The heterochromatic matchings in edge-colored bipartite graphs *

Hao Li[1,2],† Xueliang Li[3], Guizhen Liu[4], Guanghui Wang[1,4]

[1] *Laboratoire de Recherche en Informatique*
UMR 8623, C.N.R.S.-Université de Paris-sud
91405-Orsay cedex, France
e-mail: li@lri.fr, wgh@lri.fr

[2] *School of Mathematics and Statistics*
Lanzhou University
730000 Lanzhou, Gansu, China

[3] *Center for Combinatorics and LPMC*
Nankai University
Tianjin 300071, China
e-mail: lxl@nankai.edu.cn

[4] *School of Mathematics and System Science*
Shandong University
Jinan Shandong 250100, China
e-mail: gzliu@sdu.edu.cn

Abstract

Let (G, C) be an edge-colored bipartite graph with bipartition (X, Y). A heterochromatic matching of G is such a matching in which no two edges have the same color. Let $N^c(S)$ denote a maximum color neighborhood of $S \subseteq V(G)$.

*This research is supported by the NSFC(60373047 and 10471078), SRSDP(20040422004) and PDSF(2004036402) of China.

†Corresponding author.

We show that if $|N^c(S)| \geq |S|$ for all $S \subseteq X$, then G has a heterochromatic matching with cardinality at least $\lceil \frac{|X|}{3} \rceil$. We also obtain that if $|X| = |Y| = n$ and $|N^c(S)| \geq |S|$ for all $S \subseteq X$ or $S \subseteq Y$, then G has a heterochromatic matching with cardinality at least $\lceil \frac{3n}{8} \rceil$.

Keywords: heterochromatic matching, color neighborhood

1 Introduction and notation

We use [3] for terminology and notations not defined here and consider simple undirected graphs only.

Let $G = (V, E)$ be a graph. An *edge coloring* of G is a function $C : E \to N$(N is the set of nonnegative integers). If G is assigned such a coloring C, then we say that G is an *edge colored graph*, or simply *colored graph*. Denote by (G, C) the graph G together with the coloring C and by $C(e)$ the *color* of the edge $e \in E$. For a subgraph H of G, let $C(H) = \{C(e) : e \in E(H)\}$.

A subgraph H of G is called *heterochromatic*, or *rainbow*, or *colorful* if its any two edges have different colors. There are many publications studying heterochromatic subgraphs. Very often the subgraphs considered are paths, cycles, trees, etc. The heterochromatic hamiltonian cycle or path problems were studied by Hahn and Thomassen(see [9]), Rödl and Winkler(see [7]), Frieze and Reed, Albert,Frieze and Reed (see [1]), and H. Chen and X.L. Li (see [5]). For more references, see [2, 6, 9].

For an uncolored graph the following theorems are well known in matching theory and have been widely used.

Theorem 1 [10]. Let G be a bipartite graph with bipartition (X, Y). Then G contains a matching that saturates every vertex of X if and only if $|N(S)| \geq |S|$ for all $S \subseteq X$.

Theorem 2 [3]. A bipartite graph G has a perfect matching if and only if $|N(S)| \geq |S|$ for all $S \subseteq V$.

A matching is *heterochromatic* if any two edges of it have different colors. Unlike uncolored matchings for which the maximum matching problem is solvable in polynomial time (see [12]), the maximum heterochromatic matching problem is NP-complete, even for bipartite graphs (see [8]). Heterochromatic matchings have been studied for example in [11] in which by defining $N_c(S)$ (see the definition below) Hu and Li gave some sufficient conditions for the existence of perfect heterochromatic matchings in colored graphs.

Let (G, C) be a colored graph. For a vertex v of G, let $CN(v) = \{C(e) : e \text{ is incident with } v\}$ and $CN(S) = \cup_{v \in S} CN(v)$ for $S \subseteq V$. For $S \in V(G)$, denote $N_c(S)$ as one of the minimum set(s) W satisfying $W \subseteq N(S) \backslash S$ and $[CN(S) \backslash C(G[S])] \subseteq CN(W)$.

Theorem 3[11]. Let (B, C) be a colored bipartite graph with bipartition X, Y. Then, B contains a heterochromatic matching that saturates every vertex in X, if $|N_c(S)| \geq |S|$, for all $S \subseteq X$.

Theorem 4[11]. A colored graph (G, C) has a perfect heterochromatic matching, if

(1) $o(G - S) \leq |S|$, where $o(G - S)$ denotes the number of odd components in the remaining graph $G - S$, and

(2) $|N_c(S)| \geq |S|$ for all $S \subseteq V$ such that $0 \leq |S| \leq \frac{|G|}{2}$ and $|N(S) \backslash S| \geq |S|$.

We define a maximum color neighborhood and study heterochromatic matchings in colored bipartite graphs under a new condition related to maximum color neighborhoods of subsets of vertices.

Let (G, C) be a colored bipartite graph with bipartition (X, Y). For a vertex set $S \subseteq X$ or Y, a *color neighbourhood* of S is defined as a set $T \subseteq N(S)$ such that there are $|T|$ edges between S and T that are adjacent to distinct vertices of T and have distinct colors. A *maximum color neighborhood* $N^c(S)$ is a color neighborhood of S with maximum size. Given a set S and a color neighborhood T of S, denote by $C(S, T)$ a set of $|T|$ distinct colors on the $|T|$

edges between S and distinct vertices of T. Note that there might be more than one such set $C(S,T)$. If there is no ambiguity, let $C(S,T)$ be a fixed color set in the following.

Let M be a heterochromatic matching of G, we denote $b_M = \{c: c \in C(M)$ and there exists an edge $e \in E(G - V_M)$ such that $C(e) = c\}$ and denote by $(X_M \cup Y_M)$ with $X_M \subseteq X$, $Y_M \subseteq Y$, the set of vertices that are incident with the edges in M.

For a given heterochromatic matching M, an *alternating* 4-cycle AC_M is a cycle $\{xy, yx', x'y', y'x\}$ such that $C(xy) = C(x'y')$ and $C(xy') = C(x'y) \notin C(M)$, in which $xy \in E(M)$, $x'y' \in E(G - V_M)$.

The following main results are obtained in this paper.

Theorem 5. Let (G, C) be a colored bipartite graph with bipartition (X, Y) and $|N^c(S)| \geq |S|$ for all $S \subseteq X$, then G has a heterochromatic matching of cardinality at least $\lceil \frac{|X|}{3} \rceil$.

Theorem 6. Let (G, C) be a colored bipartite graph with bipartition (X, Y) and $|X| = |Y| = n$. If $|N^c(S)| \geq |S|$ for all $S \subseteq X$ or $S \subseteq Y$, then G has a heterochromatic matching of cardinality at least $\lceil \frac{3n}{8} \rceil$.

Under the conditions of Theorem 6, the following example shows that the best bound can not be better than $\lceil \frac{n}{2} \rceil$. Let $G = (X, Y)$ with $X = \{x_1, x_1, \cdots, x_{2s}\}$ and $Y = \{y_1, y_2, \cdots, y_{2s}\}$ be a bipartite graph such that $E(G) = \{x_i y_i | i = 1, 2, \cdots, 2s\} \cup \{x_{2i-1} y_{2i} | i = 1, 2, \cdots, s\} \cup \{x_{2i} y_{2i-1} | i = 1, 2, \cdots, s\}$. The edge coloring C of G is given by $C(x_{2i-1} y_{2i-1}) = C(x_{2i} y_{2i}) = 2i - 1$ and $C(x_{2i-1} y_{2i}) = C(x_{2i} y_{2i-1}) = 2i$ for $i = 1, 2, \cdots, s$. Clearly the cardinality of the maximum heterochromatic matching of (G, C) is $s = \lceil \frac{2s}{2} \rceil$. This example shows that the bound in Theorem 6 is not very far away from the best.

2 Proof of Theorem 5

Let M be a maximum heterochromatic matching of G. Put $S = X - X_M$. Let $N^c(S)$ be a maximum color neighborhood of S. And write $N^c(S) = Y_P \cup Y_Q (Y_P \cap Y_Q = \phi)$, where $C(S, Y_P) \cap C(M) = \phi$ and $C(S, Y_Q) \subseteq C(M)$. Clearly $|Y_Q| \le |M|$.

If $Y_P \not\subseteq Y_M$, then there is an edge $e \in E(X - X_M, Y - Y_M)$ and $C(e) \notin C(M)$. Hence $M \cup \{e\}$ is a heterochromatic matching with cardinality $|M| + 1$, contrary to the maximality of M.

So $Y_P \subseteq Y_M$. Since $|N^c(S)| = |Y_P| + |Y_Q| \ge |S|$, it follows that $|M| = |Y_M| \ge |Y_P| \ge |S| - |Y_Q| \ge |X| - |M| - |M|$. This gives $|M| \ge \lceil \frac{|X|}{3} \rceil$. □

3 Proof of Theorem 6

Let M be a maximum heterochromatic matching of G with $t := |M|$ such that $|b_M|$ is maximum. Assume to the contrary that $t < \frac{3n}{8}$.

Suppose that the maximum number of the vertex-disjoint AC_Ms in G is l_1. Clearly $0 \le l_1 \le t$. If $l_1 \ge 1$, assume that the i-th alternating cycle $AC_M^i = \{x_i y_i, y_i x_i^{'}, x_i^{'} y_i^{'}, y_i^{'} x_i\}$ and denote that $C(xy) = C(x_i^{'} y_i^{'}) = c_i \in C(M), C(x_i y_i^{'}) = C(x_i^{'} y_i) = c_i^{'} \notin C(M)$, where $xy \in E(M)$, and $x_i^{'} \in X - X_M, y_i^{'} \in Y - Y_M$.

Denote

$$X_{L_1} = \{x_1^{'}, x_2^{'}, \cdots, x_{l_1}^{'}\}, Y_{L_1} = \{y_1^{'}, y_2^{'}, \cdots, y_{l_1}^{'}\},$$
$$X_{M_{l_1}} = \{x_1, x_2, \cdots, x_{l_1}\} \subseteq X_M,$$
$$Y_{M_{l_1}} = \{y_1, y_2, \cdots, y_{l_1}\} \subseteq Y_M,$$

where $\{x_1 y_1, x_2 y_2, \cdots, x_{l_1} y_{l_1}\} = E(M_{l_1}) \subseteq E(M)$.

We define a procedure named *breeding* as follows.

For $i \ge 1$, if there is an edge $x^i y^i \in E(M - M_{l_1+i-1})$ such that x^i is adjacent with $y^{i'}$, y^i is adjacent with $x^{i'}$ and $C(x^{i'} y^{i'}) = C(x^i y^i)$, where $x^{i'} \in X - X_M - X_{L_1+i-1}, y^{i'} \in Y - Y_M - Y_{L_1+i-1}$; Moreover, if $C(x^i y^{i'}) \in C(M)$, then $C(x^i y^{i'}) \in C(M_{l_1+i-1})$ and if $C(x^{i'} y^i) \in C(M)$, then $C(x^{i'} y^i) \in C(M_{l_1+i-1})$; Then we do the i-th breeding (*step i*) as follows.

Denote $c_{l_1+i} = C(x^i y^i), x_{l_1+i} = x^i, y_{l_1+i} = y^i, x_{l_1+i}^{'} = x^{i'}, y_{l_1+i}^{'} = y^{i'}$. And let $X_{M_{l_1+i}} = X_{M_{l_1+i-1}} \cup \{x_{l_1+i}\}$, $Y_{M_{l_1+i}} = Y_{M_{l_1+i-1}} \cup$

$\{y_{l_1+i}\}$, $X_{L_1+i} = X_{L_1+i-1} \cup \{x^{i'}\}$, $Y_{L_1+i} = Y_{L_1+i-1} \cup \{y^{i'}\}$ and $M_{l_1+i} = M_{l_1+i-1} \cup \{x_{l_1+i}y_{l_1+i}\}$.

Suppose that after k steps, the breeding procedure stops, which means that we can not do the breeding again. Let $l = l_1 + k$ and denote $X_L = X_{L_1+k}, Y_L = Y_{L_1+k}$, $X_{M_l} = X_{M_{l_1+k}}, Y_{M_l} = Y_{M_{l_1+k}}$. Denote $C_l = \{c_1, c_2, \cdots, c_l\}$ and $C_L = \{c : c \in C(AC^i_M)$ and $c \notin C_l, 1 \le i \le l\}$. Clearly $C(M) - C(M_l) = C(M - M_l)$. Suppose $|C_L| = l'$.

Claim 1. $l' \le l$.

Proof. In each step $i (i \ge 1)$, we conclude that $C(x^i y^{i'}) \in C(M)$ or $C(x^{i'} y^i) \in C(M)$. Otherwise if $C(x^i y^{i'}) \ne C(x^{i'} y^i)$, then $M^1 = M \cup \{x^i y^{i'}, x^{i'} y^i\} - \{x^i y^i\}$ is a heterochromatic matching with size more than $|M|$, a contradiction. If $C(x^i y^{i'}) = C(x^{i'} y^i)$, then $\{x^i y^i, y^i x^{i'}, x^{i'} y^{i'}, y^{i'} x^i\}$ is an AC_M, it follows that the number of the vertex disjoint AC_Ms is at least $l_1 + 1$, a contradiction. Thus, $l' \le l$. □

Now denote $S_x = X - X_M - X_L$ and $S_y = Y - Y_M - Y_L$.

Claim 2. (2.1). For any vertex $v_x \in S_x$, if there exists an edge $v_x v_y$ such that $C(v_x v_y) \notin C(M - M_l)$, then $v_y \in Y_M - Y_{M_l}$.

(2.2). For any vertex $y \in S_y$, if there exists an edge xy such that $C(xy) \notin C(M - M_l)$, then $x \in X_M - X_{M_l}$.

Proof. By the symmetry, we only prove (2.1). Suppose, to the contrary, there exists an edge $v_x v_y$ such that $C(v_x v_y) \notin C(M - M_l)$, in which $v_x \in S_x$ and $v_y \notin Y_M - Y_{M_l}$. We distinguish the following three cases, and in each case we get a heterochromatic matching with cardinality more than $|M|$, which is a contradiction.

Case 1. $C(v_x v_y) \notin C(M)$. Then let

$$M^1 = \begin{cases} M \cup \{v_x v_y\} & v_y \in Y - Y_M; \\ M \cup \{v_x v_y, x_i' y_i'\} - \{x_i y_i\} & v_y \in Y_{M_l}, \text{w.l.o.g, suppose } v_y = y_i. \end{cases}$$

Case 2. $C(v_x v_y) \in C_l$ and $v_y \notin Y_{M_l}$. W.o.l.g, suppose $C(v_x v_y) = c_i, 1 \le i \le l$.

If $C(x_i'y_i) \notin C(M)$, then let $M^1 = M \cup \{v_xv_y, x_i'y_i\} - \{x_iy_i\}$. Otherwise, by the breeding rule, we have that $C(x_i'y_i) = c_{j_1}$ with $j_1 < i$. And if $C(x_{j_1}'y_{j_1}) \notin C(M)$, then let $M^1 = M \cup \{v_xv_y, x_i'y_i, x_{j_1}y_{j_1}\} - \{x_iy_i, x_{j_1}y_{j_1}\}$. Otherwise, we have that $C(x_{j_1}'y_{j_1}) = c_{j_2}$ with $j_2 < j_1$. Then consider $C(x_{j_2}'y_{j_2}), \cdots$. By successively doing this process, we can find $j_k(1 \le j_k < j_{k-1} < \cdots < j_1 < i)$ such that $C(x_{j_k}'y_{j_k}) \notin C(M)$, then let $M^1 = M \cup \{v_xv_y, x_i'y_i, x_{j_1}'y_{j_1}, \cdots, x_{j_{k-1}}'y_{j_{k-1}}, x_{j_k}'y_{j_k}\} - \{x_iy_i, x_{j_1}y_{j_1}, \cdots, x_{j_k}y_{j_k}\}$.

Case 3. $C(v_xv_y) \in C_l$ and $v_y \in Y_{M_l}$. Suppose $v_y = y_j, 1 \le j \le l$. Let $M' = M \cup \{x_j'y_j'\} - \{x_jy_j\}$, then use the same method as in Case 2, we can get a heterochromatic matching M^1 with size more than $|M|$. □

If $b_M \backslash C_l \neq \phi$, w.o.l.g, we assume that $b_M \backslash C_l = \{c_{l+1}, c_{l+2}, \cdots, c_{l+d}\}$ and moreover assume that $C(x_{l+i}y_{l+i}) = c_{l+i}, 1 \le i \le d$. And denote $X_{M_d} = \{x_{l+1}, x_{l+2}, \cdots, x_{l+d}\}$, $Y_{M_d} = \{y_{l+1}, y_{l+2}, \cdots, y_{l+d}\}$ and $M_d = \{x_{l+1}y_{l+1}, \cdots, x_{l+d}y_{l+d}\}$.

Claim 3. If there exists an edge v_xy_i such that $C(v_xy_i) \notin C(M - M_l)$, where $v_x \in S_x$ and $y_i \in Y_{M_d}$, then the following two conclusions hold.
(3.1). The edges in $E(G - V_M)$ with color c_i are incident with v_x.
(3.2). If there is an edge x_iv_y, where $v_y \in S_y$, then $C(x_iv_y) \in C(M - M_l)$.

Proof. Since $y_i \in Y_{M_d}$, by the definition of Y_{M_d}, there exists an edge $e \in E(G - V_M)$ such that $C(e) = C(x_iy_i)$. If e is not incident with v_x, we distinguish the following two cases.

Case 1. $C(v_xy_i) \notin C(M)$. Then let $M^1 = M \cup \{e, v_xy_i\} - \{x_iy_i\}$. Then M^1 is a heterochromatic matching M^1 with $|M^1| > |M|$, a contradiction.

Case 2. $C(v_xy_i) \in C_l$. Suppose $C(v_xv_y) = c_j, 1 \le j \le l$. Similarly as in Claim 2, we can find j_k such that $C(x_{j_k}y_{j_k}')$ (or $C(x_{j_k}'y_{j_k})) \notin C(M)$, then let $M^1 = M \cup \{e, v_xy_i, x_j'y_j (\text{or } x_jy_j'), x_{j_1}'y_{j_1} (\text{or } x_{j_1}y_{j_1}'), \cdots, x_{j_k}'y_{j_k} (\text{or } x_{j_k}y_{j_k}')\} - \{x_iy_i, x_jy_j, x_{j_1}y_{j_1}, \cdots, x_{j_k}y_{j_k}\}$. Here if e is incident with x_{j_r}', we choose the edge $x_{j_r}y_{j_r}'$, where $0 \le r \le k$ and $j_0 = j$. Thus, we get a heterochromatic matching

M^1 such that $|M^1| > |M|+1$, a contradiction. This completes the proof of (3.1).

For (3.2), suppose, to the contrary, $C(x_iv_y) \notin C(M - M_l)$. Similarly as in (3.1), the edges in $E(G - V_M)$ with color $C(x_iy_i)$ are incident with v_y. Thus $C(v_xv_y) = C(x_iy_i)$.

If $C(x_iv_y), C(v_xy_i) \notin C(M)$ and $C(x_iv_y) \neq C(v_xy_i)$, then $M^1 = M \cup \{x_iv_y, v_xy_i\} - \{x_iy_i\}$ is a heterochromatic matching such that $|M^1| > |M|$, a contradiction.

If $C(x_iv_y), C(v_xy_i) \notin C(M)$ and $C(x_iv_y) = C(v_xy_i)$, then $\{x_iy_i, y_iv_x, v_xv_y, v_yx_i\}$ is an AC_M. Thus the number of the vertex disjoint AC_M is at least $l_1 + 1$, a contradiction.

Thus we conclude that $C(x_iv_y) \in C_l$ and $C(v_xy_i) \notin C(M)$ or $C(v_xy_i) \in C_l$ and $C(x_iv_y) \notin C(M)$ or $C(x_iv_y), C(v_xy_i) \in C_l$, it follows that we can continue the breeding procedure, a contradiction. This completes the proof of (3.2). □

Let $Y_{M_{d_1}} = \{v : v \in Y_{M_d}$ and v is adjacent with a vertex $v_x \in S_x$ such that $C(v_xv) \notin C(M - M_l)\}$ and $|Y_{M_{d_1}}| = d_1$. And define $X_{d_1'} = \{v : v \in S_x$ and v is adjacent with a vertex $v_y \in Y_{M_d}$ such that $C(vv_y) \notin C(M - M_l)\}$ and $|X_{d_1'}| = d_1'$. By Claim 3, $d_1' \leq d_1$.

For a vertex $v \in V(G - V_M)$, let $b_{M_d}(v) = \{c : c \in C(M_d)$ and there exists an edge $e \in E(G - V_M)$ incident with v such that $C(e) = c\}$. And for a subset $V_1 \subseteq V(G - V_M)$, denote $b_{M_d}(V_1) = \{b_{M_d}(v) : v \in V_1\}$. Let Y' denote the set $V \subseteq S_y$ with minimum size satisfying $b_{M_d}(V) = b_{M_d}(S_y)$. Clearly, $|Y'| \leq d$. Now denote $Y'' = Y - Y_M - Y_L - Y'$.

Let $N^c(Y'')$ be a maximum color neighborhood of Y''. And assume that $N^c(Y'') = X_P \cup X_Q (X_P \cap X_Q = \phi)$, where $C(Y'', X_P) \cap (C(M) \cup C_L) = \phi$ and $C(Y'', X_Q) \subseteq C(M) \cup C_L$. Clearly $|X_Q| \leq t + l'$, then $|X_P| \geq |Y''| - |X_Q| \geq n - t - l - d - (t + l') \geq n - 2t - 2l - d$.

Let $N^c(S_x)$ be a maximum color neighborhood of S_x. And assume that $N^c(S_x) = Y_P \cup Y_Q (Y_P \cap Y_Q = \phi)$, where $C(S_x, Y_P) \cap$

$C(M-M_l)=\phi$ and $C(S_x,Y_Q)\subseteq C(M-M_l)$. Clearly $|Y_Q|\le t-l$, then $|Y_P|\ge |S_x|-|Y_Q|\ge n-t-l-(t-l)\ge n-2t$.

Claim 4. $d_1\ge 2n-5t-l$.

Proof. By Claim 2, we know that $X_P\subseteq X_M-X_{M_l}$ and $Y_P\subseteq Y_M-Y_{M_l}$. By the definition of Y'' and Claim 3, it holds that $X_P\cap X_{M_d}=\phi$. Thus $X_P\subseteq X_M-X_{M_l}-X_{M_d}$. By Claim 3 and the definition of $Y_{M_{d_1}}$, we conclude that $Y_P\cap(Y_{M_d}\backslash Y_{M_{d_1}})=\phi$. So we have that $Y_P\subseteq Y_M-Y_{M_l}-(Y_{M_d}\backslash Y_{M_{d_1}})$.

If $|X_P|+|Y_P|>|Y_M-Y_{M_l}-(Y_{M_d}\backslash Y_{M_{d_1}})|$, then there is an edge $x_jy_j\in E(M-M_l-M_d)$ such that $x_j\in X_P$ and $y_j\in Y_P$. Then there exist edges x_jv_y, v_xy_j such that $C(x_jv_y)\notin C(M)\cup C_L$ and $C(v_xy_j)\notin C(M-M_l)$, in which $v_x\in S_x$ and $v_y\in Y''$.

If $C(v_xy_j)\in C_l$, suppose that $C(v_xy_j)=c_i(1\le i\le l)$. Use the same method as in Claim 2, we can find j_k such that $C(x_{j_k}y'_{j_k})\notin C(M)$, then let $M^1=M\cup\{x_jv_y,v_xy_j,x'_iy_i,x'_{j_1}y_{j_1},\cdots,,x'_{j_k}y_{j_k}\}-\{x_jy_j,x_iy_i,x_{j_1}y_{j_1},\cdots,x_{j_k}y_{j_k}\}$. Thus M^1 is a heterochromatic matching such that $|M^1|>|M|$, which is a contradiction.

If $C(v_xy_j)\notin C_l$ and $C(v_xy_j)\neq C(x_jv_y)$. Then $M^1=M\cup\{v_xy_j,x_jv_y\}-\{x_jy_j\}$ is a heterochromatic matching with cardinality $|M|+1$, a contradiction.

So we conclude that $C(v_xy_j)=C(x_jv_y)$. Then let $M^1=M\cup\{x_jv_y\}-\{x_jy_j\}$. It follows that $|M^1|=|M|$ and $|b_{M^1}|>|b_M|$, which contradicts with the choice of M.

Thus $|X_P|+|Y_P|\le|Y_M-Y_{M_l}-(Y_{M_d}\backslash Y_{M_{d_1}})|$. It follows that $n-2t-2l-d+n-2t\le t-l-(d-d_1)$, then $d_1\ge 2n-5t-l$. □

Similarly, we define $X_{M_{d_2}}=\{v: v\in X_{M_d}$ and v is adjacent with a vertex $v_y\in S_y$ such that $C(v_yv)\notin C(M-M_l)\}$ and $|X_{M_{d_2}}|=d_2$. Similarly as Claim 4, we have that $d_2\ge 2n-5t-l$. By Claim 3, we know that $d_1+d_2\le d$.

Let $S'_x=X-X_M-X_L-X_{d'_1}$ and $N^c(S'_x)$ be a maximum color neighborhood of S'_x. And assume that $N^c(S'_x)=Y'_P\cup Y'_Q(Y'_P\cap$

$Y'_Q = \phi$), in which $C(S'_x, Y'_P) \cap C(M - M_l) = \phi$ and $C(S'_x, Y'_Q) \subseteq C(M - M_l)$. Clearly, $|Y'_Q| \leq t - l$, then $|Y'_P| \geq |S'_x| - |Y'_Q| \geq n - t - l - (t - l) - d'_1 = n - 2t - d'_1 \geq n - 2t - d_1$.

Claim 5. $|Y_M - Y_{M_l} - Y_{M_d}| \geq n - 2t - d_1$.

Proof. By Claim 2, it holds that $Y'_P \subseteq Y_M - Y_{M_l}$. And by Claim 3 and the definition of S'_x, it holds that $Y'_P \cap Y_{M_d} = \phi$. Then it follows that $Y'_P \subseteq Y_M - Y_{M_l} - Y_{M_d}$. Thus we have that $|Y_M - Y_{M_l} - Y_{M_d}| \geq |Y'_P| \geq n - 2t - d_1$. □

Now we have that

$$\begin{aligned} t &\geq |Y_M - Y_{M_l} - Y_{M_d}| + |Y_{M_d}| + |Y_{M_l}| \\ &\geq n - 2t - d_1 + d + l \\ &\geq n - 2t + 2n - 5t - l + l \\ &\geq 3n - 7t. \end{aligned}$$

Thus $t \geq \frac{3n}{8}$, which is a contradiction. The proof of Theorem 6 is complete.

References

[1] M. Albert, A. Frieze and B. Reed, Multicolored Hamilton cycles, *Electronic J. Combin.* **2** (1995), Research Paper R10.

[2] N. Alon, T. Jiang, Z. Miller and D. Pritikin, Properly colored subgraphs and rainbow subgraphs in edge-colored graphs with local constraints, *Random Struct. Algorithms* **23** (2003), No. 4, 409–433.

[3] J.A. Bondy and U.S.R. Murty. Graph Theory with Applications, Macmillan Press [M], New York, 1976.

[4] H.J. Broersma, X. Li, G. Woegingerr and S. Zhang, Paths and cycles in colored graphs, *Australian J. Combin.* **31** (2005), 297–309.

[5] H. Chen and X. Li, Long heterochromatic paths in edge-colored graphs, *The Electronic J. Combin.* **12(1)** (2005), Research Paper R33.

[6] P. Erdös and Zs. Tuza, Rainbow subgraphs in edge-colorings of complete graphs, *Ann. Discrete Math.* **55** (1993), 81–83.

[7] A.M. Frieze and B.A. Reed, Polychromatic Hamilton cycles, *Discrete Math.* **118** (1993), 69–74.

[8] M.R. Garey and D.S. Johnson, Comuters and Intractability, Freeman, New York, 1979, Pages 203. GT55: Multiple Choice Matching Problem.

[9] G. Hahn and C. Thomassen, Path and cycle sub-Ramsey numbers and edge-coloring conjecture, *Discrete Math.* **62(1)** (1986), 29–33.

[10] P. Hall, On representatives of subsets, *J. London Math. Soc.* **10** (1935), 26–30.

[11] L. Hu and X. Li, Sufficient conditions for the existence of perfect heterochromatic matcings in colored graphs, arXiv:math.Co/051160v1 24Nov 2005.

[12] E.L. Lawler, Combinatorial Optimization: Networks and Matroids, Holt, Rinehart and Winston, New York, 1976.