λ-Designs with Two Block Sizes II

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Abstract

A λ-design on v points is a set of v subsets (blocks) of a v-set such that any two distinct blocks meet in exactly λ points and not all of the blocks have the same size. Ryser's and Woodall's λ-design conjecture states that all λ-designs can be obtained from symmetric designs by a complementation procedure. In a previous paper, the author established feasibility criteria for the existence of λ-designs with two block sizes in the form of integrality conditions, equations, inequalities, and Diophantine equations involving various parameters of the designs. In that paper, these criteria and a computer were used to prove that the λ-design conjecture is true for all λ-designs with two block sizes with $\lambda \leq 90$ and $\lambda \neq 45$. In this paper, we extend these results and prove that the λ-design conjecture is also true for all λ-designs with two block sizes with $\lambda = 45$ or $91 \leq \lambda < 150$.

1 Introduction

Definition 1.1. Given integers λ and v, $0 < \lambda < v$, a λ-*design* on v points is a pair $(X, \mathcal{B})$, where X is a set of cardinality v whose elements are called *points* and $\mathcal{B}$ is a set of v subsets of X whose elements are called *blocks*, such that

(i) for all blocks $A, B \in \mathcal{B}$, $A \neq B$, $|A \cap B| = \lambda$, and

(ii) there exist blocks $A, B \in \mathcal{B}$ with $|A| \neq |B|$.

λ-designs were first defined by Ryser [15], [16] and Woodall [22]. The only known examples of λ-designs are obtained from symmetric designs by the following complementation procedure. Let $(X, \mathcal{A})$ be a symmetric (v, k, μ)-design with $\mu \neq k/2$ and fix a block $A \in \mathcal{A}$. Put $\mathcal{B} = \{A\} \cup \{A \Delta B :$

$B \in \mathcal{A}, B \neq A\}$, where Δ denotes the symmetric difference of sets (we refer to this procedure as *complementing* with respect to the block A). Then an elementary counting argument shows that $(X, \mathcal{B})$ is a λ-design on v points with one block of size k and $v-1$ blocks of size 2λ, where $\lambda = k - \mu$. Any λ-design obtained in this manner is called a *type*-1 λ-design.

The *λ-design conjecture* of Ryser [15], [16] and Woodall [22] states that all λ-designs are type-1. The conjecture was proven for $\lambda = 1$ by deBruijn and Erdős [4], for $\lambda = 2$ by Ryser [15], for $3 \leq \lambda \leq 9$ by Bridges and Kramer [1], [2], [13], for $\lambda = 10$ by Seress [18], for $\lambda = 14$ by Tsaur [3], and for $\lambda \leq 34$ by Weisz [21]. S. S. Shrikhande and Singhi [20] proved the conjecture for prime λ and Seress [19] proved it when λ is twice a prime.

Investigating the conjecture as a function of v rather than λ, Ionin and M. S. Shrikhande [10], [11] proved the conjecture for $v = p+1, 2p+1, 3p+1$, and $4p+1$, where p is any prime, Hein [9] proved it for $v = 5p+1$, where $p \not\equiv 2$ or $8 \pmod{15}$ is prime, and Fiala [5], [6] proved it for $v = 6p+1$, p any prime, and $v = 8p+1$, $p \equiv 1$ or $7 \pmod 8$ prime.

The reader interested in λ-designs should consult the last chapter of [12].

2 Preliminary results

In this section, we collect some results on λ-designs that we will need later. First, we have the following definition.

Definition 2.1. Given a λ-design $(X, \mathcal{B})$ and a point $x \in X$, the *replication number* of x, denoted by r_x, is the number of blocks $A \in \mathcal{B}$ such that $x \in A$.

Ryser [15] and Woodall [22] independently proved the following theorem concerning replication numbers of λ-designs.

Theorem 2.2. *If $(X, \mathcal{B})$ is a λ-design on v points, then there exist integers $r_1 > r_2 > 1$ such that every point has replication number r_1 or r_2 and $r_1 + r_2 = v + 1$.*

Let $(X, \mathcal{B})$ be a λ-design on v points. Then Theorem 2.2 implies that every point has replication number r_1 or r_2 for some integers $r_1 > r_2$. Therefore, the set X is partitioned into two subsets, E_1 and E_2, of points having replication numbers r_1 and r_2, respectively. Let $|E_1| = e_1$ and $|E_2| = e_2$. Additionally, each block A is partitioned into two subsets, $A' = A \cap E_1$ and $A^* = A \cap E_2$, of points having replication number r_1 and r_2, respectively.

Theorem 2.3. *Let $(X, \mathcal{B})$ be a λ-design on v points with replication numbers r_1 and r_2 and block sizes k_1 and k_2. Then $|A'|$ and $|A^*|$ depend only* *on $|A|$ and so we may denote $|A'|$ and $|A^*|$ by k_i' and k_i^*, respectively, for $|A| = k_i$, $i = 1, 2$. Moreover,*

$$e_1 = \frac{\lambda(v-1) - r_2(r_2-1)}{r_1 - r_2},$$

$$e_2 = \frac{r_1(r_1-1) - \lambda(v-1)}{r_1 - r_2},$$

$$k_1' = \frac{\lambda(v-1) - k_1(r_2-1)}{r_1 - r_2},$$

$$k_1^* = \frac{k_1(r_1-1) - \lambda(v-1)}{r_1 - r_2},$$

$$k_2' = \frac{\lambda(v-1) - k_2(r_2-1)}{r_1 - r_2},$$

and

$$k_2^* = \frac{k_2(r_1-1) - \lambda(v-1)}{r_1 - r_2}.$$

Let $(X, \mathcal{B})$ be a λ design with block sizes k_1 and k_2. Given a point x, denote by r_x' the number of blocks of size k_1 that contain x and denote by r_x^* the number of blocks of size k_2 that contain x.

Theorem 2.4. [7] *Let $(X, \mathcal{B})$ be a λ-design on v points with replication numbers r_1 and r_2, $r_1 > r_2$, v_1 blocks of size k_1, and v_2 blocks of size k_2. Then r_x' and r_x^* depend only on r_x. In addition, if we denote the number of blocks of size k_i, $i = 1, 2$, that contain a fixed point of replication number r_j, $j = 1, 2$, by r_1', r_1^*, r_2', and r_2^*, respectively, then*

$$v_1 = \frac{(k_1-\lambda)[(k_2+\lambda(v-1))(r_1-1)(r_2-1) - \lambda(k_2-\lambda)(v-1)^2]}{\lambda(k_1-k_2)(r_1-1)(r_2-1)},$$

$$v_2 = \frac{(k_2-\lambda)[\lambda(k_1-\lambda)(v-1)^2 - (k_1+\lambda(v-1))(r_1-1)(r_2-1)]}{\lambda(k_1-k_2)(r_1-1)(r_2-1)},$$

$$r_1' = \frac{(k_1-\lambda)[r_1(r_2-1) - (k_2-\lambda)(v-1)]}{(k_1-k_2)(r_2-1)},$$

$$r_1^* = \frac{(k_2-\lambda)[(k_1-\lambda)(v-1) - r_1(r_2-1)]}{(k_1-k_2)(r_2-1)},$$

$$r_2' = \frac{(k_1-\lambda)[r_2(r_1-1) - (k_2-\lambda)(v-1)]}{(k_1-k_2)(r_1-1)},$$

and

$$r_2^* = \frac{(k_2-\lambda)[(k_1-\lambda)(v-1) - r_2(r_1-1)]}{(k_1-k_2)(r_1-1)}.$$

Given two distinct points x and y, denote by r_{xy} the number of blocks containing both x and y. Also, denote by r'_{xy} the number of blocks of size k_1 that contain both x and y and denote by r^*_{xy} the number of blocks of size k_2 that contain x and y.

Theorem 2.5. [7] *Let $(X, \mathcal{B})$ be a λ-design on v points with replication numbers r_1 and r_2, $r_1 > r_2$, and block sizes k_1 and k_2. Let $x, y \in X$, $x \neq y$. Then*

$$r'_{xy} = \frac{(k_1 - \lambda)[r_{xy}(r_2 - 1) - (k_2 - \lambda)(r_1 - 1)]}{(k_1 - k_2)(r_2 - 1)}$$

and

$$r^*_{xy} = \frac{(k_2 - \lambda)[(k_1 - \lambda)(r_1 - 1) - r_{xy}(r_2 - 1)]}{(k_1 - k_2)(r_2 - 1)}$$

if $x, y \in E_1$,

$$r'_{xy} = \frac{(k_1 - \lambda)(r_{xy} - (k_2 - \lambda))}{k_1 - k_2}$$

and

$$r^*_{xy} = \frac{(k_2 - \lambda)(k_1 - \lambda - r_{xy})}{k_1 - k_2}$$

if $x \in E_1$ and $y \in E_2$, and

$$r'_{xy} = \frac{(k_1 - \lambda)[r_{xy}(r_1 - 1) - (k_2 - \lambda)(r_2 - 1)]}{(k_1 - k_2)(r_1 - 1)}$$

and

$$r^*_{xy} = \frac{(k_2 - \lambda)[(k_1 - \lambda)(r_2 - 1) - r_{xy}(r_1 - 1)]}{(k_1 - k_2)(r_1 - 1)}$$

if $x, y \in E_2$.

Theorem 2.6. [17] *Let $(X, \mathcal{B})$ be a λ-design on v points with replication numbers r_1 and r_2, $r_1 > r_2$. Let $x_1 \in E_1$ and $x_2 \in E_2$. Then more than half of the numbers in $\{r_{x_1 y_2} : y_2 \in E_2\}$ are equal to $\lceil r_1(r_2 - 1)/(v - 1) \rceil$ and more than half of the numbers in $\{r_{x_2 y_2} : y_2 \in E_2 \setminus \{x_2\}\}$ are equal to $\lceil r_2(r_2 - 1)/(v - 1) \rceil$.*

Remark 2.7. Even though r_{xy} is not necessarily constant for constant r_x and r_y, for convenience we set $r_{12} = \lceil r_1(r_2 - 1)/(v - 1) \rceil$ and $r_{22} = \lceil r_2(r_2 - 1)/(v - 1) \rceil$.

3 Sums over pairs of points

In this section, we find the values of various sums over sets of pairs of points in λ-designs with two block sizes. These results can be useful in ruling out potential designs once we have sets of all possible r'_{xy} and r^*_{xy} values.

Theorem 3.1. *Let $(X, \mathcal{B})$ be a λ-design on v points with replication numbers r_1 and r_2, $r_1 > r_2$, v_1 blocks of size k_1, and v_2 blocks of size k_2. Then*

$$\sum_{x,y \in E_1, x \neq y} r'_{xy} = v_1 k'_1 (k'_1 - 1), \tag{1}$$

$$\sum_{x,y \in E_1, x \neq y} r^*_{xy} = v_2 k'_2 (k'_2 - 1), \tag{2}$$

$$\sum_{x \in E_1, y \in E_2} r'_{xy} = v_1 k'_1 k^*_1, \tag{3}$$

$$\sum_{x \in E_1, y \in E_2} r^*_{xy} = v_2 k'_2 k^*_2, \tag{4}$$

$$\sum_{x,y \in E_2, x \neq y} r'_{xy} = v_1 k^*_1 (k^*_1 - 1), \tag{5}$$

$$\sum_{x,y \in E_2, x \neq y} r^*_{xy} = v_2 k^*_2 (k^*_2 - 1), \tag{6}$$

$$\sum_{x,y \in X, x \neq y} r'_{xy} = v_1 k_1 (k_1 - 1), \tag{7}$$

$$\sum_{x,y \in X, x \neq y} r^*_{xy} = v_2 k_2 (k_2 - 1), \tag{8}$$

$$\sum_{x,y \in X, x \neq y} (r'_{xy})^2 = \lambda v_1 (\lambda - 1)(v_1 - 1) - v_1 k_1 (k_1 - 1), \tag{9}$$

$$\sum_{x,y \in X, x \neq y} r'_{xy} r^*_{xy} = \lambda v_1 v_2 (\lambda - 1), \tag{10}$$

and

$$\sum_{x,y \in X, x \neq y} (r^*_{xy})^2 = \lambda v_2 (\lambda - 1)(v_2 - 1) - v_2 k_2 (k_2 - 1). \tag{11}$$

Proof. For $i, j, l = 1, 2$, we count in two different ways the number of triples $(x, y, A) \in E_i \times E_j \times \mathcal{B}$ such that $x \neq y$, $|A| = k_l$, and $x, y \in A$. We obtain (1), (2), (3), (4), (5), and (6).

For $i = 1, 2$, we count in two different ways the number of triples $(x, y, A) \in X^2 \times \mathcal{B}$ such that $x \neq y$, $|A| = k_i$, and $x, y \in A$. We obtain (7) and (8).

For $i, j = 1, 2$, we count in two different ways the number of quadruples $(x, y, A, B) \in X^2 \times \mathcal{B}^2$ such that $x \neq y$, $A \neq B$, $|A| = k_i$, $|B| = k_j$, and $x, y \in A \cap B$. Using (7) and (8), we obtain (9), (10), and (11). □

Theorem 3.2. *Let $(X, \mathcal{B})$ be a λ-design on v points with replication numbers r_1 and r_2, $r_1 > r_2$. Let $x_1 \in E_1$ and $x_2 \in E_2$. Then more than half of the numbers in $\{r'_{x_1y_2} : y_2 \in E_2\}$ are equal to*

$$\frac{(k_1 - \lambda)(r_{12} - (k_2 - \lambda))}{k_1 - k_2}, \tag{12}$$

more than half of the numbers in $\{r^*_{x_1y_2} : y_2 \in E_2\}$ are equal to

$$\frac{(k_2 - \lambda)(k_1 - \lambda - r_{12})}{k_1 - k_2}, \tag{13}$$

more than half of the numbers in $\{r'_{x_2y_2} : y_2 \in E_2 \setminus \{x_2\}\}$ are equal to

$$\frac{(k_1 - \lambda)[r_{22}(r_1 - 1) - (k_2 - \lambda)(r_2 - 1)]}{(k_1 - k_2)(r_1 - 1)}, \tag{14}$$

and more than half of the numbers in $\{r^*_{x_2y_2} : y_2 \in E_2 \setminus \{x_2\}\}$ are equal to

$$\frac{(k_2 - \lambda)[(k_1 - \lambda)(r_2 - 1) - r_{22}(r_1 - 1)]}{(k_1 - k_2)(r_1 - 1)}. \tag{15}$$

Proof. Apply Theorems 2.5 and 2.6. □

Remark 3.3. Even though r'_{xy} and r^*_{xy} depend on r_{xy} and not just on r_x and r_y, for convenience we denote expressions (12), (13), (14), and (15) by r'_{12}, r^*_{12}, r'_{22}, and r^*_{22}, respectively.

4 Eigenvalues

In this section, we establish some eigenvalue interlacing results that can help in ruling out possible r'_{xy} and r^*_{xy} values. First, we need the following.

Definition 4.1. Given a real symmetric $n \times n$ matrix A, we will denote the eigenvalues of A (which must be real) by $\lambda_1(A) \geq \ldots \geq \lambda_n(A)$. If B is a $m \times m$ matrix with $m \leq n$, then we say that the eigenvalues of B *interlace* the eigenvalues of A if B has only real eigenvalues and if $\lambda_i(A) \geq \lambda_i(B) \geq \lambda_{n-m+i}(A)$ for $i = 1, \ldots, m$.

Theorem 4.2. [8] *Let A be a real symmetric $n \times n$ matrix partitioned as follows:*

$$A = \begin{pmatrix} A_{1,1} & \cdots & A_{1,m} \\ \vdots & \ddots & \vdots \\ A^T_{1,m} & \cdots & A_{m,m} \end{pmatrix},$$

where $A_{i,i}$ is square for $i = 1, \ldots, m$. Let $b_{i,j}$ be the average row sum of $A_{i,j}$ for $i, j = 1, \ldots, m$. Let $B = (b_{i,j})$ (we refer to B as the quotient matrix of A with respect to the partition). Then the eigenvalues of B interlace the eigenvalues of A.

Let $(X, \mathcal{B})$ be a λ-design on v points with replication numbers r_1 and r_2, $r_1 > r_2$, v_1 blocks of size k_1, and v_2 blocks of size k_2. Let $x, y \in X$, $x \neq y$, such that $r_{xy} \neq 0$. Let N_{xy} be any $(v-2) \times r_{xy}$ matrix whose rows are indexed by the elements of $X \setminus \{x, y\}$ (points in E_1 coming first), whose columns are indexed by the elements of $\mathcal{B}$ that contain x and y (blocks of size k_1 coming first), and whose (z, A) entry is 1 if $z \in A$ and is 0 otherwise. Thus,

$$N_{xy}^T N_{xy} = \begin{pmatrix} (k_1 - \lambda) I_{r'_{xy}} + (\lambda - 2) J_{r'_{xy}} & (\lambda - 2) J_{r'_{xy}, r^*_{xy}} \\ \\ (\lambda - 2) J_{r^*_{xy}, r'_{xy}} & (k_2 - \lambda) I_{r^*_{xy}} + (\lambda - 2) J_{r^*_{xy}} \end{pmatrix},$$

where I_n denotes the $n \times n$ identity matrix, J_n denotes the $n \times n$ matrix of all 1's, and $J_{m,n}$ denotes the $m \times n$ matrix of all 1's. Let

$$A_{xy} = \begin{pmatrix} 0 & N_{xy} \\ \\ N_{xy}^T & 0 \end{pmatrix}.$$

The rows of N_{xy} can be partitioned into $E_1 \setminus \{x, y\}$ and $E_2 \setminus \{x, y\}$ and the columns of N_{xy} can be partitioned into the set of blocks of size k_1 that contain x and y and the set of blocks of size k_2 that contain x and y. This induces a partition of A_{xy} with quotient matrix B_{xy} given by

$$B_{xy} = \begin{pmatrix} 0 & 0 & \frac{r'_{xy}(k'_1 - 2)}{e_1 - 2} & \frac{r^*_{xy}(k'_2 - 2)}{e_1 - 2} \\ \\ 0 & 0 & \frac{r'_{xy} k^*_1}{e_2} & \frac{r^*_{xy} k^*_2}{e_2} \\ \\ k'_1 - 2 & k^*_1 & 0 & 0 \\ \\ k'_2 - 2 & k^*_2 & 0 & 0 \end{pmatrix}$$

if $x, y \in E_1$ and $e_1 > 2$,

$$B_{xy} = \begin{pmatrix} 0 & 0 & \frac{r'_{xy}(k'_1 - 1)}{e_1 - 1} & \frac{r^*_{xy}(k'_2 - 1)}{e_1 - 1} \\ \\ 0 & 0 & \frac{r'_{xy}(k^*_1 - 1)}{e_2 - 1} & \frac{r^*_{xy}(k^*_2 - 1)}{e_2 - 1} \\ \\ k'_1 - 1 & k^*_1 - 1 & 0 & 0 \\ \\ k'_2 - 1 & k^*_2 - 1 & 0 & 0 \end{pmatrix}$$

if $x \in E_1$, $y \in E_2$, and $e_1, e_2 > 1$, and

$$B_{xy} = \begin{pmatrix} 0 & 0 & \frac{r'_{xy}k'_1}{e_1} & \frac{r^*_{xy}k'_2}{e_1} \\ 0 & 0 & \frac{r'_{xy}(k^*_1-2)}{e_2-2} & \frac{r^*_{xy}(k^*_2-2)}{e_2-2} \\ k'_1 & k^*_1-2 & 0 & 0 \\ k'_2 & k^*_2-2 & 0 & 0 \end{pmatrix}$$

if $x, y \in E_2$ and $e_2 > 2$.

Theorem 4.3. *Let $(X, \mathcal{B})$ be a λ-design on v points with replication numbers r_1 and r_2, $r_1 > r_2$, v_1 blocks of size k_1, and v_2 blocks of size k_2. Let $x, y \in X$, $x \neq y$, such that $r_{xy} > 0$. If (i) $x, y \in E_1$ and $e_1 > 2$, (ii) $x \in E_1$, $y \in E_2$, and $e_1, e_2 > 1$, or (iii) $x, y \in E_2$ and $e_2 > 2$, then the eigenvalues of B_{xy} interlace those of A_{xy}.*

Proof. Apply Theorem 4.2. □

5 Small λ

Let $(X, \mathcal{B})$ be a λ-design on v points with replication numbers r_1 and r_2, $r_1 > r_2$, v_1 blocks of size k_1, and v_2 blocks of size k_2, $k_1 > k_2$. Let $\rho = (r_1 - 1)/(r_2 - 1) = x/y$, where $\gcd(x, y) = 1$, and $d = e_1 - r_2$. Then all of the parameters of the design discussed in this paper can be expressed in terms of λ, y, x, d, k_2, and k_1. For instance [15], [22],

$$r_1 = \lambda(\rho + 1) - (d + 1)(\rho - 1)$$

and

$$r_2 = \frac{\lambda(\rho + 1) - d(\rho - 1)}{\rho}.$$

Furthermore, it can be shown [7], [18], [20] that

$$1 \leq y \leq \lambda - 1, \tag{16}$$

$$y + 1 \leq x \leq 2(\lambda - 1), \tag{17}$$

$$\lceil \rho - \lambda(\rho + 1) \rceil \leq d \leq \lfloor \frac{(\lambda - 1)(\rho + 1)}{\rho} \rfloor, \tag{18}$$

$$\max\{\lambda + 1, r_2\} \leq k_2 \leq r_1 - 1, \tag{19}$$

and

$$k_2 + 1 \leq k_1 \leq r_1. \tag{20}$$

Therefore, for a fixed value of $\lambda > 1$, the set of 6-tuples of the form $(\lambda, y, x, d, k_2, k_1)$ corresponding to λ-designs is finite and can be generated using (16), (17), (18), (19), and (20). In [7], using an algorithm implemented in Maple [14], this set of tuples was generated for $12 \le \lambda \le 90$ and the results of [7] were used to eliminate tuples that must correspond to nonexistent or type-1 designs. All tuples except for $(45, 1, 4, 3, 81, 90)$ and $(45, 1, 4, 11, 81, 90)$ were eliminated, proving the λ-design conjecture for λ-designs with two block sizes with $\lambda \le 90$ and $\lambda \ne 45$.

This algorithm was run again for $91 \le \lambda \le 150$. For each tuple that survived the tests, we used the facts that for all different $x, y \in X$, r_{xy}, r'_{xy}, and r^*_{xy} are integers, $0 \le r_{xy} \le \min\{r_x, r_y\}$, $0 \le r'_{xy} \le \min\{r'_x, r'_y, r_{xy}\}$, and $0 \le r^*_{xy} \le \min\{r^*_x, r^*_y, r_{xy}\}$, and Theorems 2.5 and 4.3 to determine sets of all potential r'_{xy} and r^*_{xy} values in each of the three cases $(x, y) \in E_1^2$, $(x, y) \in E_1 \times E_2$, and $(x, y) \in E_2^2$. We then tried to determine if the sums of Theorem 3.1 could be written in accordance with Theorem 3.2 using only said values. If this was not possible, then the tuple could be rejected.

For example, take the tuple $(45, 1, 4, 3, 81, 90)$. For $(x, y) \in E_1^2$, $x \ne y$, it was determined that r'_{xy} must be 120, 125, 130, 135, 140, 145, 150, 155, 160, or 165. For $(x, y) \in E_1 \times E_2$, it was determined that r'_{xy} must be 35, 40, or 45. For $(x, y) \in E_2^2$, $x \ne y$, it was determined that r'_{xy} must be 0, 5, or 10. Let a_i, $i = 1, 2, \ldots, 10$, be the number of pairs $(x, y) \in E_1^2$, $x \ne y$, such that $r'_{xy} = 120 + 5(i-1)$. Let b_i, $i = 1, 2, 3$, be the number of pairs $(x, y) \in E_1 \times E_2$ such that $r'_{xy} = 35 + 5(i-1)$. Let c_i, $i = 1, 2, 3$, be the number of pairs $(x, y) \in E_2^2$, $x \ne y$, such that $r'_{xy} = 5(i-1)$. Then (5), (6), and (9) give us

$$\sum_{i=1}^{3} c_i = e_2(e_2 - 1) = 43472, \tag{21}$$

$$\sum_{i=1}^{3} 5(i-1)c_i = v_1 k_1^*(k_1^* - 1) = 413820, \tag{22}$$

and

$$\sum_{i=1}^{10}[120 + 5(i-1)]^2 a_i + 2\sum_{i=1}^{3}[35 + 5(i-1)]^2 b_i + \sum_{i=1}^{3}[5(i-1)]^2 c_i =$$

$$\lambda v_1(\lambda - 1)(v_1 - 1) - v_1 k_1(k_1 - 1) = 84400470. \tag{23}$$

Solving the linear equations (21), (22), and (23) for c_1, c_2 and c_3 we obtain for c_2

$$c_2 = 576a_1 + 625a_2 + 676a_3 + 729a_4 + 784a_5 + 841a_6 + 900a_7 + 961a_8 + 1024a_9 +$$

$$1089a_{10} + 98b_1 + 128b_2 + 162b_3 - \frac{16052454}{5},$$

which is clearly not an integer.

Since no tuple for $\lambda = 45$ or for $91 \leq \lambda < 150$ survived this additional scrutiny, we have the following result.

Theorem 5.1. *All λ-designs with two block sizes and $\lambda < 150$ are type-1.*

Remark 5.2. For $\lambda = 150$, the tuples $(150, 1, 6, 102, 225, 250)$ and $(150, 1, 6, 138, 200, 225)$ survived the first round of tests based on the results in [7]. The latter was eliminated using the results in Section 3 of the present paper, but the former could not be eliminated by any of the techniques in [7] or in this paper.

References

[1] W. G. Bridges, *Some results on λ-designs*, J. Combin. Theory **8** (1970), 350–360.

[2] W. G. Bridges and E. S. Kramer, *The determination of all λ-designs with $\lambda = 3$*, J. Combin. Theory **8** (1970), 343–349.

[3] W. G. Bridges and T. Tsaur, *Some structural characterizations of λ-designs*, Ars Combin. **44** (1996), 129–135.

[4] N. G. deBruijn and P. Erdős, *On a combinatorial problem*, Indag. Math. **10** (1948), 421–423.

[5] N. C. Fiala, *Every λ-design on $6p + 1$ points is type-1*, in: Codes and Designs, de Gruyter (2002), 109–124.

[6] N. C. Fiala, *λ-designs on $8p + 1$ points*, Ars Combin. **68** (2003), 17–32.

[7] N. C. Fiala, *λ-designs with two block sizes*, Ars Combin., to appear.

[8] W. H. Haemers, *Eigenvalue Techniques in Design and Graph Theory*, Math. Centre Tract **121**, Mathematical Centre, 1980.

[9] D. W. Hein and Y. J. Ionin, *On the λ-design conjecture for $v = 5p + 1$ points*, in: Codes and Designs, de Gruyter (2002), 145–156.

[10] Y. J. Ionin and M. S. Shrikhande, *On the λ-design conjecture*, J. Combin. Theory Ser. A **74** (1996), 100–114.

[11] Y. J. Ionin and M. S. Shrikhande, *λ-designs on $4p + 1$ points*, J. Combin. Math. Combin. Comput. **22** (1996), 135–142.

[12] Y. J. Ionin and M. S. Shrikhande, *Combinatorics of Symmetric Designs*, Cambridge Univ. Press, to appear.

[13] E. S. Kramer, *On λ-designs*, Ph. D. dissertation, Univ. of Michigan, 1969.

[14] Maple 9.5, Waterloo Maple Inc.

[15] H. J. Ryser, *An extension of a theorem of deBruijn and Erdős on combinatorial designs*, J. Algebra **10** (1968), 246–261.

[16] H. J. Ryser, *New types of combinatorial designs*, Actes Congrès Intern. Math., tome 3 (1970), 235–239.

[17] Á. Seress, *Some characterizations of type-*1 *λ-designs*, J. Combin. Theory Ser. A **52** (1989), 288–300.

[18] Á. Seress, *On λ-designs with $\lambda = 2p$*, in: Coding Theory and Design Theory, Part II, Springer-Verlag (1990), 290–303.

[19] Á. Seress, *All lambda-designs with $\lambda = 2p$ are type-*1, Designs, Codes and Cryptography **22** (2001), 5–17.

[20] N. M. Singhi and S. S. Shrikhande, *On the λ-design conjecture*, Util. Math. **9** (1976), 301–318.

[21] I. Weisz, *Lambda-designs with small lambda are type-*1, Ph. D. dissertation, The Ohio State Univ., 1995.

[22] D. R. Woodall, *Square λ-linked designs*, Proc. London Math. Soc. **20** (1970), 669–687.