identity in Lemma 9, and applying somewhat improvement to the Xu's proofs of Corollary 3.3 in [9], we can obtain the following result.

Theorem 10 *For any graph G of order n without isolated vertices, then $\lceil \frac{n}{2} \rceil \leq \gamma_S^-(G) \leq \alpha_1(G) = n - \beta_1(G)$.*

Theorem 11 *For arbitrary positive integer k, there is a connected graph G such that $\alpha_1(G) - \gamma_S^-(G) \geq k$.*

Proof. Let H be a complete graph of order $n \geq 2k$. We construct a connected graph G from H by adding n pendant edges on each vertex of H. Since G contains n^2 pendant vertices, $\alpha_1(G) \geq n^2$. Define a function $f : E(G) \to \{-1, 0, 1\}$ of G as follows:

$$f(e) = \begin{cases} -1 & e \in E(H), \\ 1 & \text{otherwise.} \end{cases}$$

Then f is an MSDF of G and $\gamma_S^-(G) \leq \sum_{e \in E(G)} f(e) = n^2 - |E(H)|$. Therefore, $\alpha_1(G) - \gamma_S^-(G) \geq |E(H)| = \frac{n(n-1)}{2} \geq k$. ∎

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