Isolated toughness and fractional (g, f)-factors of graphs *†

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Abstract: Let G be a graph with vertex set $V(G)$ and edge set $E(G)$, the isolated toughness of G is defined as $I(G) = min\{|S|/i(G-S) : S \subseteq V(G), i(G-S) \geq 2\}$ if G is not complete. Otherwise, set $I(G) = |V(G)| - 1$. Let a and b be positive integers such that $1 \leq a \leq b$, and $g(x)$ and $f(x)$ be positive integral-valued functions defined on $V(G)$ such that $a \leq g(x) \leq f(x) \leq b$. Let $h(e) \in [0, 1]$ be a function defined on $E(G)$, $d_G^h(x) = \sum_{e \in E_x} h(e)$ where $E_x = \{xy : xy \in E(G)\}$. Then $d_G^h(x)$ is called the *fractional degree* of x in G. We call h an *indictor function* if $g(x) \leq d_G^h(x) \leq f(x)$ holds for each $x \in V(G)$. Let $E^h = \{e : e \in E(G), h(e) \neq 0\}$ and G_h be a spanning subgraph of G such that $E(G_h) = E^h$. We call G_h a *fractional* (g, f)*-factor*. The main results in this paper are to present some sufficient conditions about isolated toughness for the existence of fractional (g, f)-factors. If $1 = g(x) < f(x) = b$, this condition can be improved and the improved bound is not only sharpness but also a necessary and sufficient condition for a graph to have fractional $[1, b]$-factor.

Keywords: isolated toughness, fractional (g, f)-factor, fractional $[a, b]$-factor
AMS(2001) Subject Classification: 05C70

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*This work is supported by NSFC of China (Grant number 10471078).

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1 Introduction.

All graphs considered in this paper will be simple undirected graphs. Let $G = (V(G), E(G))$ be a graph, where $V(G)$ and $E(G)$ denote the vertex set and edge set of G, respectively. We use $d_G(x)$ to denote the *degree* of x in G and $\delta(G)$ to denote the minimum vertex degree of G. For any $S \subseteq V(G)$, the subgraph of G induced by S is denoted by $G[S]$. We write $G - S$ for $G[V(G) \setminus S]$. For any $S \subseteq V(G)$, we use $i(G - S)$ to denote the number of *isolated vertices* of $G - S$. A subset I of $V(G)$ is *independent* if no two vertices of I are adjacent. The neighborhood $N_G(S)$ of a subset $S \subseteq V(G)$ is the set of vertices in $V(G) - S$ each of which is adjacent to at least one vertex in S. For $S \subseteq V(G)$ and $T \subseteq V(G)$, let $E(S,T) = \{uv \in E(G) | u \in S, v \in T\}$, $E(S) = \{uv \in E(G) | u, v \in S\}$, and $e(S,T) = |E(S,T)|$. Other notation and terminology not be defined in this paper can be found in [1].

Let $g(x) \leq f(x)$ be two nonnegative integral-valued functions defined on $V(G)$. *Fractional factors* can be considered as the generalization of factors. It defined as follows: Let $h(e) \in [0, 1]$ be a function defined on $E(G)$, $d_G^h(x) = \sum_{e \in E_x} h(e)$ where $E_x = \{xy : xy \in E(G)\}$. Then $d_G^h(x)$ is called *fractional degree* of x in G. We call h an *indictor function* if $g(x) \leq d_G^h(x) \leq f(x)$ holds for each $x \in V(G)$. Let $E^h = \{e : e \in E(G),\ h(e) \neq 0\}$ and G_h be a spanning subgraph of G such that $E(G_h) = E^h$. We call G_h a *fractional* (g, f)-*factor*. Similarly, the *fractional k-factor* and *fractional* $[a, b]$-*factor* of G can be defined as above, where a, b and k are positive integers.

A graph G has a fractional k-factor does not imply that G has a k-factor. For example, an odd cycle C is a fractional 1-factor if we set $h(e) = \frac{1}{2}$ for each $e \in C$ but not a 1-factor. Therefore, the set of fractional factors is much larger than factors.

Let $f(x)$ be any function, we define $f(S) = \sum_{x \in S} f(x)$.

A necessary and sufficient condition for a graph to have fractional 1-factors is gave in [1].

Lemma 1.1. *[1] A graph G has a fractional 1-factor if and only if $i(G - S) \leq |S|$ holds for any $S \subseteq V(G)$.*

Anstee et al. obtained the characterization of fractional (g, f)-factors.

Lemma 1.2. *[3] Let G be a graph and let $g(x) \leq f(x)$ be two nonnegative integral-valued functions defined on $V(G)$. Then G has a fractional (g, f)-factor if and only if for any $S \subseteq V(G)$,*

$$g(T) - d_{G-S}(T) \leq f(S)$$

holds, where $T = \{x : x \in V(G) - S, d_{G-S}(x) \leq g(x)\}$.

The *isolated toughness* of a graph G is introduced by Ma and Liu [4], defined as $I(G) = min\{|S|/i(G-S) : S \subseteq V(G), i(G-S) \geq 2\}$ if G is not complete; Otherwise, set $I(G) = |V(G)| - 1$, we give a sufficient condition for graphs to have fractional (g, f)-factors. By Lemma 1.1 it is clearly that a graph G has a fractional 1-factor if and only if $I(G) \geq 1$.

The main purpose in this paper is to present the following results.

Theorem 1.1. *Let* G *be a graph with* $\delta(G) \geq \frac{(a+b)^2+2(b-a)+1}{4a}$ *and the isolated toughness* $I(G) \geq \frac{(a+b)^2+2(b-a+1)}{4a}$, *where* $a \leq b$ *are positive integers. If for any two nonnegative integral-valued functions* $g(x)$ *and* $f(x)$ *defined on* $V(G)$ *such that* $a \leq g(x) \leq f(x) \leq b$, G *has fractional* (g, f)*-factors.*

We can not say the bound on $I(G)$ are best possible in theorem 1.1, however, $1 = g(x) < f(x) = b$, we obtain the following result which is much stronger than theorem 1.1, and the bound of $I(G)$ in Theorem 1.2 is not only best possible but also a necessary and sufficient.

Theorem 1.2. *Let* G *be a graph with* $\delta(G) \geq 1$ *and* $b > 1$ *be a positive integer. Then* G *has fractional* $[1, b]$*-factors if and only if* $I(G) \geq \frac{1}{b}$

For general positive integers a and $b (a \leq b)$, we obtained the results in [9].

Theorem 1.3. *Let* G *be a graph and* $a \leq b$ *be positive integers. If* $\delta(G) \geq a$ *and the isolated toughness* $I(G) \geq a - 1 + \frac{a}{b}$, *then* G *has a fractional* $[a, b]$*-factor.*

In Theorem 1.3, if $a = 1 < b$, it is the case of Theorem 1.2, if $a = b = k$, it is the case in the Corollary 1.1[9].

Corollary 1.1. *Suppose that* G *is a graph with* $\delta(G) \geq k$ *and* $I(G) \geq k$, *where* k *is a positive integer. Then* G *has a fractional* k*-factor.*

2 Proofs of Theorem 1.1and Theorem 1.2

A subset I of $V(G)$ is *an independent set* if no two vertices of I are adjacent in G and a set C of $V(G)$ is *a covering set* if every edge of G has at least one end in C. It is easy to verify that a set $I \subseteq V(G)$ is an independent set of G if and only if $V(G) - I$ is a covering set of G. To prove theorem 1.1, we need the following Lemma.

Lemma 2.1. *[7] Let H be a sub-graph of G with $1 \leq \delta(G) \leq \Delta(G) \leq b-1$ and b be a positive integer. Let $S_1, S_2, \cdots, S_{b-1}$ be a partition of the vertex set of H such that $x \in S_j$ if $d(x) \leq j$ (we allow $S_j = \emptyset$). Then there is an independent set I and a covering set C of H such that*

$$\sum_{j=1}^{b-1}(b-j)c_j \leq \sum_{j=1}^{b-1} j(b-j)i_j$$

holds. Where $|I \cap S_j| = i_j$, $|C \cap S_j| = c_j$, $j = 1, 2, \cdots, b-1$.

Proof of Theorem 1.1. Suppose there exists two positive integral-functions g and f which satisfy the conditions of the Theorem, but G does not have fractional (g, f)-factors. Then by Lemma 1.2, there exists a vertex set $S \subset V(G)$ such that

$$g(T) - d_{G-S}(T) > f(S), \tag{2.1}$$

where $T = \{x : x \in V(G) - S, d_{G-S}(x) \leq g(x)\}$.

We assume that $S \neq \emptyset$. Otherwise, there is nothing to prove since $\delta(G) \geq \frac{(a+b)^2+2(b-a)+1}{4a} > b \geq g(x)$ for any $x \in V(G)$. We also assume that $T \neq \emptyset$. Otherwise, $T = \emptyset$, then $g(T) - d_{G-S}(T) = 0 > f(S) > 0$ by (2.1), a contradiction.

For each $0 \leq i \leq b-1$, let $T^i = \{x : x \in T, d_{G-S}(x) = i\}$(we allow $T^i = \emptyset$), put $H = G[T^1 \cup T^2 \cup \cdots \cup T^{b-1}]$, T^0 is the isolated vertex set in T with $|T^0| = t_0$. For any $x \in T^i$, we have $d_H(x) \leq i$. Thus, $\{T^i : i = 1, 2, \cdots b-1\}$ is a vertex partition of H. Therefore, by Lemma 2.1, there exists an independent set I and a covering C of $V(H)$ such that

$$\sum_{j=1}^{b-1}(b-j)c_j \leq \sum_{j=1}^{b-1} j(b-j)i_j \tag{2.2}$$

holds, where $|I \cap T^j| = i_j$, $|C \cap T^j| = c_j$ $(j = 1, 2, \cdots, b-1)$.

Without loss of generalization, we choose I is the maximal independent set of H respect to C. Set $W = G - (S \cup T)$, $U = S \cup C \cup (N_{G-S}(I) \cap V(W))$. Then

$$|U| \leq |S| + \sum_{j=1}^{b-1} j i_j, \tag{2.3}$$

$$i(G-U) \geq t_0 + \sum_{j=1}^{b-1} i_j. \tag{2.4}$$

If $i(G-U) \geq 2$, by the definition of $I(G)$,

$$|U| \geq i(G-U)I(G), \tag{2.5}$$

Combine (2.3) (2.4) and (2.5), we have

$$|S| \geq \sum_{j=1}^{b-1}(I(G)-j)i_j + I(G)t_0. \tag{2.6}$$

On the other hand, Since $g(x) \leq b$ for any $x \in V(G)$, $g(T) - d_{G-S}(T) \leq b|T| - d_{G-S}(T) = \sum_{j=1}^{b-1}(b-j)p_j + bt_0 \leq \sum_{j=1}^{b-1}(b-j)i_j + \sum_{j=1}^{b-1}(b-j)c_j + bt_0$, where $p_j = |\{x : x \in V(H), d_{G-S}(x) = j\}|$. And since $f(x) \geq a$ for any $x \in V(G)$, we have $f(S) \geq a|S|$. Combining those two inequations and (2.6), we get

$$\sum_{j=1}^{b-1}(b-j)i_j + \sum_{j=1}^{b-1}(b-j)c_j + bt_0 > a|S| \geq a\sum_{j=1}^{b-1}(I(G)-j)i_j + aI(G)t_0,$$

and $aI(G) - b > 0$ since $I(G) \geq \frac{(a+b)^2+2(b-a+1)}{4a}$. Therefore,

$$\sum_{j=1}^{b-1}(b-j)c_j \geq \sum_{j=1}^{b-1}(aI(G) - aj - b + j)i_j, \tag{2.7}$$

By (2.7) and (2.2), we have

$$\sum_{j=1}^{b-1} j(b-j)i_j > \sum_{j=1}^{b-1}(aI(G) - aj - b + j)i_j.$$

This implies that $j(b-j) > aI(G) - aj - b + j$ for some $j \in \{1, 2, \cdots, b-1\}$. But for any $j \in \{1, 2, \cdots, b-1\}$, $j(b-j) + aj - j = -j^2 + (a+b-1)j \leq \frac{(a+b-1)^2}{4}$ and $aI(G) - b \geq \frac{(b+a)^2+2(b-a+1)}{4} - b = \frac{(b+a-1)^2+1}{4} > \frac{(a+b-1)^2}{4}$, a contradiction. Therefore, G has fractional (g, f)-factors.

If $i(G-U) = 0$, then by (2.4), we have $0 \geq t_0 + \sum_{j=1}^{b-1} i_j$. Clearly, $t_0 = i_j = 0$ for all $j = 1, 2, \cdots b-1$. Then $T = \emptyset$, a contradiction.

If $i(G-U) = 1$, by (2.4), we have $1 \geq t_0 + \sum_{j=1}^{b-1} i_j$. If $t_0 = i_j = 0$ for all $j = 1, 2, \cdots, b-1$, it is the case when $i(G-U) =$

0. If $t_0 = 1$, then for all $j = 1, 2, \cdots, b-1$, $i_j = 0$, and T is an isolated vertex and let $T = \{v\}$, since $a \le g(x) \le f(x) \le b$ for any $x \in V(G)$, thus $g(T) - d_{G-S}(T) = g(v) \le b$ and $f(S) \ge a|S| \ge a d_G(v) \ge a\delta(G) \ge \frac{(b+a)^2+2(b-a)+1}{4} \ge b$, a contradiction. If there is some $j_0 \in \{1, 2, \cdots, b-1\}$ such that $i_{j_0} = 1$. Then H is a complete graph. Then by (2.3), we have $|U| \le |S| + j_0$. On the other hand, $|U| \ge |S| + d_{G-S}(v) \ge \delta(G)$, therefore,

$$|S| \ge |U| - j_0 \ge \delta(G) - j_0. \tag{2.8}$$

Then

$$\begin{aligned} & g(T) - d_{G-S}(T) \le b|T| - d_{G-S}(T) \\ = & \sum_{j=1}^{b-1}(b-j)p_j \\ \le & \sum_{j=1}^{b-1}(b-j)i_j + \sum_{j=1}^{b-1}(b-j)c_j = (b-j_0)j_0 + \sum_{j=1}^{b-1}(b-j)c_j. \end{aligned}$$

And since $f(x) \ge a$ for any $x \in V(G)$, we have $f(S) \ge a|S| \ge a(|U| - j_0) \ge a(\delta(G) - j_0)$. By those inequations we get

$$\sum_{j=1}^{b-1}(b-j)c_j + (b-j_0) \ge a(\delta(G) - j_0), \tag{2.9}$$

and by (2.2),

$$\sum_{j=1}^{b-1}(b-j)c_j \le j_0(b-j_0). \tag{2.10}$$

Thus, combing (2.1) (2.9) and (2.10), we obtain

$$\begin{aligned} & a(\frac{(a+b)^2+2(b-a)+1}{4a} - j_0) - (b-j_0) \\ \le & a(\delta(G) - j_0) - (b-j_0) < j_0(b-j_0), \end{aligned}$$

a contradiction since $\frac{(a+b)^2+2(b-a)+1}{4} - b \ge -j_0^2 + (a+b-1)j_0 = -(j_0 - \frac{a+b-1}{2})^2 + \frac{(a+b-1)^2}{4}$. Hence, G has fractional (g, f)-factors.

In all the case, we proved that G has fractional (g, f)-factors. ■

The following lemma which can also be obtained from lemma 1.2 when $g(x) = a$ and $f(x) = b$.

Lemma 2.2. *Let G be a graph and $a \leq b$ be positive integers. Then G has fractional $[a,b]$-factors if and only if for any $S \subseteq V(G)$,*

$$a|T| - d_{G-S}(T) \leq b|S|.$$

where $T = \{x : x \in V(G) - S, d_{G-S}(x) \leq a-1\}$.

Proof of Theorem 1.2
We can obtain the sufficiency of the Theorem by Theorem 1.3, but we give an independent and easier proof.

Let $I(G) \geq \frac{1}{b}$. If G is a complete, then G has fractional $[1,b]$-factors. Suppose G is not complete. In order to prove G has $[1,b]$-factors, we need only to prove that $i(G-S) \leq b|S|$ for any $S \subseteq V(G)$.

If $S = \emptyset$, clearly, $i(G-S) = 0 \leq |S| = 0$ since $\delta(G) \geq 1$. So we also suppose that $S \neq \emptyset$.

If $i(G-S) \geq 2$ for some $S \subseteq V(G)$. Then $i(G-S) \leq b|S|$ by $I(G) \geq \frac{1}{b}$.

If $i(G-S) \leq 1$, Clearly $i(G-S) \leq |S|$ since $S \neq \emptyset$.

Suppose that G has a fractional $[1,b]$-factor, then by lemma 2.2, $i(G-S) \leq b|S|$ for any $S \subseteq V(G)$. If $i(G-S) \geq 2$, then $\frac{|S|}{i(G-S)} \geq \frac{1}{b}$, clearly $I(G) \geq \frac{1}{b}$. If $i(G-S) \leq 1$ for any $S \subseteq V(G)$, then G is a complete, therefore, $I(G) = |V(G)| - 1 \geq \frac{1}{b}$ by the definition of $I(G)$.

In all the case, we prove that G has fractional $[1,b]$-factors if and only if $I(G) \geq \frac{1}{b}$. ■

Remark: Although all the graphs that we considered are simple, the theorems we proved also hold for multi-graphs(without loop), since a multi-graph has the same isolated toughness as its underlying graph.

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