Two results on BCI-subset of finite groups *

Wei Jin[a] and Weijun Liu[b†]

[a]School of Mathematics, Central South University, Changsha, Hunan, 410075, P. R. China

[b]School of Science, Nantong University, Nantong, Jiangsu, 226007, P. R. China

Abstract

In paper [7], S. J. Xu and W. Jin proved that a cyclic group of order pq, for two different odd primes p and q, is a 3-BCI-group, and a finite p-group is a weak $(p-1)$-BCI-group. As a continuation of their works, in this paper, we prove that a cyclic group of order $2p$ is a 3-BCI-group, and a finite p-group is a $(p-1)$-BCI-group.

Keywords: bi-Cayley graph; BCI-subset; Isomorphism

1 Introduction

For a graph X, we use $V(X)$, $E(X)$, and $A = Aut(X)$ to denote its vertex set, edge set and the full automorphism group respectively. A graph X is said to be edge transitive if the action of A on $E(X)$ is transitive; X is said to be vertex transitive if the action of A on $V(X)$ is transitive.

For a group G, and a subset S of G such that $1 \notin S$, the Cayley digraph $X = Cay(G,S)$ of G with respect to S is defined as the graph with vertex set G and arc set $Arc(X) = \{(x, sx) | x \in G, s \in S\}$. The above subset S is called a Cayley-subset of G. Each Cayley digraph X admits $R(G)$ as a subgroup of $Aut(X)$, where $R(G)$ acts with nature action of G on X by right multiplication. If S is symmetric, that is, if $S = S^{-1} = \{s^{-1} | s \in S\}$, then (x, y) is an arc if and only if (y, x) is an arc. In this case, $Cay(G,S)$ can be viewed as an undirected graph, called a Cayley graph, simply by identifying two arcs (x, y) and (y, x) as an undirected edge.

A Cayley-subset S is called a CI-subset of G if for any Cayley-subset T, whenever $Cay(G,S) \cong Cay(G,T)$, we have $T = S^{\alpha}$ for some $\alpha \in Aut(G)$. Then the Cayley graph $Cay(G,S)$ is called a CI-graph of G. If each Cayley-subset S is a CI-subset, then G is called a CI-group. Further, for a positive integer m, if each Cayley-subset S of size at most m is a CI-subset, G is called a m-CI-group.

For a finite group G and a subset $S \subsetneq G$ (possibly, S contains the identity element), the bi-Cayley graph $BCay(G,S)$ of G with respect to S is the graph with vertex set $G \times \{0,1\}$ and with edge set $\{(x,0),(sx,1)\}, x \in G, s \in S$. Then $BCay(G,S)$ is a well-defined bipartite graph with two bipartition subsets, say $U = G \times \{0\}, W = G \times \{1\}$. Further, each $g \in G$ induces an automorphism:

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†Corresponding author: wjliu6210@126.com

$$R(g) : (x,0) \mapsto (xg,0), (x,1) \mapsto (xg,1)$$

of $BCay(G,S)$. We set $R(G) = \{R(g) | g \in G\} \leq A$. Then $R(G)$ acts regularly on both U and W. Conversely, by [1, Lemma 2.5], a bipartite graph admits a group acting regularly on both the bipartition subsets must be isomorphic to a bi-Cayley graph.

Let $BCay(G,S)$ be a bi-Cayley graph and T a subset of G. If $T = gS^\alpha$ for some $g \in G$ and $\alpha \in Aut(G)$, then $BCay(G,S) \cong BCay(G,T)$ (see [4] or [5]). In general, the converse is not necessarily holds; for example, $G=\langle a,b | a^4=b^2=1, b^{-1}ab=a^{-1}\rangle$, $S=\{1,a^2\}$ and $T=\{1,b\}$. Here we quote the following definition

Definition 1 ([7]) *Let G be a finite group, $S \subseteq G$(possibly, S contains the identity element).*

(1) *S is called a BCI-subset of G, if for any $BCay(G,S) \cong BCay(G,T)$ implies that $T = gS^\alpha$, for some $g \in G, \alpha \in Aut(G)$.*

(2) *G is called a BCI-group, if each subset $S \subseteq G$ is a BCI-subset.*

(3) *Let m be a positive integer, G is called a m-BCI-group, if each subset $S \subseteq G$ of size at most m is a BCI-subset.*

(4) *Let m be a positive integer, G is called a weak m-BCI-group, if each subset $S \subseteq G$ of size at most m such that $BCay(G,S)$ is connected and vertex transitive is a BCI-subset.*

The Cayley isomorphism problem of Cayley graphs, especially determining CI-graphs, CI-groups etc., have been an active topic in algebraic graph theory for a long time, see surveys in [2, 6] on this topic. As a generalization of the CI-property for Cayley graphs, S. J. Xu and W. Jin first gave the concept of BCI-subset for bi-Cayley graphs in [7], where they give a necessary and sufficient condition for a finite group being a 2-BCI-group, and proved that every cyclic group of order a product of two distinct odd primes is a 3-BCI-group and that every finite p-group is a weak $(p-1)$-BCI-group.

In the present paper, we shall improve two results in [7] and prove the following two results:

Theorem 1.1 *A cyclic group of order $2p$ is a 3-BCI-group, where p is a prime.*

Theorem 1.2 *A cyclic group of order p^n is a $(p-1)$-BCI-group, where p is a prime, n is a positive integer.*

2 Preliminary results

This section collects several known results which will be used in the third section.

Firstly, by [4, 5], we have the following proposition, which allows us assume that S contains the identity element of the group G if necessary when consider the bi-Cayley graph $BCay(G,S)$.

Proposition 2.1 *Let $BCay(G,S)$ be a bi-Cayley graph. Then*

$$BCay(G,S) \cong BCay(G,gS^\alpha)$$

for $g \in G$ and $\alpha \in Aut(G)$.

By [1], the bi-Cayley graph $BCay(G,S)$ is connected if and only if $\langle SS^{-1}\rangle = G$, which implies the following proposition:

Proposition 2.2 ([4]) *Let G be a finite group, and $S \subseteq G$ with $1 \in S$. Then $BCay(G,S)$ is connected if and only if $G = \langle S\rangle$.*

Elements a and b of G are said to be fused if $a = b^{\sigma}$ for some $\sigma \in Aut(G)$ and to be inverse-fused if $a = (b^{-1})^{\sigma}$ for some $\sigma \in Aut(G)$, see [3]. The following two propositions are from [7].

Proposition 2.3 ([7])

(1) *All finite groups are 1-BCI-groups.*

(2) *Finite group G is 2-BCI-group if and only if for any two elements of the same order are fused or inverse-fused.*

(3) *A cyclic group of order p is a BCI-group where p is a prime.*

Proposition 2.4 ([7]) *Let $X = \mathrm{BCay}(G,S)$ be a finite, connected, and vertex transitive bi-Cayley graph. Denote $U = G\times\{0\}$, $W = G\times\{1\}$, $A = \mathrm{Aut}(X)$, $A^{+}=\{\alpha\in A \mid U^{\alpha}=U,\ W^{\alpha}=W\}$. Assume that for each $\sigma \in Sym(V(X))$, whenever $\sigma R(G)\sigma^{-1} \leq A^{+}$, we have $\sigma R(G)\sigma^{-1}$ is conjugate in A^{+} to $R(G)$. Then S is a BCI-subset of G.*

Finally, we quote a result from [8].

Proposition 2.5 ([8]) *Let $G = \langle a,b \mid a^{2p}=b^{2}=1, b^{-1}ab=a^{-1}\rangle$ be a dihedral group of order $4p$. Assume $S \subseteq G \setminus \{1\}$ such that $|S| = 3$. Then we have S is conjugate to $\{b,a,a^{-1}\}$ or $\{b,ba,ba^{i}\}(i = 2,3,\cdots,p)$ or $\{a^{p},b,ba^{i}\}$ $(i = 1,2)$ under the action of $Aut(G)$. Further, if $\langle S\rangle = G$, then either S is a CI-subset or conjugate to $\{b,a,a^{-1}\}$ or $\{b,ba,ba^{2}\}$.*

3 Proof of Main Results

In this section, with the same notation as in Section 1 and 2, we give the proofs of Theorems 1.1 and 1.2.

Note that two graphs are isomorphic to each other if and only if there is a bijection between their connected components such that the corresponding components are isomorphic. Then the following lemma holds.

Lemma 3.1 *Let G be a finite group, and let S, T be two subsets of G. Then $BCay(G,S)\cong BCay(G,T)$ if and only if $BCay(\langle S\rangle,S) \cong BCay(\langle T\rangle,T)$.*

Proof of Theorem 1.1. Suppose $G = Z_{2p} = \langle a\rangle$ is a cyclic group of order $2p$ where p is a prime. If $p = 2$, it is easy to check that Z_4 is a 3-BCI-group. So suppose $p > 2$. Since $Aut(G)$ is transitive on the same order elements of G, then by Proposition 2.3 (1) and (2), G is a 2-BCI-group. Thus, it suffices to show each 3-subset S of G is a BCI-subset. By Proposition 2.1, we may assume $S = \{1,x,y\}$. Then

(i) $o(x) = 2, o(y) = p$;

(ii) $o(x) = 2, o(y) = 2p$;

(iii) $o(x) = o(y) = p$;

(iv) $o(x) = o(y) = 2p$;

(v) $o(x) = p, o(y) = 2p$.

If subset S in case (i), we can assume $S=\{1, a^p, a^{2n}\}$, where $n=1,2,\cdots,p-1$. Let $g = a^{-2n}$, then $gS=\{a^{-2n}, a^{p-2n}, 1\}$, where a^{-2n} has order p, a^{p-2n} has order $2p$, hence gS is contained in case (v). Similarly, we also can prove that if subset S in case either (ii) or (iv), there exists a subset T in case (v) such that $T = hS$ for some $h \in G$.

Recall that for two subsets S' and T', if there exist $g \in G$ and $\alpha \in Aut(G)$ such that $T' = gS'^{\alpha}$, then S is a BCI-subset of G if and only if T is a BCI-subset of G. Therefore without loss of generality, we can assume that subset S is contained in either case (iii) or (v).

Let $T \subseteq G, |T| = 3$. Assume that $1 \in T$ and $BCay(G, S) \cong BCay(G, T)$. Then, first, suppose that S is a subset in case (iii). Since $G \neq \langle S \rangle$, by Proposition 2.2, $BCay(G, S)$ is not connected, and so $BCay(G, T)$ is not connected too. Without loss of generality, we assume that T belongs to case (iii). Since all elements of order p are a^{2l} where $l = 1,2,\cdots,p-1$, it follows that we can suppose $S=\{1, a^{2i}, a^{2j}\}$, $T=\{1, a^{2m}, a^{2n}\}$, where $i, j, m, n \in \{1,2,\cdots,p-1\}, i \neq j, m \neq n$. Further, because $Aut(G)$ is transitive on the same order elements of G, so S is conjugate to $\{1, a^2, a^{2k}\}$, T is conjugate to $\{1, a^2, a^{2r}\}$ where $k, r \in \{2,3,\cdots,p-1\}$. Hence, we can assume $S=\{1, a^2, a^{2k}\}, T=\{1, a^2, a^{2r}\}, k, r \in \{2,3,\cdots,p-1\}$.

By the above assumption that $BCay(G, S) \cong BCay(G, T)$, then by Lemma 3.1, we have $BCay(\langle S \rangle, S) \cong BCay(\langle T \rangle, T)$ and $\langle S \rangle = \langle T \rangle \cong Z_p$. Then by Proposition 2.3 (3), cyclic group Z_p is a BCI-group. Thus there exist $g \in \langle S \rangle, \alpha \in Aut(\langle S \rangle)$ such that $S = gT^{\alpha}$. Further, since $\langle S \rangle$ is a characteristic subgroup of G, there exists $\beta \in Aut(G)$ such that $\beta|_{\langle S \rangle} = \alpha$, it follows that $S = gT^{\beta}$. Therefore S is a BCI-subset of G.

Now suppose that S is a subset of case (v). Since $G = \langle S \rangle$, by Proposition 2.2, $BCay(G, S)$ is connected, and so $BCay(G, T)$ is also connected.

Since $Aut(G)$ is transitive on the same order elements of G, so S is conjugate to one of $\{1, a, a^{2i}\}$ where $i = 1,2,\cdots,p-1$. Thus we may denote $S_i = \{1, a, a^{2i}\}$, and $X_i := BCay(G, S_i)$. Assume $\overline{G} = \langle a, b | a^{2p} = b^2 = 1, bab = a^{-1} \rangle$ is a dihedral group of order $4p$. Let $T_i = bS_i = \{b, ba, ba^{2i}\}$, $Y_i := Cay(\overline{G}, T_i)$, then define

$$\varphi : V(X_i) \mapsto V(Y_i)$$

$$(g, 0) \mapsto g$$

$$(g, 1) \mapsto bg,$$

where $g \in G, b \in \overline{G}, o(b) = 2$. It is easy to see that φ is a bijection from $V(X_i)$ to $V(Y_i)$. Further, since for each edge $\{(g,0),(sg,1)\}$ of $BCay(G, S_i)$ we have $\{(g,0),(sg,1)\}^{\varphi}$ $=\{g, bsg\}(bs \in T)$ which is an edge of $Cay(\overline{G}, T_i)$, therefore $X_i \cong Y_i$.

Further, define $\sigma : (x,0) \mapsto (x^{-1},1), (x,1) \mapsto (x^{-1},0)$ where $x \in G$, then $\sigma \in Aut(BCay(G, S_i))$, $o(\sigma) = 2$ and $R(G)^{\sigma} = R(G)$. Thus $R(G) \rtimes \langle \sigma \rangle \cong \overline{G}$. Since G char $\overline{G}$, by [7, Theorems 3.11 and 3.12], if bS is a CI-subset of $\overline{G}$, then S is a BCI-subset of G. By Proposition 2.5, all T_i are CI-subsets of $\overline{G}$ except T_1. Therefore S_i are BCI-subsets except S_1. It follows that S_1 is also a BCI-subset. □

In paper [7], authors proved that a finite p-group G is a weak $(p-1)$-BCI-group. When G is cyclic, the following proof improve the result: G is a $(p-1)$-BCI-group.

Proof of Theorem 1.2. Let $G = Z_{p^n} = \langle a \rangle$ where p is a prime, n is a positive integer. Suppose $S \subseteq G$, $|S| = m < p$ and $1 \in S$.

First, if $p = 2$, then $m = 1$, by Proposition 2.3 (1) that G is a 1-BCI-group. So we assume $p \neq 2$. Let $X = Bay(G, S)$. Denote $U = G\times\{0\}$ and $W = G\times\{1\}$ are the two bipartition parts of X. And denote $A = Aut(X)$, $A^+=\{\alpha\in A \mid U^\alpha=U,\ W^\alpha=W\}$. If $p \nmid |A^+_{(1,0)}|$, then G is the Sylow p-subgroup of A^+, by Proposition 2.4, that S is a BCI-subset of G. Thus assume that $p \mid |A^+_{(1,0)}|$. If $\langle S \rangle = G$, then by Proposition 2.2, $BCay(G, S)$ is connected, thus $p \nmid |A^+_{(1,0)}|$ a contradiction. Therefore $\langle S \rangle < G$. So $|\langle S \rangle| = p^i, i < n$ and $\langle S \rangle = \langle a^{p^j} \rangle$ where $j = 1, 2, \cdots, n-1$. Let $X_1{:=}BCay(\langle S \rangle, S)$, and $B = Aut(X_1)$. Since $m < p$, that $p \nmid |B_{(1,0)}|$, and so S is a BCI-subset of $\langle S \rangle$. For any subset T of G such that $1 \in T$ and $BCay(G, S) \cong BCay(G, T)$, by Lemma 3.1, we have $\langle S \rangle = \langle T \rangle$ and $BCay(\langle S \rangle, S) \cong BCay(\langle T \rangle, T)$. Since S is a BCI-subset of $\langle S \rangle$, there exist $g \in \langle S \rangle$ and $\alpha \in Aut(\langle S \rangle)$, such that $T = gS^\alpha$. Further, since $\langle S \rangle$ is a characteristic subgroup of G, there exists $\beta \in Aut(G)$ such that the restriction of β to $\langle S \rangle$ is equal to α. Therefore, $T = gS^\beta$ and S is a BCI-subset of G. □

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