A Note on Global Domination in Graphs

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Abstract

Let $G = (V, E)$ be a graph. A subset S of V is called a dominating set of G if every vertex in $V - S$ is adjacent to at least one vertex in S. A global dominating set is a subset S of V which is a dominating set of both G as well as its complement $\overline{G}$. The domination number (global domination number) $\gamma(\gamma_g)$ of G is the minimum cardinality of a dominating set (global dominating set) of G. In this paper we obtain a characterization of bipartite graphs with $\gamma_g = \gamma + 1$. We also characterize unicyclic graphs and bipartite graphs with $\gamma_g = \alpha_0 + 1$, where α_0 is the vertex covering number of G.

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1. Introduction

By a graph we mean a finite, undirected graph without loops or multiple edges. Terms not defined here are used in the sense of Harary [6].

Let $G = (V, E)$ be a graph. A subset S of V is called a dominating set of G if every vertex in $V - S$ is adjacent to at least one vertex in S. A dominating set S of G is called an independent dominating set of G if no two vertices of S are adjacent in G.

Sampathkumar [9] introduced the concept of global domination in graphs. Brigham and Dutton [4] introduced the concept of factor domination which includes global domination as a special case. A subset S of V is called a global dominating set of G if S is a dominating set of both G and $\overline{G}$.

The domination number γ of G is defined to be the minimum cardinality of a dominating set in G . In a similar way, we define the independent domination number γ_i and the global domination number γ_g. Several fundamental results relating to domination in graphs are given in Haynes *et al.* [7]. A survey of results on global domination is given in Brigham *et al.* [5] .

In this paper we obtain a characterization of connected bipartite graphs with $\gamma_g = \gamma + 1$. We also characterize connected unicyclic graphs and bipartite graphs with $\gamma_g = \alpha_0 + 1$, where α_0 is the vertex covering number of G. We need the following definition and theorems.

Definition 1.1. *A vertex v of a graph G which is adjacent to a pendant vertex is called a support vertex.*

Theorem 1.2. [7] *For any graph G with a pendant vertex, $\gamma_g \leq \gamma + 1$.*

Theorem 1.3. [4] *If G is a triangle-free graph, then $\gamma \leq \gamma_g \leq \gamma + 1$.*

2. Main Results

It follows from Theorem 1.3 that for any bipartite graph $\gamma_g = \gamma$ or $\gamma+1$. The following theorem gives a characterization of all bipartite graphs for which $\gamma_g = \gamma + 1$.

Theorem 2.1. *Let G be a connected bipartite graph with bipartition X, Y and $|X| \leq |Y|$. Then $\gamma_g = \gamma + 1$ if and only if either G is isomorphic to K_2 or every vertex in X is adjacent to at least two pendant vertices and there exists a vertex in Y which is adjacent to all vertices in X.*

Proof. Let $\gamma_g = \gamma + 1$. We claim that $\delta = 1$. Suppose $\delta \geq 2$. Let S be a γ-set of G. If $\gamma < |X|$, then $S \cap X \neq \phi$ and $S \cap Y \neq \phi$. If $\gamma = |X|$, since $\delta \geq 2$, it follows that $S = (X - \{u\}) \cup \{v\}$, where $u \in X$ and $v \in N(u)$ is a γ-set of G with $S \cap X \neq \phi$ and $S \cap Y \neq \phi$. Since in $\overline{G}$ any vertex in X dominates all the vertices in Y and any vertex in Y dominates all the vertices in X, it follows that S is a global dominating set of G, so that $\gamma_g = \gamma$ which is contradiction. Thus $\delta = 1$. Further X is a γ-set of G, since otherwise any γ-set S has non-empty intersection with both X and Y, so that S is a global dominating set of G and hence $\gamma_g = \gamma$, which is a contradiction. If $|X| = \gamma = 1$, then $G = K_{1,n}$ for some $n \geq 1$, which satisfies the conditions of the theorem. Suppose $\gamma \geq 2$. If there exists a vertex $u \in X$ such that $N(u)$ contains at most one pendant vertex, then $D = (X - \{u\}) \cup \{v\}$, where $v \in N(u)$ and v is chosen to be a pendant vertex, if it exits, is a global dominating set which is a contradiction. Hence every vertex of X is adjacent to at least two pendant vertices. Since X is not a global dominating set of G, there exists a vertex in Y which is adjacent to all the vertices in X. Conversely, if G satisfies the conditions of the theorem, then X is the only γ-set of G so that $\gamma_g = \gamma + 1$. □

As a corollary to Theorem 2.1 we get the following theorem of Rall [8].

Corollary 2.2. [8] *Let T be a tree. Then $\gamma_g = \gamma + 1$ if and only if T is a star or T is a tree of diameter 4 which is constructed from two or more stars, each having at least two pendant vertices, by connecting the centers of these stars to a common vertex.* □

Sampathkumar [9] has proved that for any graph G, $\gamma_g \leq \alpha_0 + 1$, where α_0 is the vertex covering number of G. We characterize unicyclic graphs and bipartite graphs with $\gamma_g = \alpha_0 + 1$.

Theorem 2.3. *Let G be a connected unicyclic graph with cycle C. Then $\gamma_g = \alpha_0 + 1$ if and only if every vertex in $V(G)\backslash V(C)$ is either a pendant vertex or a support vertex and G is isomorphic to one of the graphs given below.*

(i) $C = C_3$, every vertex not on C is a pendant vertex and exactly one vertex of C has degree 2.

(ii) $C = C_3$, there exists at least one support vertex not on C, all the support vertices not on C are adjacent to the same vertex u on C, u is not a support vertex, the other two vertices on C are support vertices and there exist at least two pendant vertices adjacent to each support vertex.

(iii) $C = C_4$, every vertex not on C is a pendant vertex, two non-adjacent vertices of C have degree 2 and the remaining two non-adjacent vertices of C have degree ≥ 4.

(iv) $C = C_4$, there exists at least one support vertex not on C, all the support vertices not on C are adjacent to the same vertex u on C, u is not a support vertex, the vertex on C which is non-adjacent to u is the only vertex on C with degree 2 and there exist at least two pendant vertices adjacent to each support vertex.

Proof. For the unicyclic graphs given in the theorem $\gamma_g = \alpha_0 + 1 = r + 3$, where r is the number of support vertices not on C.

Conversely, let G be a unicyclic graph with cycle C and $\gamma_g = \alpha_0 + 1$. Let S_1 be the set of all support vertices of G. Let S be an α_0-set of G. Without loss of generality we may assume that $S_1 \subseteq S$. Since $\gamma_g = \alpha_0 + 1$, S is not a global dominating set of G. However S is a dominating set of G. Hence S is not a dominating set of $\overline{G}$, so that there exists a vertex u which is adjacent to all the vertices of S. Since G is unicyclic, it follows that u lies on C and $|S \cap C| = 2$. Also the length of C is at most 4. Further, if there exists a vertex v not on C which is neither a pendant vertex nor a support vertex, then S contains two vertices v_1, v_2 with $d(v_1, v_2) \geq 3$ so that S is a global dominating set of G which is a contradiction. Hence every vertex not on C is either a support vertex or a pendant vertex.

Case i. Every vertex not on C is a pendant vertex.

Since S is not a global dominating set of G, it follows that exactly two vertices on C are support vertices and these two support vertices have a common neighbor and hence are non-adjacent if $C = C_4$. Also, if $C = C_4$ and if a support vertex has degree 3, then $\gamma_g = \alpha_0 = 2$ and hence it follows that every support vertex is adjacent to at least two pendant vertices. Hence G is of the form (i) or (iii).

Case ii. There exists a support vertex not on C.

Since exactly two support vertices lie on C and all the support vertices are adjacent to a common vertex u, the two vertices adjacent to u on C are support vertices. If there exists a support vertex v with exactly one pendant vertex w adjacent to it, then $(S \backslash \{v\}) \cup \{\omega\}$ is a global dominating set which is a contradiction. Hence it follows that every support vertex is adjacent to at least two pendant vertices. Hence G is of the form (ii) or (iv). □

The following theorem shows that Theorem 2.1 is true if γ is replaced by α_0.

Theorem 2.4. *Let G be a connected bipartite graph with bipartition (X, Y) and $|X| \leq |Y|$. Then $\gamma_g = \alpha_0 + 1$ if and only if either $G = K_2$ or every vertex in X is adjacent to at least two pendant vertices and there exists a vertex in Y which is adjacent to all vertices in X.*

Proof. Suppose $\gamma_g = \alpha_0 + 1$. If $\alpha_0 = 1$, then $\gamma_g = 2$ and in this case $G \cong K_{1,n}$ for some n. Suppose $\alpha_0 > 1$. If G has an α_0-set D which intersects both X and Y, then D is a global dominating set so that $\gamma_g \leq \alpha_0$, which is a contradiction. Hence X is an α_0-set in G. Since X is not a global dominating set, there exists a vertex in Y which is adjacent to all vertices in X. If there exists a vertex u in X such that $N(u)$ contains at most one pendant vertex v, then $D = (X \backslash \{u\}) \cup \{v\}$, where $v \in N(u)$ and v is chosen to be a pendant vertex, if it exists, is a global dominating set of G which is a contradiction. Hence it follows that every vertex in X is adjacent to at least two pendant vertices. The converse is obvious. □

Corollary 2.5. *Let G be a connected bipartite graph with bipartition (X,Y) and $|X| \leq |Y|$. Then the following are equivalent.*

(i) $\gamma_g = \gamma + 1$.

(ii) $\gamma_g = \alpha_0 + 1$.

(iv) $G \cong K_2$ *or every vertex in X is adjacent to at least two pendant vertices and there exists a vertex in Y which is adjacent to all vertices in X.*

Proof. Follows from Theorem 2.1 and 2.4. □

Corollary 2.6. *Let G be a connected bipartite graph with p vertices. Then $\gamma + \gamma_g = p + 1$ if and only if $G \cong K_2$.*

Proof. Since $\gamma \leq \beta_0$ and $\gamma_g \leq \alpha_0 + 1$, $\gamma + \gamma_g = p + 1$ if and only if $\gamma = \gamma_i = \beta_0$ and $\gamma_g = \alpha_0 + 1$ and hence the result follows. □

Acharya [1] introduced the concept of domsaturation number ds of a graph. The least positive integer k such that every vertex of G lies in a dominating set (connected dominating set) of cardinality k is called the domsaturation number (connected domsaturation number) of G and is denoted by $ds(G)$ ($ds_c(G)$). Several results concerning domsaturation number and connected domsaturation number are given in Arumugam *et al.* [2, 3].

The concept of domsaturation number can be naturally extended with respect to global domination.

Definition 2.7. *For any graph G, the least positive integer k such that every vertex of G lies in a global dominating set of cardinality k is called the global domsaturation number of G and is denoted by ds_g.*

If S is a γ_g-set of G, then for any vertex $u \in V \backslash S$, $S \cup \{u\}$ is a global dominating set of G and hence it follows that $ds_g = \gamma_g$ or $\gamma_g + 1$.

Theorem 2.8. *Let G be any graph with $\gamma_g = \gamma$ and $ds = \gamma + 1$. Then $ds = ds_g$.*

Proof. Since $ds = \gamma + 1$, there exists a vertex u which does not lie in any γ-set. Since $\gamma_g = \gamma$ and any γ_g-set is also a γ-set, u does not lie in any γ_g-set and hence $ds_g = \gamma_g + 1 = \gamma + 1 = ds$. □

Theorem 2.9. *For any connected bipartite graph $G \neq K_2$, $ds = ds_g$.*

Proof. Let (X, Y) be a bipartition of G and let $|X| \leq |Y|$. In view of Corollary 2.5 and Theorem 2.8, it is enough to consider the case where $\gamma_g = \gamma$ and $ds = \gamma$. In this case $|X| \geq 2$. Let $x \in X$. Since $ds = \gamma$, there exists a γ-set D such that $x \in D$. If $D \cap Y \neq \phi$, then D is a global dominating set containing x. Otherwise $D = X$. Since $ds = \gamma$, every support vertex in X is adjacent to exactly one pendant vertex. Let $y \in X, y \neq x$ and $D_1 = (X \backslash \{y\}) \cup \{v\}$ where $v \in N(y)$ and v is chosen to be a pendant vertex if y is a support vertex. Then D_1 is a global dominating set of cardinality γ_g containing x. Similarly every vertex in Y also lies in a minimum global dominating set. Hence $ds = ds_g$. □

The following are some interesting problems for further investigation.

Problem 1. Characterize graphs for which $ds_g = \gamma_g$ or $ds_g = \gamma_g + 1$.
Problem 2. Characterize graphs for which $ds = ds_g$.
Problem 3. Characterize graphs for which $\gamma_g = \alpha_0 + 1$.

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