Some Results on 4-cycle Packings

Hung-Lin Fu
Department of Applied Mathematics
National Chiao Tung University
Hsin Chu,Taiwan

Abstract

A packing of a graph G is a set of edge-disjoint 4-cycles in G and a maximum packing of G with 4-cycles is a packing which contains the largest number of 4-cycles among all packings of G. In this paper, we obtain the maximum packing of certain graphs such as $K_{2m+1} - H$ where H is a 2-regular subgraph, $K_{2m} - F$ where F is a spanning odd forest of K_{2m} and $2K_{2m} - L$ where L is a 2-regular subgraph of $2K_{2m}$.

1. Introduction

A packing of a graph G is a set of edge-disjoint 4-cycles in G and the graph induced by the edges in G but not in any 4-cycle of the packing is called the remainder graph of the packing or the leave of the packing. If a packing has a leave which has the minimum number of edges, we call it a minimum leave. A maximum packing of G (with 4-cycles) is a packing which has a minimum leave. Clearly, if $E(G)$ can be partitioned into sets which induce 4-cycles, then the leave is an empty graph and we say that G has a 4-cycle decomposition.

A 4-cycle decomposition of K_v is also known as a 4-cycle system of order v. It is folk-lore now a 4-cycle system of order v exists if and only if $v \equiv 1$ $(mod\ 8)$, and the maximum packing of K_v is also known.

Theorem 1.1.[7] The maximum packing of K_v can be depicted as in the following table.

v (mod 8)	0	1	2	3	4	5	6	7
Leave	F	$\varnothing$	F	C_3	F	E_6	F	C_5

F is a 1-factor, C_3 and C_5 are cycles of length 3 and 5 respectively, and E_6 is an even graph with 6 edges.

As to the packing of a more general graph, it is not an easy task at all. For example, let H be an arbitrary r-regular subgraph of K_{2m+1} where r is an even integer. Then, for which r, $K_{2m+1} - H$ has a 4-cycle decomposition is left unknown. So far, if $r = 2$, then we have

Theorem 1.2.[5] Let H be a 2-regular subgraph of K_{2m+1} such that

$\binom{2m+1}{2} - |E(H)|$ is a multiple of 4. Then $K_{2m+1} - H$ has a 4-cycle decomposition.

Two similar works are obtained for K_{2m}.

Theorem 1.3.[4] Let F be a spanning odd forest of K_{2m} such that $\binom{2m}{2} - |E(F)|$ is a multiple of 4. Then $K_{2m} - F$ has a 4-cycle decomposition.

Theorem 1.4.[2] Let F be a spanning odd graph with $\Delta(F) \leq 3$. Then $K_{2m} - F$ has a 4-cycle decomposition if and only if $4 | \binom{2m}{2} - |E(F)|$ and for $m = 4$, F is not one of the following two graphs in Figure 1.

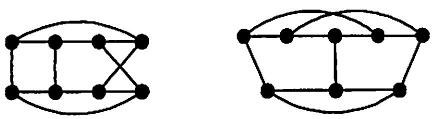

Figure 1.

In this paper, we shall extend the study of Theorem 1.2 and Theorem 1.3 to consider the maximum packing of $K_{2m+1} - H$ and $K_{2m} - F$ respectively in section 2 and 3. It is worth noting that the proofs in this paper do not use the results obtained in [4,5]. Finally in section 4, we consider $2K_{2m} - L$ where L is a 2-regular subgraph of $2K_{2m}$.

2. Packing $K_{2m+1} - H$

We start our main results with the maximum packing of $K_{2m+1} - H$ where H is a 2-regular subgraph of K_{2m+1}, i.e., H is a vertex-disjoint union of cycles. For convenience, we shall use $G_1 \vee G_2$ to denote the join of two graphs G_1 and G_2. Recall that $V(G_1 \vee G_2) = V(G_1) \cup V(G_2)$, $V(G_1) \cap V(G_2) = \emptyset$ and $E(G_1 \vee G_2) = E(G_1) \cup E(G_2) \cup \{uv | u \in V(G_1)$ and $v \in V(G_2)\}$. The following lemmas are essential to the proof of the first main theorem. Since the second one is easy to see, we omit the proof.

Lemma 2.1. $(K_2 \cup K_2) \vee C_3$, $(K_2 \cup K_2) \vee C_5$ and $(K_2 \cup K_2) \vee B$ can be packed with 4-cycles such that the leaves are C_5, C_3 and $\emptyset$ respectively. Here, B is called a bowtie which is obtained by attaching two C_3's together at a common vertex.

Proof. Let $V(K_2 \cup K_2) = \{a, b, c, d\}$, $E(K_2 \cup K_2) = \{ab, cd\}$, $C_3 = (1, 2, 3)$, $C_5 = (1, 2, 3, 4, 5)$ and $B = (1, 2, 3; 3, 4, 5)$. Then the packing of $(K_2 \cup K_2) \vee C_3$ is $\{(a, b, 2, 1),\ (b, 1, d, 3),\ (c, 1, 3, 2)\}$ with leave $(a, 2, d, c, 3)$, the packing of $(K_2 \cup K_2) \vee C_5$ is $\{(c, 1, 2, 3),\ (d, 3, 4, 5),\ (c, d, 1, 5),\ (c, 2, d, 4),\ (a, 2, b, 3),\ (a, 4, b, 5)\}$ with leave $(1, a, b)$, and the 4-cycle decomposition of

$(K_2 \cup K_2) \vee B$ is $\{(a,1,3,b),\ (c,3,5,d),\ (d,1,2,3),\ (a,3,4,5),\ (b,1,c,2),$ $(b,4,c,5),\ (a,2,d,4)\}$. ■

Lemma 2.2. Let s and t be two positive even integers. Then, $K_{s,t}$ has a 4-cycle decomposition.

It is worth of mentioning that the above lemma is a special case of the well-known Sotteau's Theorem on even cycle decomposition of complete bipartite graphs.

Theorem 2.3.[8] Let s and t be two positive even integers not less than k. Then, $K_{s,t}$ has a $2k$-cycle decomposition if and only if $2k|st$.

Now, we are ready to prove the first main result. For simplicity, we shall use bowtie B for E_6 throughout section 2 and 3.

Theorem 2.4. Let H be a 2-regular subgraph of K_{2m+1}. Then $K_{2m+1}-H$ has a maximum packing with leaves L_i if and only if $\binom{2m+1}{2} - |E(H)| \equiv i \ (mod\ 4)$. Here, $L_0 = \varnothing,\ L_1 = C_5,\ L_2 = B$ and $L_3 = C_3$.

Proof. The proof follows by Theorem 1.1 if $|E(H)| = 0$. So, let $|E(H)| > 0$ and the proof is by induction on m. First, it is easy to see that the assertion is true for small $m's$. Assume that the assertion is true for $m < k$ and let H be a 2-regular subgraph of K_{2k+1} such that $\binom{2k+1}{2} - |E(H)| \equiv i \ (mod\ 4)$ and H contains a cycle of maximum length $t+1$, $(a_0, a_1, a_2, ..., a_t)$, where $t \leq 2k$. Now, if $t = 2$, let $H' = H \setminus \{a_0, a_1\}$. Then $|E(H')| = |E(H)| - 3$. Since $\binom{2k+1}{2} - |E(H)| \equiv i \ (mod\ 4)$, $\binom{2k+1}{2} - |E(H')| \equiv \frac{(2k-1)(2k-2)}{2} - (|E(H)| - 3) = 2k^2 - 3k + 1 - |E(H)| + 3 \equiv 2k^2 + k - |E(H)| + 4 \ (mod\ 4)$. Therefore, $\binom{2k-1}{2} - |E(H')| \equiv i \ (mod\ 4)$. By the induction hypothesis, $K_{2k-1} - H'$ can be packed with leave L_i. Since $K_{2k+1} - H = K_{2,2k-2} \cup (K_{2k-1} - H')$, and $K_{2,2k-2}$ can be decomposed into 4-cycles (by Lemma 2.2), we conclude that $K_{2k+1} - H$ can be packed with leave L_i.

On the other hand, if $t \geq 5$, then delete a_0 and a_2 from H and add the edge a_3a_t to obtain a graph H'. Now, $|E(H)| = |E(H')| + 3$ and H' is a subgraph of K_{2k-1} defined on $V(K_{2k+1}) \setminus \{a_0, a_2\}$. By a similar argument as above, we have $\binom{2k-1}{2} - |E(H')| \equiv i \ (mod\ 4)$. Therefore, by the induction hypothesis, $K_{2k-1} - H'$ has a maximum packing with leave L_i. This implies that a maximum packing of $K_{2k+1} - H$ can be obtained by combining the packing of $K_{2k-1} - H'$, (a_0, a_3, a_t, a_2) and the 4-cycle decomposition of $K_{2,2k-2}$ where the two partite sets of $K_{2,2k-4}$ are $\{a_0, a_2\}$ and $V(K_{2k+1} \setminus \{a_0, a_2, a_1, a_3, a_t\})$.

It is left to consider the cases when $t = 3$ or 4. First, if $t = 3$, then set $C = (a_0, a_1, a_2, a_3)$ and $H' = H - C$. By counting, $\binom{2k+1}{2} - \binom{2k-3}{2} \equiv$

2 (*mod* 4). Therefore, if $\binom{2k+1}{2} - |E(H)| \equiv i \pmod 4$, then $\binom{2k-3}{2} - |E(H')| \equiv i+2 \pmod 4$. Since $K_{2k-3} - H'$ has a maximum packing with leave $L_{i+2(mod\ 4)}$, we conclude that $K_{2k+1} - H$ has a maximum packing with a leave which is obtained by a maximum packing of $(K_2 \cup K_2) \vee L_{i+2}$. Here, $K_2 \cup K_2$ is $\{a_0, a_2\} \cup \{a_1, a_3\}$. Now, from Lemma 2.1, $(K_2 \cup K_2) \vee L_{i+2}$ can be packed with leave L_i, thus the proof follows.

Finally, we consider the case where $t = 4$. First, delete $\{a_0, a_1, a_2, a_3\}$ from $V(K_{2k+1})$. Then, let $H' = H - (a_0, a_1, a_2, a_3, a_4)$. Hence, if $\binom{2k+1}{2} - |E(H)| \equiv i \pmod 4$, $\binom{2k-3}{2} - |E(H')| = \binom{2k-3}{2} - |E(H)| + 5 \equiv \binom{2k+1}{2} - |E(H)| + 7 \equiv i + 3 \pmod 4$. Now, by the induction hypothesis, $K_{2k-3} - H'$ has a maximum packing with leave $L_{i+3(mod\ 4)}$. Clearly, if $i = 1$, then $K_{2k-3} - H'$ has a 4-cycle decomposition. Also, $K_{4,2k-4} = (A, B)$ (where $A = \{a_0, a_1, a_2, a_3\}$ and $V(K_{2k+1}) \setminus \{a_0, a_1, a_2, a_3, a_4\}$) has a 4-cycle decomposition. Thus, $K_{2k+1} - H$ has a maximum packing with leave $(a_4, a_1, a_3, a_0, a_2)$. This is L_1 as we expect to have. On the other hand, if $i = 2$, 3 or 4, the leave will be obtained by packing the graph $(O_4 \vee L_{i+3} - K_{4,1}) \cup (a_4, a_1, a_3, a_0, a_2)$ see Figure 2 if a_4 is not on L_{i+3}. (Here, O_4 is the empty graph with four vertices a_0, a_1, a_2, a_3, and $K_{4,1}$ is defined on $\{a_0, a_1, a_2, a_3\} \cup \{x\}$, $x \in V(L_{i+3})$).

In case that a_4 is on L_{i+3}, the proof follows by taking $x = a_4$.

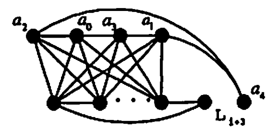

Figure 2

(i) $L_{i+3} = C_5$ $(i = 2)$

Let $C_5 = (1, 2, 3, 4, 5)$ and 5 is not adjacent to a_0, a_1, a_2 or a_3. Then the packing is $\{(a_1, 1, a_2, a_4), (a_0, 2, a_1, a_3), (a_2, 2, 1, a_0), (a_3, 1, 5, 4), (a_2, 3, a_0, 4)\}$ and its leave is $(a_3, 2, 3; 3, a_1, a_4)$ which is L_2. This concludes the proof of the case $i = 2$.

(ii) $L_{i+3} = B$ $(i = 3)$

Let $B = (1, 2, 3; 3, 4, 5)$ and 5 is not adjacent to a_0, a_1, a_2 or a_3. Then the packing is $\{(a_1, 1, a_2, a_4), (a_0, 2, a_1, a_3), (a_2, 2, 1, a_0), (a_3, 1, 3, 2), (a_1, 3, 5, 4), (a_2, 3, a_0, 4)\}$ and its leave is $(a_3, 3, 4)$ which is L_3 as we expect.

(iii) $L_{i+3} = C_3$ $(i = 4)$

Let $C_3 = (1, 2, 3)$ and 3 is not adjacent to a_0, a_1, a_2 or a_3. Now, the packing is $\{(a_2, 1, a_1, a_4), (a_0, 2, a_1, a_3), (a_2, 2, 1, a_0), (a_3, 1, 3, 2)\}$ and its leave is an empty graph, i.e. L_0. This concludes the proof of this case and the theorem. ■

3. Packing $K_{2m} - F$

A graph is an odd graph if every vertex of the graph is odd.

Theorem 3.1. Let F be a spanning odd forest of K_{2m}. Then $K_{2m} - F$ has a maximum packing with leave L_i if and only if $\binom{2m}{2} - |E(F)| \equiv i \ (mod\ 4)$ where $i = 0, 1, 2, 3$. Here, $L_0 = \emptyset, L_1 = C_5, L_2 = B$ and $L_3 = C_3$.

Proof. The proof will be by induction on m and it is easy to see the assertion is true when m is small. So, assume that the assertion is true for $m < k$ and let F be a spanning odd forest of K_{2k}. First, if all vertices of F are of degree 1, then F is a perfect matching. The proof follows by Theorem 1.1. Here, $K_{2k} - F$ has a 4-cycle decomposition. So, let v be a vertex of degree not less than 3 and u and w are two pendant vertices of F which are adjacent to v. Now, delete u and w from K_{2k} and obtain a spanning odd forest F' of K_{2k-2}. By induction, $K_{2k-2} - F'$ has a maximum packing with leave L_{i+1} provided that $\binom{2k}{2} - |E(F)| \equiv i \ (mod\ 4)$. We start with $i = 1$.

(i) $i = 1$

The leave of the maximum packing of $K_{2k-2} - F'$ is L_2 i.e. B. Let $B = (1, 2, 3; 3, 4, 5)$, then the packing of $K_2 \cup B \cup K_{2,3}$ (without using 1 and 5) is $\{(u, 3, 5, 4), (w, 2, 1, 3)\}$ with leave $(u, w, 4, 3, 2)$ which is L_1 as expected. Note that $K_{2,2k-6}$ has a 4-cycle decomposition where one partite set $V(K_{2k-2}) \setminus \{v, 2, 3, 4\}$ is of size $2k - 6$ and the other partite set is $\{u, w\}$. We shall use this result again in the following two cases where $i = 2$ and $i = 3$.

(ii) $i = 2$

The leave of the maximum packing of $K_{2k-2} - F'$ is (1,2,3). Thus the maximum packing of $K_{2k} - F$ is $\{(u, w, 3, 2)\}$ and the leave is $(w, 2, 1; 1, u, 3)$ if $v \notin \{1, 2, 3\}$. In case that $v = 3$ (W.L.O.G.), let $x \notin \{1, 2, 3\}$. Then, the packing is $\{(w, 1, 3, 2)\}$ and its leave is $(x, w, u; u,$
$1, 2)$.

(iii) $i = 3$

Since the maximum packing of $K_{2k-2} - F'$ has empty leave, the maximum packing of $K_{2k} - F$ has leave L_3 which is easy to see.

(iv) $i = 0$

The proof follows by packing $K_2 \vee C_5$ where $C_5 = (1,2,3,4,5)$. The packing is $\{(u,1,2,3),(w,3,4,5),(u,5,1,w),(u,2,w,4)\}$ with empty leave. Hence we have the proof of this case. By induction, we conclude the proof of this theorem. ■

4. Packing $2K_{2m} - L$

For convenience, we use $G_1 \Box G_2$ to denote the graph $(G_1 \vee G_2) \cup K_{|G_1|,|G_2|}$ i.e., all the edges between G_1 and G_2 will be of multiplicity 2. The following lemmas will be useful later.

Lemma 4.1. $C_3 \Box C_3$ has a 4-cycle decomposition.

Proof. Note that the first result is a special group divisible 4-cycle design, see [3]. For completeness, we give a proof here.

Let $C_3 = (1,2,3)$ and $C_3 = (4,5,6)$. Then, a 4-cycle decomposition is $\{(1,2,6,5),(2,3,5,4),(1,3,4,6),(1,4,2,5),(2,5,3,6),(1,4,3,6)\}$. ■

Lemma 4.2. $C_3 \Box C_5$, $C_3 \Box O_3$ and $C_3 \Box (D \cup O_1)$ have a maximum packing with leaves D, C_5 and C_3 respectively. Here, D is a set of double edges.

Proof.

(i) $C_3 \Box C_5$

Let $C_3 = (1,2,3)$ and $C_5 = (a,b,c,d,e)$. Then, the packing is $\{(1,2,b,a)$, $(c,1,3,2),(3,b,c,d),(2,d,e,a),(a,1,d,3),(2,c,3,e),(1,c,3,e),(2,a,3,b)$, $(1,d,2,e)\}$ with leave a set of double edges $D = \{\{1,b\},\{1,b\}\}$.

(ii) $C_3 \Box O_3$

Let $C_3 = (1,2,3)$ and $O_3 = \{a,b,c\}$ (three isolated vertices). Then the packing is $\{(a,1,2,3),(a,2,b,3),(b,1,c,2),(b,1,c,3)\}$ with leave $C_5 = (c,2,a,1,3)$.

(iii) $C_3 \Box (D \cup O_1)$

Let $C_3 = (1,2,3)$ and $D \cup O_1$ be the union of (a,c) and an isolated vertex b. Then the packing (from the above case) is $\{(a,1,2,3),(a,2,b,3)$, $(b,1,c,2),(b,1,c,3),(1,a,c,3)\}$ with leave $C_3 = (a,2,c)$. ■

Before we prove the main result, we also need the following lemma. For convenience, in what follows, we shall use $H_1 \cup H_2 \to G_1 \cup G_2$ to denote the statement that by combining two graphs H_1 and H_2 together with $V(H_1)$ and $V(H_2)$ preassigned we can decompose $H_1 \cup H_2$ into two edge-disjoint graphs G_1 and G_2. We note here that the minimum leave L_2 is no longer B in multigraphs, it is D, the double edge, see [1] for reference.

Lemma 4.3. $C_5 \cup C_5 \to C_4 \cup B$, $C_5 \cup C_5 \to C_4 \cup C_4 \cup D$, $C_5 \cup B \to C_4 \cup C_4 \cup C_3$, $C_5 \cup C_3 \to C_4 \cup C_4$, $B \cup B \to C_4 \cup C_4 \cup C_4$, $B \cup C_3 \to C_5 \cup C_4$, $C_3 \cup C_3 = B$ and $C_3 \cup C_3 \to C_4 \cup D$.

Proof. Since we can assign the vertex set of H_1 and H_2, the decomposition will be easier. For example, we can let $C_5 = (1,2,3,4,5)$ and another $C_5 = (1,3,5,2,4)$, then $C_5 \cup C_5$ is in fact K_5 and the decomposition follows. The results $C_5 \cup C_5 \to C_4 \cup C_4 \cup D$, $C_5 \cup C_3 = C_4 \cup C_4$, $B \cup C_3 = C_5 \cup C_4$, $C_3 \cup C_3 \to C_4 \cup D$ and $C_3 \cup C_3 \to B$ are also trivial. Here, we prove two nontrivial cases.

(i) $C_5 \cup B \to C_4 \cup C_4 \cup C_3$

Let $C_5 = (1,4,2,5,3)$ and $B = (1,2,3;3,4,5)$. Then the decomposition is $\{(4,1,3,2),(5,2,1,3),(3,4,5)\}$.

(ii) $B \cup B \to C_4 \cup C_4 \cup C_4$

Let one $B = (1,3,2;2,4,5)$ and the other $B = (1,2,4;4,3,5)$. Then the decomposition is $\{(2,3,5,4),(1,2,4,3),(1,4,5,2)\}$. ■

Theorem 4.4. Let L be a 2-regular subgraph of K_{2m}. Then $2K_{2m} - L$ has a maximum packing with leave L_i if and only if $2\binom{2m}{2} - |E(L)| \equiv i \ (mod\ 4)$. Here, $L_0 = \varnothing$, $L_1 = C_5$, $L_2 = D$ and $L_3 = C_3$.

Proof. The proof is by induction on m and it is easy to check that the assertion is true for small m. Let the assertion be true for $m < k$ and L be a 2-regular subgraph of K_{2k}. First, if L has a cycle C of length not less than 5, let $C = (a_0, a_1, a_2, \ldots, a_t)$, $t \geq 4$. Now, delete a_0 and a_1 from $2K_{2k}$ and add an edge a_2a_t. Clearly, we have a 2-regular graph L' of $2K_{2k-2}$. By induction, $2K_{2k-2} - L'$ has a maximum packing with leave L_i. Observe that $2\binom{2k}{2} - |E(L)| = 2\binom{2k-2}{2} - |E(L')| + (8k-6) - 2 \equiv i \ (mod\ 4)$. Therefore, the proof follows by partitioning the edges in $2K_{2k} - L$ which are not in $2K_{2k-2} - L'$ into 4-cycles. Clearly, the edges mentioned above induce the union of $(a_0, a_1, a_t, a_2), (a_0, a_t, a_1, a_2)$ and $2K_{2,2k-4}$ where the two partite sets are $\{a_0, a_1\}$ and $V(K_{2k}) \setminus \{a_0, a_1, a_2, a_t\}$. Since $2K_{2,2k-4}$ has a 4-cycle decomposition, this concludes the proof of this case.

So, we have two cases left.

(i) There exists a 4-cycle in L.

Let the 4-cycle be (a_0, a_1, a_2, a_3), and delete $\{a_0, a_1, a_2, a_3\}$ from $V(K_{2k})$. Then $L' = L - (a_0, a_1, a_2, a_3)$ is a 2-regular subgraph in $2K_{2k-4}$. Since $2\binom{2k}{2} - |E(L)| \equiv 2\binom{2k-4}{2} - |E(L')|$ $(mod\ 4), 2K_4 - (a_0, a_1, a_2, a_3)$ has a 4-cycle decomposition and $2K_{4,2k-4}$ has a 4-cycle decomposition, the leave of the maximum packing of $2K_{2k} - L$ can be obtained by the leave of the maximum packing of $2K_{2k-4} - L'$.

(ii) All cycles in L are 3-cycles.

Let one 3-cycle of L be (a_0, a_1, a_2), and delete $\{a_0, a_1, a_2\}$ from $V(K_{2k})$. Then $L' = L - (a_0, a_1, a_2)$ and L' is a 2-regular graph in $2K_{2k-3}$. Clearly, L' is also a subgraph of K_{2k-3}. Therefore, the maximum packing of $2K_{2k-3} - L'$ can be obtained from the maximum packing of K_{2k-3} and $K_{2k-3} - L'$ respectively (by Theorem 2.4) and then combine them together by using Lemma 4.3. Since, the leaves of the above packings are $\emptyset, C_5, D$ and C_3 respectively, it follows by using $C_3 \square C_5, C_3 \square (D \cup O_1), C_3 \square C_3$ and $C_3 \square O_3$ to obtain the leaves of the maximum packings of $2K_{2m} - L$. Note here that both $2K_{3,2k-8}$ and $2K_{3,2k-6}$ have a 4-cycle decomposition. By Lemma 4.1 and Lemma 4.2, we have the desired leaves. This concludes the proof. ■

Concluding Remark

It was conjectured by Nash-Williams that if H is a graph of order n with maximum degree $\Delta(H) \le \frac{n}{4}$, then $K_n - H$ has a 3-cycle decomposition if and only if each vertex of $K_n - H$ is of even degree and $3 | \binom{n}{2} - |E(H)|$. By observation, this upper bound of maximum degree may also be correct for 4-cycle decomposition. In [2], the authors presented a couple of examples, see Theorem 1.4, to show $K_8 - H$ can not be decomposed into 4-cycles where $\Delta(H) = 3$ which is larger than $\frac{n}{4}$. So, it is reasonable to pose the following conjecture on 4-cycle decomposition.

Conjecture

Let H be a graph of order n with maximum degree not greater than $\frac{n}{4}$. Then $K_n - H$ can be decomposed into 4-cycles if and only if $K_n - H$ is an even graph and $4 | \binom{n}{2} - |E(H)|$.

Acknowledgement

The author would like to express his appreciation to the referee for his

(or her) helpful comments.

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