

# Reinforcing the number of disjoint spanning trees

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## Abstract

The **spanning tree packing number** of a connected graph  $G$ , denoted by  $\tau(G)$ , is the maximum number of edge-disjoint spanning trees of  $G$ . In this paper, we determine the minimum number of edges that must be added to  $G$  so that the resulting graph has spanning tree packing number at least  $k$ , for a given value of  $k$ .

**Key words.** Edge-disjoint spanning trees, spanning tree packing numbers, edge arboricity

## 1. Introduction.

We shall use the notation of Bondy and Murty [1], except defined otherwise. We allow graphs to have multiple edges but not loops. Let  $G$  be a graph. The set  $E(G^c)$  denotes the collection of edges that are not in  $E(G)$  but both ends of each member in  $E(G^c)$  are in  $V(G)$ . A

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maximal connected subgraph of  $G$  is called a component of  $G$ . The number of components of  $G$  is denoted by  $\omega(G)$ . Let  $L$  and  $H$  be two subgraphs of  $G$  with  $V(L) \cap V(H) \neq \emptyset$ . Define  $L \cap H$  to be a subgraph of  $G$  with  $V(L \cap H) = V(L) \cap V(H)$  and  $E(L \cap H) = E(L) \cap E(H)$ . For a set of edges  $E \subseteq E(G)$ , we define the **contraction**  $G/E$  to be the graph obtained from  $G$  by contracting the edges in  $E$  and deleting all resulting loops. If  $H$  is a connected subgraph of  $G$ , then  $G/H$  denotes  $G/E(H)$ . The maximum number of edge-disjoint spanning trees in  $G$  is called the **spanning tree packing number** of  $G$  (a recent survey on spanning tree packing number can be found in [7]), and is denoted by  $\tau(G)$ . For convenience, we define  $G/\emptyset = G$  and define  $\tau(K_1) = \infty$ . The set of all positive integers is denoted by  $\mathbb{N}$ .

In [6], Payan considered the following problem: Find an edge  $e \in E(G)$  and an edge  $e' \in E(G^c)$  such that  $G - e + e'$  is closer to having  $k$  edge-disjoint spanning trees than  $G$  does. A partial solution of this problem has been found in [3], and the general case remains open.

In this paper, we consider a problem with a similar nature: for a graph  $G$ , and a given integer  $k > \tau(G)$ , find the minimum number of edges  $X \subseteq E(G^c)$  such that  $\tau(G + X) \geq k$ .

We use decomposition and contraction methods to approach the problem. This decomposition is described in Section 2. The main result is proved in Section 3.

## 2. Some properties involving $\tau(G)$ .

Let  $X$  be a nonempty set. A **partition**  $(P_1, P_2, \dots, P_m)$  of  $X$  satisfies:

- (a)  $P_i \neq \emptyset$ ,  $1 \leq i \leq m$ ;
- (b)  $P_i \cap P_j = \emptyset$ ,  $i \neq j$  and  $1 \leq i, j \leq m$ ;
- (c)  $\bigcup_{i=1}^m P_i = X$ .

For an integer  $r \geq 1$ , let  $\mathcal{T}_r$  denote the family of all graphs  $G$  with  $\tau(G) \geq r$ . Lemma 2.1 below summarizes some observations.

**Lemma 2.1** Let  $G$  be a connected graph, and let  $r, r'$  be integers with  $r' \geq r > 0$ .

(i) Let  $H$  be a subgraph of  $G$  and  $H \in \mathcal{T}_{r'}$ . Then  $G/H \in \mathcal{T}_r$  if and only if  $G \in \mathcal{T}_r$ .

(ii) If  $G \in \mathcal{T}_r$ , and if  $e \in E(G^c)$ , then  $G + e \in \mathcal{T}_r$ .

(iii) If  $G \in \mathcal{T}_r$  and if  $e \in E(G)$ , then  $G/e \in \mathcal{T}_r$ .

(iv) If  $H_1$  and  $H_2$  are two subgraphs of  $G$  such that  $H_1, H_2 \in \mathcal{T}_r$  and  $V(H_1) \cap V(H_2) \neq \emptyset$ , then  $H_1 \cup H_2 \in \mathcal{T}_r$ .

**Proof:** (i) Since  $H \in \mathcal{T}_{r'}$  and since  $r' \geq r$ ,  $H$  has  $r$  edge-disjoint spanning trees  $T_1, \dots, T_r$ . Since  $G/H \in \mathcal{T}_r$ ,  $G/H$  has disjoint spanning trees  $T'_1, \dots, T'_r$ . Note that each  $T''_i = G[E(T_i) \cup E(T'_i)]$  is a spanning tree of  $G$ , and so  $G \in \mathcal{T}_r$ .

Conversely, suppose that  $G$  has  $r$  edge-disjoint spanning trees, say  $T_1, T_2, \dots$ , and  $T_r$ . Then  $T_i/(E(T_i) \cap E(H))$  is a spanning connected subgraph of  $G/H$  ( $1 \leq i \leq r$ ), and so  $G/H$  has  $r$  edge-disjoint spanning trees. Thus,  $G/H \in \mathcal{T}_r$ .

(ii) Any spanning tree of  $G$  is also a spanning tree of  $G + e$ .

(iii) Let  $T_1, \dots, T_r$  be edge-disjoint spanning trees of  $G$ . Let  $T'_i = T_i$  if  $e \notin E(T_i)$  and  $T'_i = T_i/e$  if  $e \in E(T_i)$ , for  $1 \leq i \leq r$ . Then  $T'_1, \dots, T'_r$  are edge-disjoint spanning subgraphs of  $G/e$ , and so  $G/e \in \mathcal{T}_r$ .

(iv) Let  $G = H_1 \cup H_2$ . Since  $H_1 \in \mathcal{T}_r$ , and by Lemma 2.1(iii),  $G/H_2 \in \mathcal{T}_r$ . Since  $H_2 \in \mathcal{T}_r$ , and by Lemma 2.1(i),  $G = H_1 \cup H_2 \in \mathcal{T}_r$ .  $\square$

Let  $G$  be a nontrivial connected graph. For any  $r \in \mathbb{N}$ , a nontrivial subgraph  $H$  of  $G$  is called  $r$ -maximal if  $H \in \mathcal{T}_r$  and if there is no subgraph  $K$  of  $G$ , such that  $K$  contains  $H$  properly and that  $K \in \mathcal{T}_r$ . An  $r$ -maximal subgraph  $H$  of  $G$  is called an  $r$ -region if  $r = \tau(H)$  (See Example 3.5). Call a subgraph  $H$  of  $G$  a region if  $H$  is an  $r$ -region for some integer  $r$ . Define  $\xi(G) = \max\{r \mid G \text{ has a subgraph as an } r\text{-region}\}$ .

**Lemma 2.2** Let  $H$  be a nontrivial subgraph of  $G$ . If  $\tau(H) = r$ ,

then there is always a region  $L$  of  $G$  with  $E(H) \subseteq E(L)$  and with  $\tau(L) \geq r$ .

**Proof:** Let  $L$  be the union of all  $r$ -regions of  $G$  each of which contains  $H$ . Then by Lemma 2.1(iv)  $L \in \mathcal{T}_r$ , and so  $L$  is  $\tau(L)$ -maximal.  $\square$

**Lemma 2.3** Let  $r', r \in \mathbf{N}$ , let  $H$  be an  $r'$ -region of  $G$ , and let  $K$  be an  $r$ -region of  $G$ . One of the following holds:

- (i)  $V(H) \cap V(K) = \emptyset$ ,
- (ii)  $r' = r$  and  $H = K$ ,
- (iii)  $r' > r$  and  $H$  is a nonspanning subgraph of  $K$ ,
- (iv)  $r' < r$  and  $H$  contains  $K$  as a nonspanning subgraph.

**Proof:** Suppose that Lemma 2.3(i) does not hold, and so  $V(H) \cap V(K) \neq \emptyset$ . Without loss of generality, we assume  $r' \geq r$ . By Lemma 2.1(i),  $H \cup K \in \mathcal{T}_r$ . Since  $K$  is an  $r$ -region,  $H \cup K$  is a subgraph of  $K$ , and so  $H$  is a subgraph of  $K$ . This implies (ii)-(iv) of Lemma 2.3.  $\square$

**Theorem 2.4** Let  $G$  be a nontrivial connected graph. Then

(a) there exist an integer  $m \in \mathbf{N}$ , and an  $m$ -tuple  $(i_1, i_2, \dots, i_m)$  of integers in  $\mathbf{N}$  with

$$\tau(G) = i_1 < i_2 < \dots < i_m = \xi(G), \quad (1)$$

and a sequence of edge subsets

$$E_m \subset \dots \subset E_2 \subset E_1 = E(G); \quad (2)$$

such that each component of the induced subgraphs  $G[E_j]$  is an  $r$ -region of  $G$  for some  $r \in \mathbf{N}$  with  $r \geq i_j$  ( $1 \leq j \leq m$ ), and such that at least one component  $H$  in  $G[E_j]$  is an  $i_j$ -region of  $G$ ;

(b) if  $H$  is a subgraph of  $G$  with  $\tau(H) \geq i_j$ , then  $E(H) \subseteq E_j$ ;

(c) the integer  $m$  and the sequences (1) and (2) are uniquely determined by  $G$ .

**Proof:** Let  $\mathcal{R}(G)$  denote the collection of all regions of  $G$ . By Lemma 2.2,  $\mathcal{R}(G)$  is not empty. Since  $G$  is a finite graph,

$$|\mathcal{R}(G)| \text{ is finite.} \quad (3)$$

Define  $sp(G)$  as

$$sp(G) = \{\tau(H) : H \in \mathcal{R}(G) \text{ is nontrivial}\}.$$

By (3),  $|sp(G)|$  is finite. Since  $G \in \mathcal{R}(G)$ ,  $|sp(G)| \geq 1$ . Let  $m = |sp(G)|$ . We may assume that  $sp(G) = \{i_1, i_2, \dots, i_m\}$  with  $i_1 < i_2 < \dots < i_m$ . By Lemma 2.1(i), we have

$$\tau(G) = i_1. \quad (4)$$

For each  $j \in \{1, 2, \dots, m\}$ , define

$$E_j = \bigcup_{\tau(H) \geq i_j} E(H). \quad (5)$$

By the definition of  $T_r$ ,

$$T_{i_1} \supset T_{i_2} \supset \dots \supset T_{i_m}. \quad (6)$$

Hence by (5) and (6),

$$E_1 \supseteq E_2 \supseteq \dots \supseteq E_m. \quad (7)$$

By (4),

$$E_1 = \bigcup_{\tau(H) \geq i_1} E(H) = \bigcup_{\tau(H) \geq \tau(G)} E(H) = E(G). \quad (8)$$

Fix  $j \in \{1, 2, \dots, m-1\}$ . Since  $i_j \in sp(G)$ , there is an  $i_j$ -region  $K$  of  $G$ . Since  $\tau(K) = i_j < i_{j+1}$ ,  $E(K) - E_{j+1} \neq \emptyset$ . Hence,  $E_j \neq E_{j+1}$ , and so (1) and (2) hold.

Fix  $j \in \{1, 2, \dots, m\}$ . We prove the following claim first.

**Claim A** Every component of  $G[E_j]$  is an  $r$ -region of  $G$ , for some  $r \geq i_j$ , where  $1 \leq j \leq m$ .

Let  $H$  be a nontrivial component of  $G[E_j]$ . By (5), we may assume that there are  $s$  regions  $H_t$ , ( $1 \leq t \leq s$ ) such that each  $H_t$  is an  $r_t$ -region, for some  $r_t \geq i_j$ , and such that

$$E(H) = \bigcup_{t=1}^s E(H_t).$$

Without loss of generality, we may assume that

$$r_1 \leq r_2 \leq \dots \leq r_s.$$

Since  $H$  is connected, if  $s \geq 2$ , then  $H_1$  must share a common vertex with some  $H_i$  for some  $i \geq 2$ , and so by Lemma 2.1(iv),  $H_1 \cup H_i \in \mathcal{T}_{r_1}$ , contrary to the fact that  $H_1$  is  $r_1$ -maximal. Hence, we must have  $s = 1$ . Thus, Claim A is proved.

What is left is to show that  $G[E_j]$  contains an  $i_j$ -region of  $G$ . Since  $i_j \in sp(G)$ , there is an  $i_j$ -region  $H$  of  $G$ . By (5),  $E(H) \subseteq E_j$ . We claim that  $H$  is a component of  $G[E_j]$ . Since  $H$  is connected,  $H$  is in a component  $K$  of  $G[E_j]$ . By Claim A,  $K$  is an  $r$ -region with  $r \geq i_j$ . It follows by Lemma 2.3 that  $H = K$ . Thus the claim follows and so (a) of Theorem 2.4 must hold. Theorem 2.4(b) follows from Lemma 2.2 and the proof above.

Since  $\mathcal{R}(G)$  and  $sp(G)$  are uniquely determined by  $G$ , the integer  $m$ , the  $m$ -tuple  $(i_1, i_2, \dots, i_m)$  and the sequence (2) are all uniquely determined by  $G$ . Therefore (c) of Theorem 2.4 follows. This proves Theorem 2.4.  $\square$

**Corollary 2.5** If  $(i_1, i_2, \dots, i_m)$  is the tuple determined by  $G$  as defined in Theorem 2.4, then  $(i_1, i_2, \dots, i_{m-1})$  is the tuple determined by  $G/E_m$ . In particular,  $i_{m-1} = \xi(G/E_m)$ .

**Proof:** By Theorem 2.4, we know that each component of  $G[E_m]$  is an  $i_m$ -region. The corollary follows from Lemma 2.1(i) and the definition of  $G/E_m$ .  $\square$

**Proposition 2.6** Let  $r', r \in \mathbb{N}$  and let  $H$  be an  $r'$ -region of  $G$  and  $K$  be an  $r$ -region of  $G$ .

(i) If  $V(H) \cap V(K) = \emptyset$ , then  $(G/H)[E(K)]$  is also an  $r$ -region of  $G/H$ ;

(ii) If  $K$  contains  $H$  as a nonspanning subgraph, then  $r' > r$  and  $K/H$  is an  $r$ -region of  $G/H$ .

**Proof:** Let  $v_H$  denote the vertex of  $G/H$  onto which the subgraph  $H$  is contracted.

(i) Suppose that  $V(H) \cap V(K) = \emptyset$ . Then  $K$  is a subgraph of  $G/H$ . If  $K$  is not a region of  $G/H$ , then  $G/H$  has a region  $L'$  with  $\tau(L') \geq r$  and  $K \subset L'$ . If  $v_H \notin V(L')$ , then  $L'$  is a subgraph of  $G$ , contrary to the fact that  $K$  is a region of  $G$ . Hence,  $v_H \in V(L')$ . Let  $L = G[E(L') \cup E(H)]$ . Then  $L$  is a subgraph of  $G$  containing both  $K$  and  $L'$ . If  $r' \geq r$ , then by Lemma 2.1(i),  $\tau(L) \geq r$  and so  $K$  is not a region, a contradiction. Similarly, if  $r \geq r'$ , then  $\tau(L) \geq r'$  and so  $H$  is not a region, a contradiction. These contradictions establish Proposition 2.6(i).

(ii) Now suppose that  $K$  contains  $H$  as a nonspanning subgraph. By Lemma 2.3(iii),  $r' > r$ . By Lemma 2.1(i),  $\tau(K/H) \geq r$ . If  $G/H$  has a region  $L'$  containing  $K/H$  with  $\tau(L') \geq r$ , then by Lemma 2.1(i),  $L = G[E(L')]$  is a subgraph of  $G$  containing  $K$  with  $\tau(L) \geq r = \tau(K)$ . Since  $K$  is a region,  $K = L$ , and so  $K/H = L'$ . This proves that  $K/H$  is an  $r$ -region of  $G/H$ .  $\square$

**Corollary 2.7** Let  $r', r \in \mathbb{N}$  and let  $H$  be an  $r'$ -region of  $G$ , and denote by  $v_H$  the vertex in  $G/H$  to which  $H$  is contracted.

(i) If  $K$  is an  $r$ -region of  $G/H$  not containing  $v_H$ , then  $G[E(K)]$  is an  $r$ -region of  $G$  disjoint from  $H$ .

(ii) If  $K$  is an  $r$ -region of  $G/H$  containing  $v_H$ , and if  $r' > r$ , then  $G[E(K) \cup E(H)]$  is an  $r$ -region of  $G$ .

**Proof:** (i) Suppose that  $v_H \notin V(K)$ . Then  $G[E(K)] \cong K$ , and so  $K$  can be regarded as a subgraph of  $G$  disjoint from  $H$ . Since  $\tau(K) = r$ ,  $G$  has an  $s$ -region  $L$  containing  $K$  as a subgraph, where  $s \geq r$ . Then  $L$  (if  $V(L) \cap V(H) = \emptyset$ ) or  $L/(L \cap H)$  (if  $V(L) \cap V(H) \neq \emptyset$ ) is a region of  $G/H$  containing  $K$ , by Proposition 2.6, and so we must have  $L = K$ .

(ii) Let  $K'' = G[E(K) \cup E(H)]$  with  $\tau(K'') = s$ . By  $r' > r$ , both  $K \in \mathcal{T}_r$  and  $H \in \mathcal{T}_{r'} \subset \mathcal{T}_r$ , and so by Lemma 2.1(i),  $K'' \in \mathcal{T}_r$ . This implies  $s \geq r$ . By Lemma 2.2, there is a region  $L$  of  $G$  containing  $K''$  as a subgraph with  $\tau(L) \geq s \geq r$ . Note that  $H$  is a nonspanning subgraph of  $L$ . Apply Proposition 2.6(ii) to  $L$  and  $H$  to conclude that  $L/H$  is a  $\tau(L)$ -region of  $G/H$  containing  $K$ . Then apply Lemma

2.3 to  $L/H$  and  $K$  to conclude that  $r \geq \tau(L)$ , where equality holds if and only if  $K = L/H$ . It follows that  $r = s = \tau(L)$  and  $K = L/H$ , and so  $K'' = L$  is an  $r$ -region of  $G$ .  $\square$

### 3. The Main results.

Let  $G$  be a graph. The edge arboricity of  $G$ ,  $a(G)$ , is the minimum number of edge-disjoint spanning forests whose union is  $G$ .

**Theorem 3.1** (Nash-Williams [4], [5], Tutte [8]) Let  $G$  be a graph and let  $k$  be an integer. Then

(i)  $a(G) = \max_{H \subseteq G} \left\lceil \frac{|E(H)|}{|V(H)| - 1} \right\rceil$ , where the maximum is taken over all induced subgraphs  $H$  of  $G$  with  $|V(H)| \neq 2$ .

(ii) If  $|E(G)| \geq k(|V(G)| - 1)$ , then  $G$  has a subgraph  $H$  with  $\tau(H) \geq k$ .

(iii)  $\tau(G) = \left\lfloor \min_{X \subseteq E(G)} \frac{|X|}{\omega(G - X) - \omega(G)} \right\rfloor$ , where the minimum is over all subsets  $X \subseteq E(G)$  such that  $\omega(G - X) > \omega(G)$ .

**Corollary 3.2**  $a(G) \geq i_m \geq a(G) - 1$ .

**Proof:** Let  $L$  be a component of  $G[E_m]$ . By Theorem 2.4, every component of  $G[E_m]$  has  $i_m$  edge-disjoint spanning trees, and no nontrivial subgraph of  $G$  with  $i_m + 1$  edge-disjoint spanning trees. Thus by Theorem 3.1,  $i_m = \tau(L) \leq |E(L)| / (|V(L)| - 1) \leq a(G)$ .

By Theorem 3.1(i), there is a subgraph  $H$  of  $G$  such that  $|E(H)| \geq (a(G) - 1)(|V(H)| - 1)$ . Therefore, by Theorem 3.1(ii),  $H$  (and so  $G$ ) has a subgraph  $H'$  with  $\tau(H') \geq (a(G) - 1)$ , and so by Lemma 2.2,  $G$  has a region  $K$  with  $E(H') \subseteq E(K)$  and with  $\tau(K) \geq (a(G) - 1)$ . By the definition of  $i_m$  in the proof of Theorem 2.4, we know that  $i_m \geq a(G) - 1$ .  $\square$

**Lemma 3.3** Let  $G$  be a graph and let  $k$  be an integer with  $k > \tau(G)$ . If  $k \geq a(G)$ , then one can find  $X \subseteq E(G^c)$  with  $|X| = k(|V(G)| - 1) - |E(G)|$  such that  $G + X$  is the union of  $k$  edge-disjoint spanning trees.



**Proof:** By  $a(G) \leq k$ , there are edge-disjoint spanning forests  $F_1, \dots, F_k$  such that  $G = \cup_{i=1}^k F_i$ . Set  $X_0 = \emptyset$ . For each  $i$ , ( $1 \leq i \leq k$ ), there is an edge set  $X_i \subset E((G + (\cup_{j=0}^{i-1} X_j))^c)$  such that  $F_i + X_i$  is a tree. Let  $X = \cup_{j=1}^k X_j$ . Then  $G + X$  is the union of  $k$  edge-disjoint spanning trees, and so  $|E(G)| + |X| = k(|V(G)| - 1)$ .  $\square$

Let  $G$  be a graph and let  $k \geq \tau(G)$  be an integer. Let  $f(G, k)$  denote the minimum number of edges that must be added to  $G$  so that the resulting graph has  $k$  edge-disjoint spanning trees. By Theorem 2.4,  $G$  has a decomposition satisfying (1) and (2). If  $k \leq i_m$ , define  $i(k) = \min\{i_j : i_j \geq k \text{ and } i_j \in sp(G)\}$ ; if  $k > i_m$ , define  $i(k) = \infty$ , and define  $E_\infty = \emptyset$ . Let  $c_k(G)$  be the number of components of  $G[E_{i(k)}]$ , and let  $w_k(G) = |V(G[E_{i(k)}])|$ . Note that  $c_k(G) = w_k(G) = 0$  if  $i(k) = \infty$ .

**Theorem 3.4** Let  $G$  be a graph and let  $k > \tau(G)$ . Then

$$f(G, k) = k(|V(G)| - w_k(G) + c_k(G) - 1) - (|E(G)| - |E_{i(k)}|).$$

**Proof:** If  $k > a(G)$ , then by Corollary 3.2 and the definition of  $i(k)$ , we have  $i(k) = \infty$ , and so  $c_k(G) = w_k(G) = 0$ . Thus, by Lemma 3.3,  $f(G, k) = k(|V(G)| - 1) - |E(G)|$ . Theorem 3.3 holds in this case. In the following we assume that  $k \leq a(G)$ .

By Theorem 2.4,  $G$  has a decomposition satisfying (1) and (2). Let  $G' = G/E_{i(k)}$ . Then

$$|V(G')| = |V(G)| - (w_k(G) - c_k(G)) \text{ and } |E(G')| = |E(G)| - |E_{i(k)}|. \quad (9)$$

**Claim**  $a(G') \leq k$ .

Suppose that  $a(G') > k$ . By Corollary 3.2, we know that  $G'$  has an  $r$ -region  $L'$  with  $r \geq k$ . Let  $H_1, \dots, H_c$  be the components of  $G[E_{i(k)}]$ , and let  $v_i$  denote the vertex in  $G'$  to which  $H_i$  is contracted. By Theorem 2.4,

$$\tau(H_i) \geq k, \text{ for every } i = 1, 2, \dots, c. \quad (10)$$

If  $L'$  does not contain any  $v_i$ , then by Corollary 2.7(i),  $L' = G[E(L')]$  is an  $r$ -region of  $G$ . Since  $r \geq k$ , and by Theorem 2.4,  $E(L') \subseteq E_{i(k)}$ , then  $L'$  cannot be a subgraph of  $G'$ , a contradiction.

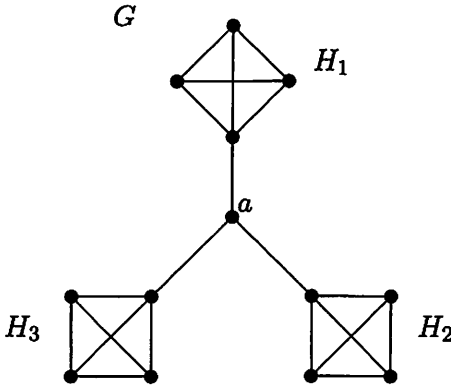
Hence, we may assume that  $v_1, \dots, v_t \in V(L')$  and  $v_i \notin V(L')$  for  $i \geq t + 1$ . Let  $L = G[E(L') \cup (\cup_{i=1}^t E(H_i))]$ , and let  $k' = \min_{1 \leq i \leq t} \tau(H_i)$ . Then by the definition of  $E_{i(k)}$ ,  $k' \geq k$ . Thus,  $\min\{r, k'\} \geq k$ . By Lemma 2.1(i),  $L$  is a subgraph of  $G$  containing  $H_1, \dots, H_t$  as subgraphs, and  $\tau(L) \geq \min\{r, k'\} \geq k$ . Therefore, by the definition of  $E_{i(k)}$ , by  $\tau(L) \geq k$ , and by Theorem 2.4,  $E(L) \subset E_{i(k)}$ , contrary to the assumption that  $L'$  is a subgraph of  $G'$ . Thus, the claim follows.

By the Claim,  $G'$  satisfies the hypothesis of Lemma 3.3, and so by Lemma 3.3, there is an edge subset  $X \subset E((G')^c)$  with

$$|X| = k(|V(G')| - 1) - |E(G')|, \quad (11)$$

such that  $G' + X$  is the union of  $k$  edge-disjoint spanning trees. Thus, the number of additional edges represented in (11) is the minimum number of edges that must be added to  $G'$  to have  $k$  edge-disjoint spanning trees. Note that  $G' + X = G/E_{i(k)} + X \cong (G + X)/E_{i(k)}$ , and each component of  $G[E_{i(k)}]$  is an  $r$ -region of  $G$  with  $r \geq i(k) \geq k$ . By Lemma 2.1(i),  $\tau(G + X) \geq k$ . By (9) and (11), Theorem 3.4 follows.  $\square$

**Example 3.5** Let  $V(K_{1,3}) = \{a, v_1, v_2, v_3\}$ , where  $d(a) = 3$ , and  $d(v_i) = 1$  ( $1 \leq i \leq 3$ ). Let  $G$  be a graph obtained from  $K_{1,3}$  by replacing each  $v_i$  in  $K_{1,3}$  by  $H_i = K_4$  ( $1 \leq i \leq 3$ ) as shown below. Obviously,  $\tau(G) = 1$ , and  $G$  itself is a 1-region. Only  $H_1, H_2$  and  $H_3$  are 2-regions in  $G$ . If  $r \geq 3$ ,  $G$  has no  $r$ -region, and so  $\xi(G) = 2$ . Therefore,  $sp(G) = \{1, 2\}$ . Thus, as stated in Theorem 2.4,  $1 = i_1 < 2 = i_2$  are the integers uniquely determined by  $G$ . And  $E_2 = \bigcup_{i=1}^3 E(H_i) \subset E_1 = E(G)$  are the edge subsets uniquely determined by  $G$ .



Let  $k = 2$ . Then  $i(k) = 2$ , and so  $|E_{i(k)}| = |E_2| = 18$ ,  $c_k(G) = 3$ , and  $w_k(G) = |V(G[E_2])| = 12$ . By Theorem 3.4, the minimum number of edges that must be added to  $G$  so that the resulting graph has 2 edge-disjoint spanning trees is

$$\begin{aligned}
 f(G, 2) &= k(|V(G)| - w_k(G) + c_k(G) - 1) - (|E(G)| - |E_{i(k)}|) \\
 &= 2(13 - 12 + 3 - 1) - (21 - 18) = 3.
 \end{aligned}$$

Note that there are more than one way to select three edges to add to  $G$  so that the resulting graph has 2 edge-disjoint spanning trees. In fact, we can choose an arbitrary vertex  $v_{H_i}$  from  $V(H_i)$  ( $1 \leq i \leq 3$ ), and let  $e_1 = v_{H_1}v_{H_2}$ ,  $e_2 = v_{H_1}v_{H_3}$ , and  $e_3 = v_{H_2}v_{H_3}$  be the three new edges. Then the new graph obtained from  $G$  by adding  $e_1, e_2$ , and  $e_3$  has 2 edge-disjoint spanning trees.

**Remark.** From Theorem 3.4 above, one can see that for a given graph  $G$  and a given integer  $k$ , the main task to find  $f(G, k)$  is to find  $E_{i(k)}$ . Hobbs [2] developed a polynomial-time algorithm to compute the number  $i_m$  and to locate the subset  $E_m$  as defined in Theorem 2.4. As long as  $E_m \neq E(G)$  and  $i_m \geq k$ , by Corollary 2.5, one can apply Hobbs' algorithm to the contraction  $G/E_m$ . There are at most  $m$  iterations before  $E_{i(k)}$  is found. Once  $E_{i(k)}$  is found, it is easy to compute  $c_k(G)$  and  $w_k(G)$ , and so by Theorem 3.4 to compute

$f(G, k)$ . Thus, this gives a polynomial-time algorithm to compute  $f(G, k)$ .

In the following, we shall derive a different expression, a min-max formula, for  $f(G, k)$ .

Define, for each subset  $X \subseteq E(G)$ ,

$$f_k(G, X) = k[\omega(G - X) - 1] - |X|,$$

and

$$F_k(G) = \max_{X \subseteq E(G)} \{f_k(G, X)\}. \quad (12)$$

Note that  $F_k(G) \geq f_k(G, \emptyset) \geq 0$ , and that  $F_k(K_1) = 0$ , for any  $k \geq 1$ . We shall show in Theorem 3.10 that  $F_k(G) = f(G, k)$ .

**Lemma 3.6** Assume that  $X \subseteq E(G)$  is an edge-subset with  $f_k(G, X) = F_k(G)$ , and that  $H$  is a component of  $G - X$ . If  $X_H \subseteq E(H)$  is an edge-subset, then

$$f_k(G, X \cup H_X) = f_k(G, X) + f_k(H, X_H). \quad (13)$$

**Proof:** Let  $X$ ,  $H$  and  $X_H$  be as assumed. Then

$$\begin{aligned} f_k(G, X \cup H_X) &= k[\omega(G - X \cup H_X) - 1] - |X| - |X_H| \\ &= k[\omega(G - X) - 1 + \omega(H - X_H) - 1] - |X| - |X_H| \\ &= f_k(G, X) + f_k(H, X_H). \end{aligned}$$

**Corollary 3.7** If  $X \subseteq E(G)$  satisfies  $F_k(G) = f_k(G, X)$ , then for every component  $H$  of  $G - X$ ,  $F_k(H) = 0$ . In particular,  $\tau(H) \geq k$ .

**Proof:** By Lemma 3.6, for any  $X_H \subseteq E(H)$ ,  $f_k(H, X_H) = f_k(G, X \cup X_H) - F_k(G) \leq 0$ , and so  $F_k(H) = \max_{X_H \subseteq E(H)} \{f_k(H, X_H)\} = 0$ .

To prove that  $\tau(H) \geq k$ , we may assume that  $H \neq K_1$  since

$\tau(K_1) = \infty$ . By the definition of  $f_k(H, X_H)$ ,  $F_k(H) =$

$\max_{X_H \subseteq E(H)} \{f_k(H, X_H)\} = 0$  implies that

$$\max_{X_H \subseteq E(H)} \{k[\omega(H - X_H) - 1] - |X_H|\} = 0.$$

Therefore, for any  $X_H \subseteq E(H)$  with  $\omega(H - X_H) > 1$ ,

$$\frac{|X_H|}{\omega(H - X_H) - 1} \geq k.$$

By Theorem 3.1(iii),  $\tau(H) \geq k$ .  $\square$

**Lemma 3.8** If  $G$  is connected, and if  $F_k(G) = f_k(G, E(G))$ , then  $a(G) \leq k$ .

**Proof:** Let  $H$  be an induced subgraph of  $G$ . Define  $E_H = E(G) - E(H)$ . Since the components of  $G - E_H$  are  $H$  and  $|V(G)| - |V(H)|$  isolated vertices,

$$\omega(G - E_H) = |V(G)| - |V(H)| + \omega(H). \quad (14)$$

By (12),

$$\begin{aligned} F_k(G) &\geq f_k(G, E_H) \geq k(\omega(G - E_H) - 1) - |E_H| \\ &= k(|V(G)| - |V(H)| + \omega(H) - 1) - |E(G)| + |E(H)| \\ &\geq k(|V(G)| - |V(H)|) - |E(G)| + |E(H)| \\ &= k(|V(G)| - 1) - |E(G)| + k - (k|V(H)| - |E(H)|) \\ &= f_k(G, E(G)) - [k(|V(H)| - 1) - |E(H)|] \\ &= F_k(G) - [k(|V(H)| - 1) - |E(H)|]. \end{aligned}$$

It follows that

$$0 \geq -k(|V(H)| - 1) + |E(H)|,$$

and so

$$\frac{|E(H)|}{|V(H)| - 1} \leq k.$$

By Theorem 3.1(i),  $a(G) \leq k$ .  $\square$ .

**Lemma 3.9** Let  $G$  be a graph and let  $E_0 \subseteq E(G)$  be such that  $f_k(G, E_0) = F_k(G)$ . Let  $G_0 = G/(E(G) - E_0)$ . Then

$$f_k(G_0, E_0) = F_k(G_0) = F_k(G).$$

**Proof:** Note that  $\omega(G - E_0) = \omega(G_0 - E_0)$ , and so by the assumption that  $f_k(G, E_0) = F_k(G)$ , we have

$$F_k(G_0) \geq f_k(G_0, E_0) = f_k(G, E_0) = F_k(G).$$

Choose  $E_1 \subseteq E_0$ , such that  $F_k(G_0) = f_k(G_0, E_1)$ . Then since  $E_1 \subseteq E_0$ ,  $\omega(G - E_1) = \omega(G_0 - E_1)$ , and so

$$F_k(G) \geq f_k(G, E_1) = f_k(G_0, E_1) = F_k(G_0). \quad \square$$

Next we prove a min-max theorem.

**Theorem 3.10**  $F_k(G) = f(G, k)$ .

**Proof:** Let  $E_0 \subseteq E(G)$  be an edge subset of  $E(G)$  such that  $f_k(G, E_0) = F_k(G)$ , and let  $G_0 = G/(E(G) - E_0)$  as defined in Lemma 3.9. By Lemma 3.9,  $f_k(G_0, E(G_0)) = F_k(G_0) = F_k(G)$ .

By Lemma 3.8,  $a(G_0) \leq k$ . Hence,  $G_0$  is an edge-disjoint union of  $k$  spanning forests  $F_1, F_2, \dots, F_k$  of  $G_0$ . Let  $|E(F_i)| = |V(G_0)| - 1 - s_i$ , where  $s_i \geq 0$  and  $1 \leq i \leq k$ . Then one can add  $s_i$  edges to  $F_i$  to form a spanning tree of  $G_0$ . Therefore, by adding an edge set  $X$  with  $\sum_{i=1}^k s_i$  edges to  $G_0$ , the resulting graph  $G_0 + X$  has  $k$  edge-disjoint spanning trees. Note that  $|E(G_0)| = \sum_{i=1}^k |E(F_i)| = k(|V(G_0)| - 1) - \sum_{i=1}^k s_i$ . Since  $F_k(G) = F_k(G_0) = f_k(G_0, E(G_0)) = k(|V(G_0)| - 1) - |E(G_0)|$ ,  $F_k(G) = F_k(G_0) = \sum_{i=1}^k s_i$ . This shows that

$$F_k(G) = F_k(G_0) = f(G_0, k). \quad (15)$$

Let  $H_1, H_2, \dots, H_c$  be the components of  $G - E_0$ . By Corollary 3.7,  $\tau(H_i) \geq k$ ,  $1 \leq i \leq c$ . Note that  $G_0 + X = (G + X)/(E(G) - E_0) = (G + X)/\bigcup_{i=1}^c H_i = ((G + X)/\bigcup_{i=2}^c H_i)/H_1$ . By repeatedly

applying Lemma 2.1(i), we have  $\tau(G + X) \geq k$ , and so  $F_k(G) \geq f(G, k)$ .

Conversely, let  $X$  be a set of  $f(G, k)$  edges that must be added to  $G$  such that  $\tau(G + X) \geq k$ . Let  $W_i = X \cap E(H_i^c)$ , and let  $H_i' = H_i + W_i$ . Since  $\tau(H_i) \geq k$ , by Lemma 2.1(ii),  $\tau(H_i') \geq k$ . Let  $X_1 = \cup_{i=1}^c W_i$ , and  $X_0 = X - X_1$ . Then  $G_0 + X_0 = (G + X) / ((E(G) + X) - (E_0 + X_0)) = (G + X) / \cup_i^c H_i'$ . By Lemma 2.1(i),  $\tau(G_0 + X_0) \geq k$ . Therefore, by (15)

$$F_k(G) = f(G_0, k) \leq |X_0| \leq |X| = f(G, k). \quad \square$$

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# The heterochromatic matchings in edge-colored bipartite graphs \*

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## Abstract

Let  $(G, C)$  be an edge-colored bipartite graph with bipartition  $(X, Y)$ . A heterochromatic matching of  $G$  is such a matching in which no two edges have the same color. Let  $N^c(S)$  denote a maximum color neighborhood of  $S \subseteq V(G)$ .

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