# NORM ESTIMATES OF WEIGHTED COMPOSITION OPERATORS BETWEEN BLOCH-TYPE SPACES

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**Abstract**

We give some estimates of the norm of weighted composition operators from $\alpha$-Bloch spaces to Bloch-type spaces on the unit ball in $^n$.

## 1. INTRODUCTION AND PRELIMINARIES

Let $\mathbb{B} = \mathbb{B}^n$ be the unit ball in the complex vector space $\mathbb{C}^n$, $\mathbb{D} = \mathbb{B}^1$ the unit disk in $\mathbb{C}$, and $H(\mathbb{B})$ the class of all holomorphic functions on $\mathbb{B}$. Let $\mu(z)$ be a positive continuous function on $\mathbb{B}$ (*weight*) and $\nabla f$ the gradient of the function $f$. The Bloch-type space $\mathcal{B}_\mu = \mathcal{B}_\mu(\mathbb{B})$ consists of all $f \in H(\mathbb{B})$ such that

$$b_\mu(f) := \sup_{z\in\mathbb{B}} \mu(z)|\nabla f(z)| < \infty.$$

With the norm $\|f\|_{\mathcal{B}_\mu} = |f(0)| + b_\mu(f)$, $\mathcal{B}_\mu$ is a Banach space. For $\mu(z) = (1-|z|)^\alpha$, $\alpha > 0$, we get the $\alpha$-Bloch space $\mathcal{B}^\alpha = \mathcal{B}^\alpha(\mathbb{B})$, and the quantity $b_\mu(f)$ and the norm $\|f\|_{\mathcal{B}_\mu}$ are denoted respectively by $b_\alpha(f)$ and $\|f\|_{\mathcal{B}^\alpha}$.

The little Bloch-type space $\mathcal{B}_{\mu,0} = \mathcal{B}_{\mu,0}(\mathbb{B})$ is a subspace of $\mathcal{B}_\mu$ consisting of all $f \in H(\mathbb{B})$ such that $\lim_{|z|\to 1} \mu(z)|\nabla f(z)| = 0$. For $\mu(z) = (1-|z|)^\alpha$, $\alpha > 0$, we get the little $\alpha$-Bloch space space $\mathcal{B}_0^\alpha = \mathcal{B}_0^\alpha(\mathbb{B})$.

Let $u \in H(\mathbb{B})$ and $\varphi = (\varphi_1, \dots, \varphi_n)$ be a holomorphic self-map of $\mathbb{B}$. For $f \in H(\mathbb{B})$ the weighted composition operator is defined by $(uC_\varphi f)(z) = u(z)f(\varphi(z))$. It is of interest to provide function theoretic characterizations when $u$ and $\varphi$ induce bounded or compact weighted composition operators on spaces of holomorphic functions. For some recent results related to the case of the unit ball or to Bloch-type spaces, see, e.g., [1]-[10], [12]-[24] and the references therein.

One of interesting problems is to calculate the norm of the operator $uC_\varphi$. Majority of papers in the area only find asymptotics of the norm of certain linear operators. There are a few papers which calculate the norm of these operators. Recently in [14] we calculated $\|uC_\varphi\|_{\mathcal{B}^1(\text{or } \mathcal{B}_0^1)\to H_\mu^\infty}$, which motivated us to find the norms of weighted composition and other closely related operators between various spaces of holomorphic functions (see [15], [16], [17], [18], [21]). Motivated by [1] and [22], here we estimate the norm $\|uC_\varphi\|_{\mathcal{B}^\alpha (\text{or } \mathcal{B}_0^\alpha)\to\mathcal{B}_\mu}$, $\alpha \neq 1$.

We need the next auxiliary result (see also [10] and [11] for related estimates).

**Lemma 1.** *Let $f \in \mathcal{B}^\alpha(\mathbb{B})$, $\alpha \neq 1$. Then the following inequality holds*

$$|f(z)| \leq |f(0)| + \frac{b_\alpha(f)}{\alpha - 1}\left(\frac{1}{(1-|z|)^{\alpha-1}} - 1\right). \tag{1}$$

*Proof.* Since $\alpha \neq 1$, we have

$$\begin{aligned} |f(z) - f(0)| &= \left|\int_0^1 \frac{d}{dt}(f(tz))dt\right| = \left|\int_0^1 \langle \nabla f(tz), \bar{z}\rangle dt\right| \\ &\leq b_\alpha(f)\int_0^1 \frac{|z|dt}{(1-|z|t)^\alpha} = \frac{b_\alpha(f)}{\alpha - 1}\left(\frac{1}{(1-|z|)^{\alpha-1}} - 1\right). \end{aligned}$$ □

## Main result

Before we formulate the main result, we introduce some notation. Let

$$I_1^{(n)} = \sup_{z\in\mathbb{B}} \frac{\mu(z)|\nabla u(z)|}{\alpha - 1}\left(\frac{1}{(1-|\varphi(z)|)^{\alpha-1}} - 1\right) \text{ and } I_2^{(n)} = \sup_{z\in\mathbb{B}} \frac{\mu(z)|u(z)||D\varphi(z)|}{(1-|\varphi(z)|)^\alpha},$$

where $|D\varphi(z)|^2 = \sum_{j=1}^n |\nabla\varphi_j(z)|^2$.

**Theorem 1.** *Suppose $\varphi$ is a holomorphic self-map of $\mathbb{B}$, $u \in H(\mathbb{B})$, $\alpha \in (0,\infty)\setminus\{1\}$, and $\mu$ is a weight on $\mathbb{B}$. Then the following inequalities hold*

$$\begin{aligned} &\max\left\{\|u\|_{\mathcal{B}_\mu}, \frac{|u(0)|}{\alpha-1}\left(\frac{1}{(1-|\varphi(0)|)^{\alpha-1}} - 1\right)\right\} \leq \|uC_\varphi\|_{\mathcal{B}_0^\alpha\to\mathcal{B}_\mu} \leq \|uC_\varphi\|_{\mathcal{B}^\alpha\to\mathcal{B}_\mu} \\ &\leq \max\left\{\|u\|_{\mathcal{B}_\mu}, \frac{|u(0)|}{\alpha-1}\left(\frac{1}{(1-|\varphi(0)|)^{\alpha-1}} - 1\right) + I_1^{(n)} + I_2^{(n)}\right\}. \end{aligned} \tag{2}$$

*Proof.* Set $f_0(z) \equiv 1$. Then $\|f_0\|_{\mathcal{B}^\alpha} = 1$ and $f \in \mathcal{B}_0^\alpha$. Hence we have

$$\|uC_\varphi\|_{\mathcal{B}_0^\alpha\to\mathcal{B}_\mu} = \|f_0\|_{\mathcal{B}^\alpha}\|uC_\varphi\|_{\mathcal{B}_0^\alpha\to\mathcal{B}_\mu} \geq \|uC_\varphi f_0\|_{\mathcal{B}_\mu} = \|u\|_{\mathcal{B}_\mu}. \tag{3}$$

For each $w \in \mathbb{B}$ set $f_w(z) = \frac{1}{\alpha-1}\left(\frac{1}{(1-\langle z,w\rangle)^{\alpha-1}} - 1\right)$. Since $f_w(0) = 0$ and

$$(1-|z|)^\alpha|\nabla f_w(z)| = \frac{(1-|z|)^\alpha|w|}{|1-\langle z,w\rangle|^\alpha} \leq \frac{(1-|z|)^\alpha}{(1-|w||z|)^\alpha} \leq \min\left\{1, \frac{(1-|z|)^\alpha}{(1-|w|)^\alpha}\right\},$$

it follows that $\sup_{w\in\mathbb{B}}\|f_w\|_{\mathcal{B}^\alpha} \leq 1$, and $f_w \in \mathcal{B}_0^\alpha$ for each fixed $w \in \mathbb{B}$.

If $\varphi(0) \neq 0$, then for every $r \in (0,1)$ we have

$$\begin{aligned} \|uC_\varphi\|_{\mathcal{B}_0^\alpha\to\mathcal{B}_\mu} &\geq \|uC_\varphi f_{r\frac{\varphi(0)}{|\varphi(0)|}}\|_{\mathcal{B}_\mu} \geq |u(0)||f_{r\frac{\varphi(0)}{|\varphi(0)|}}(\varphi(0))| \\ &\geq \frac{|u(0)|}{\alpha-1}\left(\frac{1}{(1-r|\varphi(0)|)^{\alpha-1}} - 1\right). \end{aligned} \tag{4}$$

If $\varphi(w) = 0$, then (4) obviously holds. Letting $r \to 1^-$ in (4), we get

$$\|uC_\varphi\|_{\mathcal{B}_0^\alpha \to \mathcal{B}_\mu} \geq \frac{|u(0)|}{\alpha - 1}\left(\frac{1}{(1 - |\varphi(0)|)^{\alpha-1}} - 1\right). \tag{5}$$

From (3) and (5) the first inequality in (2) follows.

If $f \in \mathcal{B}^\alpha$, by the Cauchy-Schwarz inequality and Lemma 1 we have that

$$\begin{aligned}
\|uC_\varphi f\|_{\mathcal{B}_\mu} &\leq |u(0)||f(\varphi(0))| + \sup_{z\in\mathbf{D}} \mu(z)|\nabla u(z)|\,|f(\varphi(z))| \\
&\quad + \sup_{z\in\mathbf{B}} \mu(z)|u(z)||D\varphi(z)||\nabla f(\varphi(z))| \\
&\leq |u(0)|\left(|f(0)| + \frac{b_\alpha(f)}{\alpha-1}\left(\frac{1}{(1-|\varphi(0)|)^{\alpha-1}} - 1\right)\right) \\
&\quad + \sup_{z\in\mathbf{B}} \mu(z)|\nabla u(z)|\left(|f(0)| + \frac{b_\alpha(f)}{\alpha-1}\left(\frac{1}{(1-|\varphi(z)|)^{\alpha-1}} - 1\right)\right) \\
&\quad + b_\alpha(f) \sup_{z\in\mathbf{B}} \frac{\mu(z)|u(z)||D\varphi(z)|}{(1-|\varphi(z)|)^\alpha} \\
&\leq \|f\|_{\mathcal{B}^\alpha} \max\left\{\|u\|_{\mathcal{B}_\mu}, \frac{|u(0)|}{\alpha-1}\left(\frac{1}{(1-|\varphi(0)|)^{\alpha-1}} - 1\right) + I_1^{(n)} + I_2^{(n)}\right\},
\end{aligned}$$

from which the third inequality in (2) follows.

From above mentioned inequalities and the following obvious inequality $\|uC_\varphi\|_{\mathcal{B}_0^\alpha \to \mathcal{B}_\mu} \leq \|uC_\varphi\|_{\mathcal{B}^\alpha \to \mathcal{B}_\mu}$, all the relationships in (2) follow, as claimed. □

Note that for $n = 1$

$$I_1^{(1)} = \sup_{z\in\mathbf{D}} \frac{\mu(z)|u'(z)|}{\alpha-1}\left(\frac{1}{(1-|\varphi(z)|)^{\alpha-1}} - 1\right) \quad \text{and} \quad I_2^{(1)} = \sup_{z\in\mathbf{D}} \frac{\mu(z)|u(z)\varphi'(z)|}{(1-|\varphi(z)|)^\alpha}.$$

Hence, from Theorem 1 we obtain the following corollary:

**Corollary 1.** *Suppose $\varphi$ is a holomorphic self-map of $\mathbb{D}$, $u \in H(\mathbb{D})$, $\alpha \in (0,\infty) \setminus \{1\}$, and $\mu$ is a weight on $\mathbb{D}$. Then the following inequalities hold*

$$\begin{aligned}
&\max\left\{\|u\|_{\mathcal{B}_\mu}, \frac{|u(0)|}{\alpha-1}\left(\frac{1}{(1-|\varphi(0)|)^{\alpha-1}} - 1\right)\right\} \leq \|uC_\varphi\|_{\mathcal{B}_0^\alpha \to \mathcal{B}_\mu} \\
&\leq \|uC_\varphi\|_{\mathcal{B}^\alpha \to \mathcal{B}_\mu} \leq \max\left\{\|u\|_{\mathcal{B}_\mu}, \frac{|u(0)|}{\alpha-1}\left(\frac{1}{(1-|\varphi(0)|)^{\alpha-1}} - 1\right) + I_1^{(1)} + I_2^{(1)}\right\}.
\end{aligned}$$

**Remark 1.** Theorem 1 can be regarded as a complement and an extension of Theorems 2.1 and 2.2 in [1], since the case $\alpha = 1$, $\mu(z) = 1 - |z|^2$, on the unit disk $\mathbb{D}$ was considered therein. Note that unlike the norm on the Bloch space $\mathcal{B}^1(\mathbb{D})$ in the present paper, in [1] the authors used a slightly different norm, that is, $\|f\|'_{\mathcal{B}} = |f(0)| + \sup_{z\in\mathbf{D}}(1-|z|^2)|f'(z)|$, which is more suitable for the case $\alpha = 1$. On the other hand, some recent investigations of ours show that from the practical point of view, for the case $\alpha \neq 1$, the norm $\|f\|_{\mathcal{B}^\alpha} = |f(0)| + \sup_{z\in\mathbf{D}}(1-|z|)^\alpha|f'(z)|$ is more suitable (see also Lemma 1 above).

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