MATRIX REPRESENTATION OF THE SECOND ORDER RECURRENCE $\{u_{kn}\}$

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ABSTRACT. In this note, we consider a generalized Fibonacci sequence $\{u_n\}$. Then give a generating matrix for the terms of sequence $\{u_{kn}\}$ for a positive integer k. With the aid of this matrix, we derive some new combinatorial identites for the sequence $\{u_{kn}\}$.

1. INTRODUCTION

Let r be a nonzero integer such that $D = \sqrt{r^2+4} \neq 0$. The generalized Fibonacci and Lucas sequences $\{u_n\}$ and $\{v_n\}$ are defined by the following equations

$$u_{n+1} = ru_n + u_{n-1} \tag{1.1}$$

and

$$v_{n+1} = rv_n + v_{n-1} \tag{1.2}$$

where $u_0 = 0$, $u_1 = 1$ and $v_0 = 2$, $v_1 = r$, respectively.

When $r = 1$, $u_n = F_n$ (the nth Fibonacci number) and $v_n = L_n$ (the nth Lucas number).

If α and β are the roots of the equation $x^2 - rx - 1 = 0$, then the Binet formulas of the sequences $\{u_n\}$ and $\{v_n\}$ have the forms

$$u_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \text{ and } v_n = \alpha^n + \beta^n,$$

respectively.

Matrix methods are very convenient for deriving certain properties of linear recurrence sequences. Some authors have used matrix methods or other methods to derive some identities, combinatorial representations of linear recurrence relations etc. [3, 5, 9, 13, 14, 15, 18, 22]. In [23], the author formulate the nth power of an arbitrary 2×2 matrix. In [1], the author considers functions over 2×2 matrices other than addition and multiplication and then he proves that any positive integer power of such a matrix could be expressed as a linear combination of the matrix and the identity matrix. In [3], the author considers a 2×2 companion matrix and

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he derives some known relations involving Fibonacci numbers as well as many new relations. In [18], the author gives a new formula for the nth power of an arbitrary 2×2 matrix and derive various matrix identities and formulae for the nth power of particular matrices to obtain various combinatorial identities.

Recently some authors gave an interesting relationships between the specially multiplicative functions and the second order recurrence $\{u_n\}$ by considering their matrix representations. It is worth noting that the specially multiplicative functions satisfy a matrix recurrence relation similar to the sequence $\{u_n\}$. For more details, we refer to [7, 19]. For similar connections between kth order linear recurrences and rational arithmetical functions are also derived in [21, 12].

In [16], the authors derive the following recurrence relations for the sequences $\{u_{kn}\}$ and $\{v_{kn}\}$ for $k \geq 0$ and $n > 1$,

$$u_{kn} = v_k u_{k(n-1)} + (-1)^{k+1} u_{k(n-2)} \tag{1.3}$$

and

$$v_{kn} = v_k v_{k(n-1)} + (-1)^{k+1} v_{k(n-2)}$$

where the initial conditions of the sequences $\{u_{kn}\}$ and $\{v_{kn}\}$ are 0 and u_k, and 2 and v_k, respectively.

If $\alpha(k)$ and $\beta(k)$ are the roots of equation $x^2 - v_k x + (-1)^k = 0$, then the Binet formulas of the sequences $\{u_{kn}\}$ and $\{v_{kn}\}$ are given by

$$u_{kn} = u_k \frac{\alpha(k)^n - \beta(k)^n}{\alpha(k) - \beta(k)} \text{ and } v_{kn} = \alpha(k)^n + \beta(k)^n,$$

respectively. It is clear that $\alpha(1) = \alpha$ and $\beta(1) = \beta$. From the Binet formulas, one can see that

$$u_{-kn} = (-1)^{kn+1} u_{kn} \text{ and } u_{2kn} = v_{kn} u_{kn}. \tag{1.4}$$

2. Matrix representations for the sequence $\{u_{kn}\}$

In this section, we define a 2×2 matrix A and then we give some new results for the sequence $\{u_{kn}\}$ by matrix methods.

Define the 2×2 matrix A as follows:

$$A = \begin{bmatrix} v_k & (-1)^{k+1} \\ 1 & 0 \end{bmatrix}.$$

By an inductive argument and using (1.3), we get

$$A^n = \frac{1}{u_k} \begin{bmatrix} u_{k(n+1)} & (-1)^{k+1} u_{kn} \\ u_{kn} & (-1)^{k+1} u_{k(n-1)} \end{bmatrix}.$$

Clearly the matrix A^n satisfies the recurrence relation: for $n > 0$

$$A^{n+1} = v_k A^n + (-1)^{k+1} A^{n-1}, \tag{2.1}$$

where $A^0 = I$, $A^1 = A$.

If we use the equation (2.1), we can write

$$v_k A^{n+1} - \left(v_k^2 - (-1)^k\right) A^n + A^{n-2} = 0. \tag{2.2}$$

From the (1,1)-entries of the matrix equation (2.2), we get

$$v_k \frac{u_{k(n+2)}}{u_k} - \left(v_k^2 - (-1)^k\right) \frac{u_{k(n+1)}}{u_k} + \frac{u_{k(n-1)}}{u_k} = 0. \tag{2.3}$$

Thus we have

$$v_k = \left(\left(v_k^2 - (-1)^k\right) u_{k(n+1)} - u_{k(n-1)}\right) / u_{k(n+2)}. \tag{2.4}$$

The simple form of equation (2.4) can be found in [10, 11].

For $n \geq 0$, if we consider the fact that $\det(A^n) = (\det A)^n$, then we obtain the generalized Cassini identity:

$$u_{k(n+1)} u_{k(n-1)} - u_{kn}^2 = \begin{cases} (-1)^{kn} u_k^2 & \text{if } k \text{ is odd,} \\ (-1)^{kn+1} u_k^2 & \text{if } k \text{ is even.} \end{cases}$$

For example, for $k = r = 1$, we get $F_{n+1}F_{n-1} - F_n^2 = (-1)^n$ (see page 74, [17]).

By the Binet formulas, one can see that

$$u_{k(n+1)} + (-1)^{k+1} u_{k(n-1)} = u_k v_{kn}. \tag{2.5}$$

The eigenvalues of A^n are the roots of the equation

$$u_k^2 \lambda^2 - u_k (u_{k(n+1)} + (-1)^{k+1} u_{k(n-1)}) \lambda + (-1)^{kn} u_k^2 = 0$$

or by (2.5), we may rewrite it as

$$\lambda^2 - v_{kn} \lambda + (-1)^{kn} = 0.$$

Thus the characteristic roots of A^n are given by

$$\lambda_{1,2} = \left(v_{kn} \pm \sqrt{v_{kn}^2 + 4(-1)^{kn+1}}\right) / 2.$$

Now we shall derive some results for $\{u_{kn}\}$ by matrix methods.

Theorem 1. *For all $n, m \in Z$,*

$$u_k u_{k(n+m)} = u_{km} u_{k(n+1)} + (-1)^{k+1} u_{k(m-1)} u_{kn}. \tag{2.6}$$

Proof. Since $A^{n+m} = A^n A^m$ and after some simplifications, we obtain

$$A^{n+m} = \frac{u_{km}}{u_k^2} \begin{bmatrix} u_{k(n+2)} & (-1)^{k+1} u_{k(n+1)} \\ u_{k(n+1)} & (-1)^{k+1} u_{kn} \end{bmatrix} + \frac{u_{k(m-1)}}{u_k^2} \begin{bmatrix} (-1)^{k+1} u_{k(n+1)} & u_{kn} \\ (-1)^{k+1} u_{kn} & u_{k(n-1)} \end{bmatrix}.$$

Thus we obtain

$$u_k A^{n+m} = u_{km} A^{n+1} + (-1)^{k+1} u_{k(m-1)} A^n \tag{2.7}$$

which, by equating (2, 1)-entries of (2.7), gives the conclusion. □

When $m = n$ in (2.6), we obtain

$$u_k u_{2kn} = u_{kn}\left(u_{k(n+1)} + (-1)^{k+1} u_{k(n-1)}\right) \tag{2.8}$$

which, by (1.4), gives the equality in (2.5) for $u_{kn} \neq 0$. By equating (1, 1)-entries of the equality (2.7) and by taking $m = n$, we obtain

$$u_k u_{k(2n+1)} = u_{k(n+1)}^2 + (-1)^{k+1} u_{kn}^2. \tag{2.9}$$

From the above results, we obtain

$$u_k A^{2n} = u_{kn} A^{n+1} + (-1)^{k+1} u_{k(n-1)} A^n$$

and so the general cases of the well known divisibility properties for the Fibonacci numbers (see [17]):

$$u_{kn} \mid u_{2kn} \text{ and } v_{kn} \mid u_{2kn}. \tag{2.10}$$

Theorem 2. *For $k > 0$ and $n \in Z$,*

$$u_{k(2n+1)} + u_{k(2n-1)} = \frac{1}{u_k}\begin{cases} u_{k(n+1)}^2 + 2u_{kn}^2 + u_{k(n-1)}^2 & \text{if } k \text{ is odd,} \\ u_{k(n+1)}^2 - u_{k(n-1)}^2 & \text{if } k \text{ is even,} \end{cases} \tag{2.11}$$

and

$$u_{k(2n+1)} - u_{k(2n-1)} = \frac{1}{u_k}\begin{cases} u_{k(n+1)}^2 - u_{k(n-1)}^2 & \text{if } k \text{ is odd,} \\ u_{k(n+1)}^2 - 2u_{kn}^2 + u_{k(n-1)}^2 & \text{if } k \text{ is even.} \end{cases} \tag{2.12}$$

Proof. Considering the (1, 1) and (2, 2)- entries of the matrix equation $A^{2n} = (A^n)^2$, we get

$$u_k u_{k(2n+1)} = u_{k(n+1)}^2 + (-1)^{k+1} u_{kn}^2, \tag{2.13}$$

$$u_k u_{k(2n-1)} = u_{kn}^2 + (-1)^{k+1} u_{k(n-1)}^2. \tag{2.14}$$

By adding and substracting of (2.13) and (2.14) side by side, we have the conclusion.

□

Corollary 1. *For $k > 0$ and $n \in Z$,*

$$u_{2k} u_{2kn} = u_{k(n+1)}^2 - u_{k(n-1)}^2. \tag{2.15}$$

Proof. If we combine the equalities (2.11) and (2.12) , then we can write

$$u_k\left(u_{k(2n+1)} + (-1)^k u_{k(2n-1)}\right) = u_{k(n+1)}^2 - u_{k(n-1)}^2. \tag{2.16}$$

Using the recurrence relation $\{u_{kn}\}$ in (2.16), we obtain

$$u_k\left(v_k u_{2kn}\right) = u_{k(n+1)}^2 - u_{k(n-1)}^2. \tag{2.17}$$

The conclusion is clear from (1.4). □

For any integer p, $A^{2n} = A^{n+p}A^{n-p}$. Here if we consider the $(2,1)$-entries of the product $A^{n+p}A^{n-p}$ and the matrix A^{2n}, we get

$$u_k u_{2kn} = u_{k(n+p)}u_{k(n-p+1)} + (-1)^{k+1}u_{k(n+p-1)}u_{k(n-p)}.$$

Since $\det A \neq 0$, we can write the matrix A^{2n} as $A^{2n+m}A^{-m}$ and then by equating $(2,1)$-entries of this equation, we have

$$u_k u_{2kn} = u_{k(2n+m)}u_{-k(m-1)} + (-1)^{k+1}u_{k(2n+m-1)})u_{-km}. \tag{2.18}$$

Since $u_{-km} = (-1)^{km+1}u_{km}$, we can write

$$u_k u_{2kn} = u_{k(2n+m)}u_{k(1-m)} + (-1)^{k(m+1)}u_{k(2n+m-1)})u_{km}. \tag{2.19}$$

By a similar argument, we may obtain

$$\begin{aligned} & u_{k(2n+m)}u_{k(1-m)} + (-1)^{k(m+1)}u_{k(2n+m-1)}u_{km} \\ = \ & u_{k(n+m)}u_{k(n-m+1)} + (-1)^{(k+1)}u_{k(n+m-1)}u_{k(n-m)}. \end{aligned} \tag{2.20}$$

3. Some new combinatorial representations for $\{u_{kn}\}$

In this section, we consider the binomial expansion of A^n for some n and then derive some new combinatorial representations for the sequence $\{u_{kn}\}$.

Theorem 3. *For $n > 0$,*

$$\sum_{t=1}^{n} \binom{n}{t}(-1)^{n-t+1}v_k^t u_{k(n-t)} = \begin{cases} 0 & \text{if } n \text{ is odd,} \\ 2u_{kn} & \text{if } n \text{ is even,} \end{cases} \tag{3.1}$$

and

$$\sum_{t=1}^{n} \binom{n}{t}(-1)^{n(k+1)-t}v_k^t u_{k(n-t)} = \begin{cases} 0 & \text{if } n \text{ is odd,} \\ 2u_{-kn} & \text{if } n \text{ is even.} \end{cases} \tag{3.2}$$

Proof. If we consider the matrix relation

$$\begin{aligned} A^n &= \left(v_k I + (-1)^{k+1} A^{-1}\right)^n \\ &= \sum_{t=0}^{n} \binom{n}{t}(-1)^{(k+1)(n-t)}v_k^t A^{-n+t}, \end{aligned} \tag{3.3}$$

and equating $(2,1)$-entries of the equality (3.3), we get

$$u_{kn} = \sum_{t=0}^{n} \binom{n}{t}(-1)^{n-t+1}v_k^t u_{k(n-t)}.$$

Thus one can easily obtain (3.1).

Similarly, equating the $(1,2)$-entries of the equality (3.3), we get (3.2). This completes the proof. □

Theorem 4. *For $n, k > 0$,*

$$u_{2kn} = \sum_{t=0}^{n} \binom{n}{t}(-1)^{(k+1)(n-t)}v_k^t u_{kt}.$$

Proof. If we write A^{2n} in the form

$$A^{2n} = \sum_{t=0}^{n} \binom{n}{t}(-1)^{(k+1)(n-t)} v_k^t A^t, \tag{3.4}$$

then, by equating the (2, 1)-entries of the equality (3.4), we have the conclusion. □

Theorem 5. *For $n > 0$,*

$$u_{k(2n+1)} = \sum_{t=0}^{n} \binom{n+1}{t+1}(-1)^{(k+1)(n-t)} v_k^{t+1} u_{kt} + (-1)^{n(k+1)} u_k. \tag{3.5}$$

Proof. Since $A^{2n+1} = AA^{2n}$ for any $n \in Z$ and using (3.4), we get

$$A^{2n+1} = \sum_{t=0}^{n} \binom{n}{t}(-1)^{(n-t)(k+1)} v_k^t A^{t+1}, \tag{3.6}$$

which yields

$$u_{k(2n+1)} = \sum_{t=0}^{n} \binom{n}{t}(-1)^{(n-t)(k+1)} v_k^t u_{k(t+1)},$$

which yields

$$u_{k(2n+1)} = \sum_{t=0}^{n} \binom{n+1}{t+1}(-1)^{(k+1)(n-t)} v_k^{t+1} u_{kt} + (-1)^{n(k+1)} u_k.$$

Thus the proof is complete. □

Relation (3.5) can also be obtained by iterating $A^{2n+1} = v_k A^{2n} + (-1)^{k+1} A^{2n-1}$. Thus we get

$$\begin{aligned} A^{2n+1} &= v_k A^{2n} + (-1)^{k+1} A^{2n-1} \\ &= v_k A^{2n} + (-1)^{k+1}(v_k A^{2n-2} + (-1)^{k+1} A^{2n-3}) \\ &\vdots \\ &= v_k \sum_{p=0}^{n} \left(\sum_{t=0}^{n} \binom{n-t}{p} \right) (-1)^{(k+1)(n-p)} v_k^p A^p + (-1)^{(k+1)(n+1)} A^{-1}. \end{aligned}$$

From [20], it is well known that

$$\sum_{t=0}^{n} \binom{n-t}{p} = \binom{n+1}{p+1},$$

so we obtain

$$A^{2n+1} = \sum_{p=0}^{n} \binom{n+1}{p+1}(-1)^{(k+1)(n-p)} v_k^{p+1} A^p + (-1)^{(k+1)(n+1)} A^{-1}. \tag{3.7}$$

From (3.7), we get (3.5).

Theorem 6. *For $n > 0$ and $t, s \in Z$,*

$$u_{k(tn+s)} = \sum_{p=0}^{n} \binom{n}{t}(-1)^{(n-p)(k+1)} v_k^p u_{k((t-2)n+s+p)} \tag{3.8}$$

Proof. We consider the following matrix relation:

$$\begin{aligned} A^{tn+s} &= A^{n(t-2)+s} A^{2n} = A^{n(t-2)+s} \sum_{p=0}^{n} \binom{n}{p}(-1)^{(n-p)(k+1)} v_k^p A^p \\ &= \sum_{p=0}^{n} \binom{n}{p}(-1)^{(n-p)(k+1)} v_k^p A^{(t-2)n+s+p} \end{aligned}$$

which gives us (3.8). Thus the proof is complete. □

Theorem 7. *Let k be an odd integer and $p > 0$,*

$$u_{k(pn+1)} + u_{kpn} = v_k \sum_{t=1}^{pn} u_{kt} + u_k. \tag{3.9}$$

Proof. For $p > 0$, consider

$$A^{pn} - I = (A - I)(A^{pn-1} + A^{pn-2} + ... + A^2 + A + I). \tag{3.10}$$

Thus

$$\begin{aligned} (A+I)(A^{pn} - I) &= (A^2 - I)(A^{pn-1} + A^{pn-2} + ... + A^2 + A + I) \\ &= v_k A(A^{pn-1} + A^{pn-2} + ... + A^2 + A + I) \\ &= v_k \sum_{t=1}^{pn} A^t. \end{aligned}$$

Equating the $(2,1)$-entries of this matrix relation, we get equation (3.9). This completes the proof. □

Theorem 8. *For $n > 0$,*

$$v_k \sum_{t=1}^{n} u_{2kt} = \begin{cases} u_{k(2n+1)} + u_k + 2\sum_{t=1}^{n-1} u_{k(2t+1)} & \text{if } k \text{ is even,} \\ u_{k(2n+1)} - u_k & \text{if } k \text{ is odd,} \end{cases} \tag{3.11}$$

and

$$v_k \sum_{t=1}^{n} u_{k(2t-1)} = \begin{cases} u_{2kn} + 2\sum_{t=1}^{n-1} u_{2kt} & \text{if } k \text{ is even,} \\ u_{2kn} & \text{if } k \text{ is odd.} \end{cases} \tag{3.12}$$

Proof. By the recurrence relation of $\{A^n\}$, we can write

$$\begin{aligned} v_k \sum_{t=1}^{n} A^{2t} + (-1)^{(k+1)} \sum_{t=1}^{n} A^{2t-1} &= \sum_{t=1}^{n} A^{2t+1} \\ v_k \sum_{t=1}^{n} A^{2t} + (-1)^{(k+1)} (A + \sum_{t=1}^{n-1} A^{2t+1}) &= \sum_{t=1}^{n-1} A^{2t+1} + A^{2n+1} \end{aligned}$$

Considering required entries of the equation just above, the proof is complete. □

For arbitrary integers p and q such that $p^2 + 4q \neq 0$, the sequence $\{w_n\}$ is defined by

$$w_n = pw_{n-1} + qw_{n-2}$$

for $w_0 = a$, $w_1 = b$ and for $n > 1$.

In [10], the authors considered the sequence $\{w_n\}$ and gave the following result:

$$\begin{aligned} w_n(a,b,p,q) &= a\sum_{t=0}^{\lfloor n/2 \rfloor} (-1)^t \binom{n-t}{t} p^{n-2t} q^t \\ &\quad + (b - pa) \sum_{t=0}^{\lfloor (n-1)/2 \rfloor} (-1)^t \binom{n-1-t}{t} p^{n-1-2t} q^t \end{aligned} \tag{3.13}$$

Then we have the following result.

Corollary 2. *For $n, k > 0$,*

$$u_{kn} = u_k \sum_{t=0}^{\lfloor (n-1)/2 \rfloor} \binom{n-1-t}{t} (-1)^{t(k+1)} v_k^{n-1-2t}$$

Proof. The proof directly follows from (3.13). □

Let C be an arbitrary 2×2 matrix, T and D denote the trace and determinant of C, respectively. For the distinct eigenvalues α and β of matrix C, the following result can be found in [18, 6]:

Lemma 1. *If*

$$z_n := \frac{\alpha^n - \beta^n}{\alpha - \beta} = \frac{1}{2^{n-1}} \sum_{m=0}^{\lfloor (n-1)/2 \rfloor} \binom{n}{2m+1} T^{n-2m-1} \left(T^2 - 4D\right)^m,$$

then $C^n = z_n C - z_{n-1} D I_2$, *where* I_2 *is the identity matrix of order* 2.

As a consequence of Lemma 1, we obtain that

$$u_{kn} = \frac{u_k}{2^{n-1}} \sum_{i=0}^{\lfloor (n-1)/2 \rfloor} \binom{n}{2i+1} v_k^{n-2i-1} \left(v_k^2 - 4(-1)^k\right)^i. \tag{3.14}$$

From also [18], let g be a complex number such that $g^2 + Tg + D \neq 0, g \neq 0$ and let n be a positive integer. Then

$$C^n = \left(\frac{gD}{g^2 + Tg + D}\right)^n \sum_{t=0}^{2n} \sum_{i=0}^{t} \binom{n}{i} \binom{n}{t-i} \left(\frac{D}{g^2}\right)^i \left(\frac{g}{D}\right)^t C^t. \tag{3.15}$$

Therefore we get the following result of equality (3.15).

Theorem 9. *For* $n > 0$ *and any every complex number* g *different from* $0, 1/2$ *and* $\left(v_k \pm \sqrt{v_k^2 + 4(-1)^{k+1}}\right)/2$,

$$u_{kn} = \frac{g^n (-1)^{kn}}{(g^2 + v_k g + (-1)^k)^n} \sum_{t=0}^{2n} \sum_{i=0}^{t} \binom{n}{i} \binom{n}{t-i} (-1)^{k(i-t)} g^{t-2i} u_{kt}.$$

Let the $k \times k$ companion matrix is as follows:

$$A_k = \begin{bmatrix} c_1 & c_2 & \dots & c_{k-1} & c_k \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{bmatrix}.$$

Consider the kth order recurrence $\{z_n\}$ defined by $z_n = c_1 z_{n-1} + c_2 z_{n-2} + \ldots + c_k z_{n-k}$ for $n > k-1$ and z_i' s are arbitrary for $0 \leq i \leq k-1$. Thus it follows that

$$\left[z_{n+k-1}, z_{n+k-2}, \ldots, z_n\right]^T = A_k^n \left[z_{k-1}, z_{k-2}, \ldots, z_0\right]^T.$$

Thus the power A_k^n determines the solution to the recurrence $\{z_n\}$ in terms of the initial conditions $z_0, z_1, \ldots, z_{k-1}$.

We find the following Theorem in [4].

Theorem 10. *The* (i,j) *entry* $a_{ij}^{(n)}$ *in the matrix* A_k^n *is given by the following formula:*

$$a_{ij}^{(n)} = \sum_{(t_1, t_2, \ldots t_k)} \frac{t_j + t_{j+1} + \ldots + t_k}{t_1 + t_2 + \ldots + t_k} \times \binom{t_1 + t_2 + \ldots + t_k}{t_1, t_2, \ldots, t_k} c_1^{t_1} \ldots c_k^{t_k} \tag{3.16}$$

where the summation is over nonnegative integers satisfying $t_1+2t_2+\ldots+kt_k=n-i+j$ *and the coefficients in (3.16) is defined to be* 1 *if* $n=i-j$.

When $a_1=v_k$ and $a_2=(-1)^{k+1}$ in the sequence $\{z_n\}$ and considering the recurrence $\{u_{kn}\}$ and its companion matrix, we give the following Corollary.

Corollary 3. *For* $n,k>0$,

$$u_{kn}=u_k\sum_{(t_1,t_2)}\binom{t_1+t_2}{t_1,t_2}v_k^{t_1}(-1)^{t_2k}$$

where the summation is over nonnegative integers satisfying $t_1+2t_2=n-1$ *and the coefficients in (3.4) is defined to be* 1 *if* $n=i-j$.

Similarly one can find the basic multinomial formula for kth order linear recurrences in [8]. Also applying the results of recent studies [2, 18] on the similar topics, many various combinatorial representations for the recurrences $\{u_{kn}\}$ and $\{v_{kn}\}$ can be derived.

We should note that one can apply many earlier results for our matrix A to obtain some different results. Also considering the results of the present paper, many analogue formulas for the generalized Lucas sequence $\{v_{kn}\}$ can be derived by considering and extending the simple relation between the vector of Lucas sequence and generating matrix of Fibonacci sequences:

$$\begin{bmatrix} L_{n+1} \\ L_n \end{bmatrix}=\begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

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NEWTON'S IDENTITIES

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The elementary symmetric polynomials in the n commuting variables $X_1, ..., X_n$ are defined

$$\sigma_1 = \sum_{1 \leq i \leq n} X_i,\ \sigma_2 = \sum_{1 \leq i < j \leq n} X_i X_j,\ \sigma_3 = \sum_{1 \leq i < j < k \leq n} X_i X_j X_k, ...,\ \sigma_n = X_1 X_2 \cdots X_n.$$

For $k \geq 1$ let $\rho_k = X_1^k + X_2^k + \cdots X_n^k$. With this notation Netown's identities read

$$0 = \rho_k - \sigma_1 \rho_{k-1} + \sigma_2 \rho_{k-2} - \sigma_3 \rho_{k-3} + \cdots$$
$$\cdots + (-1)^{k-1} \sigma_{k-1} \rho_1 + (-1)^k k \sigma_k, \quad 1 \leq k \leq n, \ (1)$$

$$0 = \rho_{j+n} - \sigma_1 \rho_{j+n-1} + \sigma_2 \rho_{j+n-2} - \sigma_3 \rho_{j+n-3} + \cdots$$
$$\cdots + (-1)^{n-1} \sigma_{n-1} \rho_{j+1} + (-1)^n \sigma_n \rho_j, \quad j \geq 1 \ (2).$$

Various classical proofs are known, and new ones are still being published. The most recent seems to be [M]. We furnish the following extremely simple proof.

Let f denote the right hand side of (1). Then f is symmetric and homogeneous of degree k. Thus $f = 0$ if and only if no monomial $X_1^{e_1} X_2^{e_2} \cdots X_n^{e_n}$, with $e_1 + e_2 + \cdots + e_n = k$ and $e_1 \geq e_2 \geq \cdots \geq e_n$, appears in f. The only such monomials possibly present in f are

$$X_1^k, X_1^{k-1} X_2, X_1^{k-2} X_2 X_3, ..., X_1^2 X_2 X_3 \cdots X_{k-1}, X_1 X_2 X_3 \cdots X_k.$$

Now X_1^k appears once in ρ_k and $\sigma_1 \rho_{k-1}$, without appearing in any other summand of f. As ρ_k and $\sigma_1 \rho_{k-1}$ have opposite signs in f, the coefficient of X_1^k in f is 0. Likewise if $1 \leq i \leq k-2$ then $X_1^{k-i} X_2 \cdots X_{i+1}$ appears once in $\sigma_i \rho_{k-i}$ and $\sigma_{i+1} \rho_{k-i-1}$, and does not appear in any other summand of f. Since $\sigma_i \rho_{k-i}$ and $\sigma_{i+1} \rho_{k-i-1}$ have opposite signs in f, the coefficient of $X_1^{k-i} X_2 \cdots X_{i+1}$ in f is 0. Finally $X_1 X_2 X_3 \cdots X_k$ appears once in σ_k, k times in $\sigma_{k-1} \rho_1$, namely in the forms $(X_2 X_3 \cdots X_k) X_1, (X_1 X_3 \cdots X_k) X_2, ...,$ $(X_1 X_2 \cdots X_{k-1}) X_k$, and does not appear in any other summand of f. The definition of f ensures that the coefficient of $X_1 X_2 X_3 \cdots X_k$ in f is 0.

For completeness we include the standard proof of (2). Let X be another variable and let $g(X) = (X - X_1)(X - X_2) \cdots (X - X_n)$. According to the very definition of the elementary symmetric polynomials we have $g(X) = X^n - \sigma_1 X^{n-1} + \cdots + (-1)^{n-1} \sigma_{n-1} X + (-1)^n \sigma_n$, so

$$0 = g(X_i) = X_i^n - \sigma_1 X_i^{n-1} + \cdots + (-1)^{n-1} \sigma_{n-1} X_i + (-1)^n \sigma_n, \quad 1 \leq i \leq n.$$

Multiplying this by X_i^j and adding the resulting n equations, we obtain (2).

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