$(3,2)^*$-choosability of triangle-free toroidal graphs

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Abstract

A list-assignment L to the vertices of G is an assignment of a set $L(v)$ of colors to vertex v for every $v \in V(G)$. An $(L,d)^*$-coloring is a mapping ϕ that assigns a color $\phi(v) \in L(v)$ to each vertex $v \in V(G)$ such that at most d neighbors of v receive color $\phi(v)$. A graph is called $(k,d)^*$-choosable, if G admits an $(L,d)^*$-coloring for every list assignment L with $|L(v)| \geq k$ for all $v \in V(G)$. In this note, it is proved that: (1) every toroidal graph containing neither adjacent 3-cycles nor 5-cycles, is $(3,2)^*$-choosable; (2) every toroidal graph without 3-cycles, is $(3,2)^*$-choosable.

Key Words: improper choosability; toroidal graph; cycle.
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1 Introduction

Graphs considered in this paper are finite, simple, and undirected.

A torus is a closed surface (compact, connected 2-manifold without boundary) that is a sphere with a unique handle, and a toroidal graph is a graph embedable in the torus. For a toroidal graph G, we still use G to denote an embedding of G in the torus.

Let V, E and F denote the set of vertices, edges and faces of G, respectively. A face of an embedded graph is said to be incident with the edges and vertices on its boundary. Two faces are adjacent if they share common edges. In particular, two adjacent 3-faces are often referred as adjacent triangles. The degree of a face f of G, denoted by $d_G(f)$(or simply $d(f)$), is the length of the walk bounding the face.

A k-vertex (or k-face) is a vertex (or a face) of degree k, a k^--vertex (or k^--face) is a vertex (or a face) of degree at most k, and a k^+-vertex (or k^+-face) is defined similarly.

For $f \in F(G)$, we write $f = [u_1 u_2 \cdots u_n]$ if $u_1, u_2, \cdots, u_n$ are on the boundary of f in the clockwise order. A k-face is called an $(m_1, m_2, \cdots, m_k)$-face if $d(u_i) = m_i$ for $i = 1, 2, \cdots, k$. We use $F_k(x)$ and $V_k(x)$ to denote the set of all k-faces and k-vertices that are incident or adjacent to x, respectively.

A k-coloring of G is a mapping ϕ from $V(G)$ to a set of size k such that $\phi(x) \neq \phi(y)$ for any adjacent vertices x and y. A graph is k-colorable if it has a k-coloring.

A graph G is k-colorable with deficiency d, or simply $(k,d)^*$-colorable, if the vertices of G can be colored with k colors so that each vertex has at most d neighbors receiving the same color as itself. An $(k,0)^*$-coloring is an ordinary k-coloring. Given a list assignment L, an L-coloring with deficiency d, or an $(L,d)^*$-coloring of G, is a mapping $\phi : V(G) \rightarrow \bigcup_{v \in V(G)} L(v)$ such that $\phi(v) \in L(v)$ and every vertex has at most d neighbors receiving the same color as itself. A graph G is called $(k,d)^*$-choosable, if there exists an $(L,d)^*$-coloring for every list assignment L with $|L(v)| = k$ for all $v \in V(G)$.

The concept of list improper coloring was first introduced by Škrekovski [8], and Eaton and Hull [5], independently. They proved that every planar graph is $(3,2)^*$-choosable and every outerplanar graph is $(2,2)^*$-choosable. Let $g(G)$ denote the girth of a graph G, i.e, the length of a shortest cycle. Škrekovski [10] proved that every planar graph G is $(2,1)^*$-choosable if $g(G) \geq 9$, $(2,2)^*$-choosable if $g(G) \geq 7$, $(2,3)^*$-choosable if $g(G) \geq 6$, and $(2,d)^*$-choosable if $g(G) \geq 5$ and $d \geq 4$. In [5], Škrekovski proved that every plane graph without 3-cycles is $(3,1)^*$-choosable. In [7] it was proved that every planar graph without 4-cycles and l-cycles for some $l \in \{5,6,7\}$ is $(3,1)^*$-choosable. In [14], Dong and Xu proved that every plane graph without 4-cycles and l-cycles for some $l \in \{8,9\}$ is $(3,1)^*$-choosable. In [12], it is proved that every plane graph without neither adjacent triangles nor 5-cycles is $(3,1)^*$-colorable. Interested readers may read [6], [11] for more results and references.

For toroidal graphs, in [3], it is proved that every toroidal graph is $(5,1)^*$- and $(3,2)^*$-colorable. In [13], Xu and Zhang proved that every toroidal graph without adjacent triangles is $(4,1)^*$-choosable.

In this note, we make some investigations on $(3,2)^*$-choosability of toroidal graphs.

2 Toroidal graphs without triangles

We first prove a lemma on the structure of triangle-free toroidal graphs.

Lemma 2.1. *Let G be a triangle-free toroidal graph. Suppose that $\delta(G) \geq 3$, each 3-vertex is adjacent to only 5^+-vertices, and each 5-vertex is adjacent to at most three 3-vertices. Then*

(1) *$\Delta(G) \leq 6$, and G contains no 5^+-faces.*

(2) *Each 6-vertex of G is adjacent to six 3-vertices if $\Delta(G) = 6$; and each 5-vertex is adjacent to exact three 3-vertices if $\Delta(G) = 5$.*

Proof : Assume to the contrary that the theorem does not hold.

In the beginning, each vertex v is assigned a charge $w(v) = d(v) - 4$ and each face f is assigned a charge $w(f) = d(f) - 4$. By applying Eulers formula $|V| + |F| - |E| = 0$ for toroidal graphs, we have

$$\Sigma_{x \in V(G) \cup F(G)} w(x) \leq 0.$$

If we obtain a new weight $w^*(x)$ for all $x \in V \cup F$ by transferring weights from one element to another, then we also have $\sum w^*(x) \leq 0$. If these transfers result in $w^*(x) > 0$ for all $x \in V \cup F$, then we get a contradiction and the theorem is proved.

The new charge function $w^*(x)$ is obtained by following discharging rules given below:

(R_1) : For every vertex v with $d(v) \geq 5$, we transfer $\frac{1}{3}$ from v to each incident 3-vertex.

Let v be a k-vertex of G. If $k = 3$, $w^*(v) = 3 - 4 + 3 \times \frac{1}{3} = 0$.
If $k = 4$, $w^*(v) = w(v) = 4 - 4 = 0$.
If $k = 5$, $w^*(v) \geq 5 - 4 - 3 \times \frac{1}{3} = 0$.
If $k = 6$, we have $w^*(v) \geq k - 4 - 6 \times \frac{1}{3} = 0$.
If $k \geq 7$, we have $w^*(v) = k - 4 - k \times \frac{1}{3} > 0$.
Let f be a h-face of G. Since G is triangle-free, $h \geq 4$, $w^*(f) \geq 0$.

Thus, $w^*(x) \geq 0$ for each $x \in V(G) \cup F(G)$. If G contains a 7^+-vertex or a 5^+-face, $\Sigma_{x \in V(G) \cup F(G)} w^*(x) > 0$. If $\Delta = 6$ and there exists a 6-vertex x adjacent to at most five 3-vertices, $\Sigma_{x \in V(G) \cup F(G)} w^*(x) > 0$. If $\Delta = 5$ and there exists a 5-vertex x adjacent to at most two 3-vertices, $w^*(x) > 0$. Also we have $\Sigma_{x \in V(G) \cup F(G)} w(x)^* > 0$. This contradiction completes the proof. ∎

Theorem 2.1. *Every triangle-free toroidal graph is $(3, 2)^*$-choosable.*

Proof : Assume to the contrary. Let G be a counterexample with the fewest vertices, i.e., there exists a list assignment L with $|L(v)| = 3$ for all

$v \in V(G)$ such that G is not $(L,2)^*$-choosable, but any proper subgraph of G is.

If $\delta(G) < 3$, let v be a 2^--vertex of G. Then, $G\backslash\{v\}$ is $(3,2)^*$-choosable by the choice of G. Since in any $(L,2)^*$-coloring of $G-v$, there must exist a color in $L(v)$ that is not used by any neighbors of v, any $(L,2)^*$-coloring of $G-v$ can be extended to a $(L,2)^*$-coloring of G, a contradiction. So we assume that $\delta(G) \geq 3$.

If G contains two adjacent 3-vertices, say u and v, then by the choice of G, $G\backslash\{u,v\}$ is $(3,2)^*$-choosable. In any $(L,2)^*$-coloring of $G\backslash\{u,v\}$, there exists a color in $L(u)$ that is not used by any neighbors of u in $G\backslash\{u,v\}$, and the same holds for v. Applying the same argument as above, we see that G is $(L,2)^*$-choosable, a contradiction.

If G contains a 3-vertex u adjacent to a 4-vertex v, there is an $(L,2)^*$-coloring of $G\backslash\{u\}$. If v has two neighbors which received the same color as itself, then the number of distinct colors used by the neighbors of v in $G\backslash\{u\}$ is at most 2. Then we recolor v such that the color of v is distinct to those of it neighbors. So we assume that in any $(L,2)$-coloring of $G\backslash\{u\}$ at most one neighbor of v receives the same color as v. We may color u so that, among all neighbors of u in G, only v may have the same color as u. to establish an $(L,2)^*$-coloring of G.

If G contains a face f with the boundary $v_1v_2\cdots v_3v_4\ v_k$, where $d(v_i) = 4, 1 \leq i \leq k$. Let $H = G\backslash V(f)$. By the choice of G, H admits an $(L,2)^*$-coloring ϕ. For $w \in V(f)$, let $L'(w) = L(w)\backslash\{\phi(v)|v \in N_H(w)\}$. Then, $|L'(w)| \geq 1$. It is easy to verify that $v_1v_2\cdots v_3v_4v_k$ admits an $(L',2)^*$-coloring. This together with ϕ leads to an $(L,2)^*$-coloring of G.

Now we suppose that $\delta \geq 3$ and $\Delta \geq 5$, each 3-vertex is adjacent to 5^+-vertices. If G contains a 5-vertex v adjacent at least four 3-vertices, let x_i for $i = 1,2,3,4,5$, be the neighbors of v in clockwise and $d(x_i) = 3$ for $i = 1,2,3,4$. By the choice of G, $G' = G\backslash\{v,x_1,x_2,x_3,x_4\}$ is $(3,2)^*$-choosable. Let φ' be an $(L,2)^*$-coloring of G'. Take any list assignment $|L(v)| = 3$, let $L'(u) = L(u)$ for $u \in V(G)\backslash\{v,x_1,x_2,x_3,x_4\}$ and $L'(u) = L(u)\backslash\varphi'(u)$ for $u \in \{v,x_1,x_2,x_3,x_4\}$. $L'(v) \geq 2$ and $L'(x_i) \geq 1$ for $i = 1,2,3,4$. We color x_i for $i = 1,2,3,4$ with a color in $L'(x_i)$. Then, there exists a color $\alpha \in L'(v)$ that is assigned to at most two of x_1,x_2,x_3,x_4. By coloring v with α, we get an $(L,2)^*$-coloring of G.

Suppose that $\delta \geq 3$ and each 3-vertex is adjacent to 5^+-vertices and each 5-vertex is adjacent to at most three 3-vertices, by lemma 2.1, G contains neither 7^+-vertices nor 5^+-faces. So we assume that G contains only 6^--vertices and each face of G is degree 4.

If $\Delta = 6$, then each 6-vertex of G is adjacent to six 3-vertices. Assume that v is a 6-vertex and x_i for $i = 1,2,3,4,5,6$, be the neighbors of v in clockwise and $d(x_i) = 3$ for $i = 1,2,3,4,5,6$. By the choice of G, $G' = G\backslash\{v,N(v)\}$ is $(3,2)^*$-choosable. Let φ' be an $(L,2)^*$-coloring of

G'. Let $L'(u) = L(u)$ for $u \in V(G)\backslash\{v, x_1, x_2, x_3, x_4, x_5, x_6\}$ and $L'(u) = L(u)\backslash\varphi'(u)$ for $u \in \{v, x_1, x_2, x_3, x_4x_5, x_6\}$. It is easy to extend an $(L, 2)^*$-coloring of $G\backslash\{v, x_1, x_2, x_3, x_4, x_5, x_6\}$ to an $(L', 2)^*$-coloring of G.

Finally assume that $\Delta = 5$. By Lemma 2.1, each 5-vertex of G is adjacent to exact three 3-vertices. Since each 3-vertex is adjacent to 5-vertices, G contains a cycle $C = v_1v_2 \cdots v_i \cdots v_{2k-1}v_{2k}$ such that $d(v_i) = 5$ if i is odd and $d(v_i) = 3$ if i is even. We assume that w_i is the other neighbor of v_i for even i. By the minimality of G, $H = G\backslash\{v_2, v_4, \cdots, v_{2k}\}$ admits an $(L, 2)^*$-coloring φ'. Note that $d_H(v_i) = 3$ for odd i, and we can always color them such that the color $\varphi'(v_i)$ appears at most once in those of $N_H(v_i)$.

then we color v_i for even i with a color in $L(v_i)\backslash\{\varphi'(v_{i-1}), \varphi'(w_i)\}$ to extend φ' to the whole graph G. This completes the proof of Theorem 2.1. ∎

3 Toroidal graphs without adjacent triangles

Let $\mathcal{G}$ denote the family of toroidal graphs which contain neither adjacent 3-cycles nor 5-cycles, Our next main result shows that every graph in $\mathcal{G}$ is $(3, 2)^*$-choosable.

In this section, a 4-face is called bad if it contains two 3-vertices. A face f is called light if each vertex on its boundary has degree at most 4.

Lemma 3.1. *Let G be a graph in $\mathcal{G}$. Then one of the following must hold:*

(1) $\delta(G) < 3$.

(2) *G contains a 3-vertex adjacent to some 4^--vertex.*

(3) *G contains a 5-vertex adjacent to at least four 3-vertices.*

(4) *G contains a 6-vertex adjacent to just six 3-vertices.*

(5) *G contains a light face.*

(6) *G contains a $(3, 5, 3, 5)$- face.*

Proof : Assume to the contrary that the conclusion does not hold.

In the beginning, each vertex v is assigned a charge $w(v) = d(v) - 4$ and each face f is assigned a charge $w(f) = d(f) - 4$. By applying Eulers formula $|V| + |F| - |E| = 0$ for toroidal graphs, we have

$$\Sigma_{x \in V(G) \cup F(G)} w(x) \leq 0.$$

If we obtain a new weight $w^*(x)$ for all $x \in V \cup F$ by transferring weights from one element to another, then we also have $\sum w^*(x) \leq 0$. If these transfers result in $w^*(x) > 0$ for all $x \in V \cup F$, then we get a contradiction and the conclusions proved.

Let r_v be the number of 3-faces incident with v.

By the choice of G, we have the following observations.

(O_1): G contains neither adjacent 3-faces nor a 3-face adjacent to a 4-face.

(O_2): $r_v \leq 2$ if $d(v) = 5$.

(O_3): $|V_3(f)| \leq \lfloor d(f)/2 \rfloor$ for all $f \in F(G)$.

The new charge function $w^*(x)$ is obtained by following discharging rules given below:

(R_1) : For every vertex v with $d(v) \geq 5$, we transfer $\frac{1}{3}$ from v to each incident 3-face.

(R_2) : For every vertex v with $d(v) \geq 5$, we transfer $\frac{1}{3}$ to each bad 4-face, transfer $\frac{1}{6}$ to each incident 4-face containing just two 5^+-vertices and a unique 3-vertex, $\frac{1}{9}$ to each incident 4-face containing three 5^+-vertex and a unique 3-vertex.

(R_3) : For every 4^--face f, we transfer $\frac{1}{3}$ from f to each incident 3-vertex.

(R_4) : For every face f with $d(f) \geq 6$, we transfer $\frac{1}{3}$ from f to each incident 3-vertex, and $\frac{1}{3}$ from f to each adjacent 3-face.

Let v be a k-vertex of G. If $k = 3$, by R_3 and $R4$, $w^*(v) = 3-4+3\times\frac{1}{3} = 0$.

If $k = 4$, $w^*(v) = w(v) = 4 - 4 = 0$.

If $k = 6$, by the choice of G, each 6-vertex is adjacent to at most five 3-vertices, and $r_v \leq 3$. We have $w^*(v) \geq 6-4-r_v\times\frac{1}{3}-(6-2r_v-1)\frac{1}{6} > 0$ by $R4$.

If $k \geq 7$, we have $w^*(v) = k - 4 - k \times \frac{1}{3} > 0$ by $R4$.

Suppose that $k = 5$. Let v be a 5-vertex and x_1, x_2, x_3, x_4, x_5 be the neighbors of v in clockwise. By (O_2), we have $w^*(v) = 5 - 4 - 2 \times \frac{1}{3} > 0$ if $r_v = 2$, since v is not adjacent to any 4-faces at this situation.

If $r_v = 1$, $w^*(v) = 5 - 4 - 1 \times \frac{1}{3} - 2 \times \frac{1}{3} \geq 0$. Moreover, $w^*(v) = 5 - 4 - 1 \times \frac{1}{3} - 2 \times \frac{1}{6} > 0$ if v is not incident to any bad 4-faces.

If $r_v = 0$, since v is adjacent to at most three 3-vertices, v is incident to at most two bad 4-faces. By $R1$ and $R2$, we have the following cases:

(1) v is incident to two bad 4-faces. Obviously, the two bad 4-faces are adjacent and v is adjacent to three 3-vertices. Assume that $d(x_1) = d(x_2) = d(x_3) = 3$. If $d(x_4) = d(x_5) = 4$, $w^*(v) \geq 5-4-2\times\frac{1}{3}-2\times\frac{1}{6} \geq 0$. If $d(x_4) = 4$ and $d(x_5) \geq 5$, $w^*(v) \geq 5 - 4 - 2 \times \frac{1}{3} - 1 \times \frac{1}{6} - 1 \times \frac{1}{9} \geq 0$. If $d(x_4) \geq 5$ and $d(x_5) \geq 5$, $w^*(v) \geq 5 - 4 - 2 \times \frac{1}{3} - 3 \times \frac{1}{9} \geq 0$.

(2) v is incident to just one bad 4-face. $w^*(v) \geq 5-4-1\times\frac{1}{3}-4\times\frac{1}{6} \geq 0$.

(3) v is not incident to any bad 4-faces. $w^*(v) \geq 5 - 4 - 5 \times \frac{1}{6} > 0$.

Let f be an h-face of G.

If $h \geq 6$, by (O_3), (R_3) and (R_4), $w^*(f) = h-4-|V_3(f)| \times \frac{1}{3} - |F_3(f)| \times \frac{1}{3} \geq 0$, noting that $|V_3(f)| + |F_3(f)| \leq h$.

If $h = 3$. By assumption, G contains no light faces. f is incident with at least one 5^+-vertex, and at most one 3-vertex. By $R1$, $R3$ and $R4$, $w^*(f) = w(f) + 4 \times \frac{1}{3} - \frac{1}{3} = 0$ if $|V_3(f)| = 1$. If f is not incident with any 3-vertex, then $w^*(f) \geq w(f) + 3 \times \frac{1}{3} + \frac{1}{3} > 0$.

If $h = 4$, by the choice of G and (O_3), $|V_3(f)| \leq 2$ and $|F_3(f)| = 0$. By $R2$ and $R3$, we have the following:

If $|V_3(f)| = 2$, since G contains no light faces, f is incident to two 5^+-vertices, we have $w^*(f) \geq h - 4 - 2 \times \frac{1}{3} + 2 \times \frac{1}{3} \geq 0$.

If $|V_3(f)| = 1$, since G contains no light faces, f is incident to at least two 5^+-vertices, we have $w^*(f) \geq 4 - 4 - 1 \times \frac{1}{3} + 2 \times \frac{1}{6} \geq 0$ or $w^*(f) \geq 4 - 4 - 1 \times \frac{1}{3} + 3 \times \frac{1}{9} \geq 0$.

If $|V_3(f)| = 0$, $w^*(f) \geq 4 - 4 = 0$.

Thus, $w^*(x) \geq 0$ for each $x \in V(G) \cup F(G)$. If G contains 6^+-vertices, $w^*(x) > 0$, the conclusion is proved. So we assume that $\Delta(G) = 5$. By the hypothesis, G contains no $(3,5,3,5)$-face, G contains no bad 4-faces. Let v^* be any 5-vertex, $w^*(v^*) > 0$.

We have $0 < \Sigma_{x \in V \cup F}\, w^*(x) = \Sigma_{x \in V \cup F}\, w(x) \leq 0$, which completes the proof. ∎

Theorem 3.1. *Every graph in* $\mathcal{G}$ *is* $(3,2)^*$*-choosable.*

Proof : Assume to the contrary. Let G be a counterexample with the fewest vertices, i.e., there exists a list assignment L with $|L(v)| = 3$ for all $v \in V(G)$ such that G is not $(L,2)^*$-choosable, but any proper subgraph of G is.

If $\delta(G) < 3$, let v be a 2-vertex of G. Then, $G\backslash\{v\}$ is $(3,2)^*$-choosable by the choice of G. Since in any $(L,2)^*$-coloring of $G - v$, there must exist a color in $L(v)$ that is not used by any neighbors of v, any $(L,2)^*$-coloring of $G - v$ can be extended to a $(L,2)^*$-coloring of G, a contradiction. So we assume that $\delta(G) \geq 3$.

If G contains two adjacent 3-vertices, say u and v, then by the choice of G, $G\backslash\{u,v\}$ is $(3,2)^*$-choosable. In any $(L,2)^*$-coloring of $G\backslash\{u,v\}$, there exists a color in $L(u)$ that is not used by any neighbors of u in $G\backslash\{u,v\}$, and the same holds for v. Applying the same argument as the above, we see that G is $(L,2)^*$-choosable, a contradiction.

If G contains a 3-vertex u adjacent to a 4-vertex v,there is an $(L,2)^*$-coloring of $G\backslash\{u\}$. If v has two neighbors which received the same color with itself, then the number of distinct colors used by the neighbors of v in $G\backslash\{u\}$ is at most 2. Then we recolor v such that the color of v is distinct to those of it neighbors. We may color u so that, among all neighbors of u in G, only v may have the same color as u. to establish an $(L,2)^*$-coloring of G.

If G contains a 5-vertex v adjacent at least four 3-vertices, let x_i for $i = 1, 2, 3, 4, 5$, be the neighbors of v in clockwise and $d(x_i) = 3$ for $i = 1, 2, 3, 4$. By the choice of G, $G\backslash\{v, x_1, x_2, x_3, x_4\}$ is $(3, 2)^*$-choosable. Take any list assignment $|L(v)| = 3$, let $L'(u) = L(u)$ for $u \in V(G)\backslash\{v, x_1, x_2, x_3, x_4\}$ and $L'(u) = L(u) - A(u)$ for $u \in \{v, x_1, x_2, x_3, x_4\}$, where $A(x_i)$ denotes the colors assigned to the neighbors in the $(L, 2)^*$-coloring of $G\backslash\{v, x_1, x_2, x_3, x_4\}$. $L'(v) \geq 2$ and $L(x_i) \geq 1$ for $i = 1, 2, 3, 4$. It is easy to extend an $(L, 2)^*$-coloring of $G\backslash\{v, x_1, x_2, x_3, x_4\}$ to an $(L', 2)^*$-coloring of G.

If G contains a 6-vertex adjacent to six 3-vertices, assume that v is a 6-vertex and x_i for $i = 1, 2, 3, 4, 5, 6$, be the neighbors of v in clockwise and $d(x_i) = 3$ for $i = 1, 2, 3, 4, 5, 6$. By the choice of G, $G\backslash\{v, N(v)\}$ is $(3, 2)^*$-choosable. Take any list assignment $|L(v)| = 3$, let $L'(u) = L(u)$ for $u \in V(G)\backslash\{v, x_1, x_2, x_3, x_4, x_5, x_6\}$ and $L'(u) = L(u) - A(u)$ for $u \in \{v, x_1, x_2, x_3, x_4 x_5, x_6\}$, where $A(x_i)$ denotes the colors assigned to the neighbors in the $(L, 2)^*$-coloring of $G\backslash\{v, x_1, x_2, x_3, x_4, x_5, x_6\}$. $L'(v) \geq 2$ and $L(x_i) \geq 1$ for $i = 1, 2, 3, 4, 5, 6$. It is easy to extend an $(L, 2)^*$-coloring of $G\backslash\{v, x_1, x_2, x_3, x_4, x_5, x_6\}$ to an $(L', 2)^*$-coloring of G.

Now we assume that G contains a light face f with the boundary $v_1 v_2 \cdots v_3 v_4\ v_k$, where $d(v_i) \leq 4, 1 \leq i \leq k$. Let $H = G\backslash V(f)$. By the choice of G, H admits an $(L, 2)^*$-coloring ϕ. For $w \in V(f)$, let $L'(w) = L(w)\backslash\{\phi(v)|v \in N_H(w)\}$. Then, $|L'(w)| \geq 1$. It is easy to verify that $v_1 v_2 \cdots v_3 v_4 v_k$ admits an $(L', 2)^*$-coloring. This together with ϕ leads to an $(L, 2)^*$-coloring of G.

Finally assume that $f = [v x_2 u x_1]$ be a bad 4-face of G, and let v, u be 5-vertices and $d(x_i) = 3$ for $i = 1, 2$. Assume $N(x_1) = \{u, v, w_1\}$, $N(x_2) = \{u, v, w_2\}$. By the choice of G, $H = G\backslash\{x_1, x_2\}$, H admits an $(L, 2)^*$-coloring ϕ. Note that $d_H(u) = d_H(v) = 3$, and we can always color u (and v) such that the color $\phi(u)$ (and $\phi(v)$) appears at most once in those of $N_H(u)$ (and $N_H(v)$).

If $\phi(u) = \phi(v)$, then we color x_i with a color in $L(x_i)\backslash\{\phi(u), \phi(w_i)\}$ for $i = 1, 2$ to extend ϕ to the whole graph G. If $\phi(u) \neq \phi(v)$, we can color x_1 with a color in $L(x_1)\backslash\{\phi(u), \phi(w_1)\}$ and x_2 with a color in $L(x_2)\backslash\{\phi(v), \phi(w_2)\}$ to extend ϕ to the whole graph G.

This contradiction completes the proof of the theorem. ∎

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