On Uniquely 4-List Colorable Complete Multipartite Graphs*

Yanning Wang[a,b], Yufa Shen[c,†], Guoping Zheng[c], Wenjie He[d]

[a]College of Science, Yanshan University,
Qinhuangdao 066004, P.R. China

[b]Key Lab of Industrial Computer Control Engineering of Hebei Province,
Institute of Electrical Engineering, Yanshan University,
Qinhuangdao, Hebei 066004, China

[c]Department of Mathematics and Physics,
Hebei Normal University of Science and Technology,
Qinhuangdao 066004, P.R. China

[d]Applied Mathematics Institute, Hebei University of Technology,
Tianjin 300401, P.R.China

Mathematics Subject Classifications: 05C15.

Abstract

A graph G is called *uniquely k-list colorable*, or $UkLC$ for short, if it admits a *k-list assignment* L such that G has a *unique L-coloring*. A graph G is said to have *the property* $M(k)$ (M for Marshal Hall) if and only if it is not $UkLC$. The *m-number* of a graph G, denoted by $m(G)$, is defined to be the least integer k such that G has the property $M(k)$. After M. Mahdian and E.S. Mohmoodian characterized the $U2LC$ graphs, M. Ghebled and E.S. Mohmoodian characterized the $U3LC$ graphs for complete multipartite graphs except for nine graphs in 2001. Recently, W. He et al. verified all the nine graphs are not $U3LC$ graphs. Namely, the $U3LC$ complete multipartite graphs are completely characterized. In this paper, complete multipartite graphs whose m–number are equal to 4 are researched and the $U4LC$ complete multipartite graphs, which have at least 6 parts, are characterized except for finitely many of them. At the same time, we give some results about some complete multipartite graphs whose

*This research was supported by Science Foundation of the Education Department of Hebei Province, P.R.China (2005108), and the National Natural Science Foundation of China (10871058).

†Corresponding author. *E-mail:* syf030514@163.com (Yufa Shen).

number of parts is smaller than 6.

Keywords: List coloring, $U4LC$ graphs, property $M(4)$, complete multipartite graphs

1 Introduction

We consider undirected, finite, simple graphs. For the necessary definitions and notation we refer the reader to standard texts, such as [1]. In this paper, we use the notation K_{s*r} for a complete r-partite graph in which each part is of size s. Notation such as $K_{s*r,t}$, etc, are used similarly.

For a graph $G = (V, E)$ and each vertex $u \in V(G)$, let $L(v)$ denote a list of colors available for v. $L = \{L(u)|u \in V(G)\}$ is said to be *a list assignment* of G. If $|L(v)| = k$ for all $v \in V(G)$, L is called *k-list assignment* of G. A *list coloring* from a given collection of lists is a proper coloring c such that $c(v)$ is chosen from $L(v)$. We will refer to such a coloring as an *L-coloring.*

The idea of list colorings of graphs is due independently to V.G. Vizing [13] and to P. Erdős, A.L. Rubin, and H. Taylar [3]. For a survey on list coloring we refer the interested reader to D.R. Woodall [14] and Ch. Eslahchi, M. Ghebleh, and H. Hajiabolhassan [4]. Here we mention some definitions and results about list colorings which are referred throughout the paper.

The concept of *unique list coloring* was introduced by J.H. Dinitz and W.J. Martin [2] and independently by E.S. Mahmoodian and M. Mahdian [10], which can be used to study defining set of k–coloring [11] and critical sets in Latin squares [8]. Let G be a graph with n vertices and suppose that for each vertex v in G, there exists a list of k colors $L(v)$, such that there exists a unique L-coloring for G, then G is called *uniquely k-list colorable graph* or a *UkLC graph* for short. For a graph G, it is said to have *the property* $M(k)$ (M for Marshal Hall) if and only if it is not uniquely k-list colorable. So G has the property $M(k)$ if for any collection of lists assigned to its vertices, each of size k, either there is no list coloring for G or there exist two list colorings. The m-number of a graph G, denoted by $m(G)$, is defined to be the least integer k such that G has the property $M(k)$.

In 1999, M. Mahdian and E.S. Mahmoodian characterized uniquely 2-list colorable graphs. They showed that

Theorem 1.1 ([9]). *A connected graph has the property* $M(2)$ *if and only if every block of* G *is either a cycle, a complete graph, or a complete bipartite graph.*

It seems that characterizing $UkLC$ graphs for any k is not easy. M. Ghebleh and E.S. Mahmoodian have characterized $U3LC$ graphs for complete multipartite graphs except for finitely many of them. They showed that

Theorem 1.2 ([5]). *The graphs* $K_{3,3,3}$, $K_{2,4,4}$, $K_{2,3,5}$, $K_{2,2,9}$, $K_{1,2,2,2}$, $K_{1,1,2,3}$, $K_{1,1,1,2,2}$, $K_{1*4,6}$, $K_{1*5,5}$, *and* $K_{1*6,4}$ *are U3LC.*

Theorem 1.3 ([5]). *Let G be a complete multipartite graph that is not* $K_{2,2,r}$, *for* $r = 4, 5, \dots, 8$, $K_{2,3,4}$, $K_{1*4,4}$, $K_{1*4,5}$, *or* $K_{1*5,4}$ *then G is U3LC if and only if it has one of the graphs in Theorem 1.2 as an induced subgraph.*

Recently, Wenjie He et al. [6, 7, 15, 12] researched the exempted graphs in Theorem 1.3 and they have showed that all these exempted graphs have the property $M(3)$. Then they gave an improved version of Theorem 1.3 as follows.

Theorem 1.4 ([12]). *Let G be a complete multipartite graph, then G is U3LC if and only if it has one of the graphs in Theorem 1.2 as an induced subgraph.*

In this paper, complete multipartite graphs whose m–number are equal to 4 are researched and the $U4LC$ complete multipartite graphs, which have at least 6 parts, are characterized except for finitely many of them. At the same time, the property $M(4)$ of some complete multipartite graphs whose number of parts is smaller than 6 are investigated. In section 2, we give some propositions about complete multipartite graphs whose m–number are equal to 4, and these propositions will pave the way to prove our main results. In section 3 and section 4, we investigate $U4LC$ complete k-partite graphs for $k \geq 7$ and for $k = 6$ respectively. In section 5, we give some results about of some complete k-partite($k \leq 5$) graphs which will pave the way in characterization of $U4LC$ complete k-partite($k \leq 5$) graphs. In section 6, we give some open problems.

2 Some propositions about complete multipartite graphs whose m–number are equal to 4

In order to characterize $U4LC$ complete multipartite graphs by using the method of characterizing $U3LC$ complete multipartite graphs in [5], it is clear that we must determine all the maximal complete multipartite graphs G with m–number 4, where the maximality means that, for any complete multipartite graphs H, if G is a proper induced subgraph of H, then $m(H) > 4$.

As a preparation, in the following we give some complete multipartite graphs whose m–number are equal to 4.

Lemma 2.1 ([5]). *If G is a complete multipartite graph which has an induced $UkLC$ subgraph, then G is $UkLC$.*

Lemma 2.2 ([10]). *If L is a k-list assignment to the vertices in the graph G, and G has a unique L-coloring, then $|\bigcup_v L(v)| \geq k+1$ and all these colors are used in the unique L-coloring of G.*

Lemma 2.3 ([5]). *For any integer s, r, $K_{1,s,r}$ has the property $M(3)$.*

Lemma 2.4 ([5]). *If G has n vertices, then $m(G) \leq \lceil n/3 \rceil + 1$.*

Proposition 2.1. *If G is a graph which has the property $M(3)$, and v is a vertex which does not belongs to $V(G)$, then the union graph $G \vee \{v\}$ has the property $M(4)$.*

Proof. Let $H = G \vee \{v\}$. Suppose L is a 4-list assignment to H which induced a list coloring c. Let $c(v) = a$. We introduce a 3-list assignment L' to G as follows. For every vertex u in G, if $a \in L(u)$ then $L'(u) = L(u)\backslash\{a\}$, otherwise $L'(u) = L(u)\backslash\{b\}$ where $b \in L(u)$ and $b \neq c(u)$. Since L induce a list coloring c for H, G has exactly one L'-coloring, namely the restriction of c on G. By the property $M(3)$ of G, we can obtain a new L'–coloring of G, which can be extended to H. Thus, H has the property $M(4)$. □

Proposition 2.2. *For every $r \geq 1$, $K_{1*r,5}$ has the property $M(4)$, and if $r \geq 5$, $m(K_{1*r,5}) = 4$.*

Proof. For $G = K_{1*r,5}$, denote its $r+1$ parts by $V_i = \{v_i\}$ for $i = 1, 2, \ldots, r$ and $U = \{u_1, u_2, \ldots, u_5\}$, and let c be a 4-list coloring of G with a given 4-list assignment L.

Case 1. $c(u_1), c(u_2), \ldots, c(u_5)$ are pairwise different.

Add new edges between any two vertices in $\{u_1, u_2, \ldots, u_5\}$, the resulting graph is $G' = K_{r+5}$. Note that c is also a proper L–coloring of G', and G' has the property $M(2)$ by Theorem 1.1, so we can obtain another coloring of G', which is a legal L–coloring for G, and hence G has the property $M(4)$.

Case 2. In $\{u_1, u_2, \ldots, u_5\}$, there are at least two assigned a common color.

Let the common color be a and $X = \{u | c(u) = a, u \in \{u_1, u_2, \ldots, u_5\}\}$. Consider the graph $H = G - X$, clearly, H is a subgraph of $K_{1*r,3}$. Let $L'(v) = L(v)\backslash\{a\}$ for every $v \in V(H)$. It is obvious that $|L'(v)| \geq 3$ for every $v \in V(G)$ and the restriction of c on G' is a L'-coloring of G'. By Theorem 1.4, $K_{1*r,3}$ has the property $M(3)$, so H also has the property $M(3)$. By the property $M(3)$ of H, we can obtain a new L'–coloring of H, which can be extended to G. Thus, G has the property $M(4)$.

If $r \geq 5$, G is $U3LC$ by Theorem 1.4. Therefore $m(K_{1*r,5}) = 4$. □

Proposition 2.3. *For every $r \geq 1$, $K_{1*5,r}$ has the property $M(4)$, and if $r \geq 5$, $m(K_{1*5,r}) = 4$.*

Proof. For $G = K_{1*5,r}$, denote its 6 parts by $V_i = \{v_i\}$ for $i = 1, 2, \ldots, 5$ and $U = \{u_1, u_2, \ldots, u_r\}$, and let c be a 4-list coloring of G with a given 4-list assignment L. Put $C = \{c(u_i) | i = 1, 2, \ldots, r\}$. Without loss of generality, write $c(v_i) = i$ for $i = 1, 2, \ldots, 5$. If there are two vertices u_i and u_j such that $c(u_i) \in L(u_j) \backslash c(u_j)$ $(i, j = 1, 2, \ldots, r)$, a new L-coloring can be found easily. Now it is supposed that $c(u_i) \notin L(u_j) \backslash c(u_j)$ for each i, j $(i, j = 1, 2, \ldots, r)$. By Lemma 2.2, we know that $(L(u_i) \backslash c(u_i)) \subset \{c(v_1), c(v_2), \ldots, c(v_5)\} = \{1, 2, \ldots, 5\}$.

Case 1. $|L(v_i) \cap C| \leq 2$ for any $i = 1, 2, 3, 4, 5$.

Denote $G' = G[v_1, v_2, \ldots, v_5]$ and let $L'(v_i) = L(v_i) \backslash (L(v_i) \cap C)$ for every v_i for $i = 1, 2, \ldots, 5$. Clearly, $|L'(v_i)| \geq 2$. As $G' = K_5$ and K_5 has the property $M(2)$ by Theorem 1.1, we can obtain a new L'–coloring of G', which can be extended to G. Thus, G has the property $M(4)$.

Case 2. In $\{v_1, v_2, \ldots, v_5\}$, there exists at least one vertex v such that $|L(v) \cap C| \geq 3$.

Without loss of generality, say $|L(v_1) \cap C| \geq 3$, namely, $L(v_1) = \{1, a_1, b_1, c_1\}$, where $a_1, b_1, c_1 \in C$ and they are pairwise different. Denote $C_1 = \{1\} \cup C$ and consider $L(v_2), L(v_3), L(v_4)$ and $L(v_5)$. If $|L(v_i) \cap C_1| \leq 2$ for any $i = 2, 3, 4, 5$, similar to the discussion in Case 1, by the property $M(2)$ of K_4 we can obtain a new L''–coloring of $G'' = G[v_2, v_3, v_4, v_5] = K_4$, which can be extended to G. Thus, G has the property $M(4)$. Otherwise, there exists at least one vertex v in $\{v_2, \ldots, v_5\}$ such that $|L(v) \cap C_1| \geq 3$. Without loss of generality, say $|L(v_2) \cap C_1| \geq 3$. Namely, $L(v_2) = \{2, a_2, b_2, c_2\}$, where $a_2 \in C_1$, $b_2, c_2 \in C$. Denote $C_2 = \{1, 2\} \cup C$ and consider $L(v_3), L(v_4)$ and $L(v_5)$. If $|L(v_i) \cap C_2| \leq 2$ for any $i = 3, 4, 5$, similar to the discussion above, by the property $M(2)$ of K_3 we can obtain a new L'''–coloring of $G''' = G[v_3, v_4, v_5] = K_3$, which can be extended to G. Thus, G has the property $M(4)$. If in $\{v_3, v_4, v_5\}$, there exists at least one vertex v such that $|L(v) \cap C_2| \geq 3$, without loss of generality, say $|L(v_3) \cap C_2| \geq 3$. Namely, $L(v_3) = \{3, a_3, b_3, c_3\}$, where $\{a_3, b_3\} \subset C_2$, $c_3 \in C$. It is clear that we can choose a color from $\{b_2, c_2\}$, say c_2, such that $c_2 \neq c_3$. Furthermore we can choose a color from $\{a_1, b_1, c_1\}$, say c_1, such that c_3, c_2 and c_1 are pairwise different. Note that $(L(u_i) \backslash c(u_i)) \subset \{1, 2, 3, 4, 5\}$ for $i = 1, 2, \ldots, r$, we have $(L(u_i) \backslash c(u_i)) \cap \{1, 2, 3\} \neq \Phi$ for $i = 1, 2, \ldots, r$. So we can let $c'(v_4) = c(v_4)$, $c'(v_5) = c(v_5)$, $c'(v_1) = c_1$, $c'(v_2) = c_2$, $c'(v_3) = c_3$ and $c'(u_i) \in (\{1, 2, 3\} \cap (L(u_i) \backslash c(u_i)))$ for $i = 1, 2, \ldots, r$. Thus c' is a new L-coloring of G and hence G has the property $M(4)$.

If $r \geq 5$, $K_{1*5,r}$ is $U3LC$ by Theorem 1.4. Therefore $m(K_{1*r,5}) = 4$. □

Proposition 2.4. *For every $r \geq 1$, $K_{1*r,3,3}$ has the property $M(4)$, and if $r \geq 2$, $m(K_{1*r,3,3}) = 4$.*

Proof. For $G = K_{1*r,3,3}$, let $V_i = \{v_i\}, i = 1, 2, \ldots, r$, W, U be the two parts of G which contain three vertices. Suppose that, for each $v \in V(G)$, there is assigned a color list $L(v)$ of size 4 and G has an L-coloring c.

Case 1. $|\{c(w) : w \in W\}| \leq 2$ or $|\{c(u) : u \in U\}| \leq 2$.

Without loss of generality, it is supposed that $|\{c(w) : w \in W\}| \leq 2$. Then there exist at least two vertices in W which are assigned the same color. Say the color is a, and denote $X = \{w | c(w) = a, w \in W\}$ and $G' = G - X$. Then it is obvious that G' is a subgraph of $K_{1*(r+1),3}$. For every $v \in V(G')$, let $L'(v) = L(v) \backslash \{a\}$. Clearly, $|L'(v)| \geq 3$ for every $v \in V(G')$ and the restriction of c on G' is an L'-coloring of G'. By the property $M(3)$ of $K_{1*(r+1),3}$ (by Theorem 1.4), we can obtain a new L'-coloring of G' which is extendible to G.

Case 2. $|\{c(u) : u \in W\}| = |\{c(u) : u \in U\}| = 3$.

Add new edges between any two vertices in W and any two vertices in U. The resulting graph is K_{s+6}, which has the property $M(4)$ by the Theorem 1.1. So we obtain a new L-coloring which is a legal L-coloring for G. In sum, G has the property $M(4)$.

If $r \geq 2$, G is a $U3LC$ graph by Theorem 1.4, so its m-number is equal to 4. □

Proposition 2.5. $m(K_{1*5,2,4}) = 4$.

Proof. For $G = K_{1*5,2,4}$, denote its 7 parts by $V_i = \{v_i\}$ for $i = 1, 2, \ldots, 5$, $U = \{u_1, u_2\}$ and $W = \{w_1, w_2, w_3, w_4\}$. Let c be a 4-list coloring of G with a given 4-list assignment L.

Case 1. $c(u_1) = c(u_2)$.

Let $G' = G - \{u_1, u_2\}$, $L'(v) = L(v) \backslash c(u_1)$ for every $v \in V(G')$. Clearly, $|L'(v)| \geq 3$ for every $v \in V(G')$. Note that $K_{1*5,4}$ has the property $M(3)$ by Theorem 1.4, then we can obtain a new L'–coloring of G', which can be extended to G. Thus, G has the property $M(4)$.

Case 2. $c(u_1) \neq c(u_2)$.

Add an edge between u_1 and u_2, and the resulting graph is $G' = K_{1*7,4}$. Since $K_{1*7,5}$ has the property $M(4)$ by Proposition 2.2, then $K_{1*7,4}$ has the property $M(4)$ by Lemma 2.1. Thus we can obtain an L–coloring of G', which is a legal L–coloring for G. Thus, G has the property $M(4)$.

Note that $K_{1*5,2,4}$ is a $U3LC$ graph by Theorem 1.4, then we know that $m(K_{1*5,2,4}) = 4$. □

Proposition 2.6. $m(K_{1*4,3,4}) = m(K_{1*3,2,2,2}) = m(K_{1*4,2,5}) = 4$.

Proof. Obviously, $K_{1*4,3,4}$, $K_{1*3,2,2,2}$ and $K_{1*4,2,5}$ are $U3LC$ from Theorem 1.4, then it is only to prove that they have the property $M(4)$. From Lemma 2.4, we have that $m(K_{1*3,2,2,2}) \leq \lceil \frac{9}{3} \rceil + 1 = 4$. Namely $K_{1*3,2,2,2}$ has the property $M(4)$. From Proposition 2.2 and Theorem 1.4, we know that $K_{1*6,5}$ has the property $M(4)$ and $K_{1*4,5}$ has the property $M(3)$. With a similar way with the way used in Proposition 2.5, we obtain that $K_{1*4,2,5}$ has the property $M(4)$. From Proposition 2.2, Lemma 2.1 and Theorem 1.4, we know that $K_{1*7,4}$ has the property $M(4)$ and $K_{1*5,4}$ has the property $M(3)$. With a similar way with the way used in Proposition 2.4, we obtain that $K_{1*4,3,4}$ has the property $M(4)$. □

3 On complete k-partite($k \geq 7$) graphs

In this section, the $U4LC$ complete k-partite($k \geq 7$) graphs are characterized except for finitely many of them.

Theorem 3.1. *The graphs $K_{1*4,2,2,2}$, $K_{1*5,2,5}$, $K_{1*6,2,4}$, $K_{1*6,20}$ and $K_{1*12,6}$ are $U4LC$.*

Proof. 1) For $K_{1*4,2,2,2}$, let $\{v_1\}$, $\{v_2\}$, $\{v_3\}$, $\{v_4\}$, $\{v_5, u_5\}$, $\{v_6, u_6\}$, $\{v_7, u_7\}$ be the parts. We assign the following lists for the vertices of $K_{1*4,2,2,2}$: $L(v_i) = \{i, 5, 6, 7\}$ for i=1,2,3,4, $L(v_i) = \{4, 5, 6, 7\}$ for i=5,6,7 and $L(u_i) = \{1, 2, 3, i\}$ for i=5,6,7. It is obvious that every part must take a common color, then the last three parts must take 5, 6 and 7, respectively. Hence the colors of other vertices are determined uniquely in turn.

2) Consider the following 4-list assignment for $K_{1*5,2,5}$:

$$\{\{\underline{1}678\}, \{\underline{2}678\}, \{\underline{3}678\}, \{\underline{4}678\}, \{\underline{5}678\},$$
$$\{123\underline{6}, 45\underline{6}\underline{7}\},$$
$$\{126\underline{7}, 456\underline{7}, 236\underline{8}, 134\underline{8}, 156\underline{8}\}\}.$$

Obviously, the last part must take two colors, then the sixth part must take one color, and 6 is the unique choice. Therefore the last part has to take the colors 7 and 8. After colors 6, 7 and 8 are used, the colors of other vertices are determined uniquely afterwards(the colors in the list coloring are marked by underlines).

3) Consider the following 4-list assignment for $K_{1*6,2,4}$:

$$\{\{\underline{1}789\}, \{\underline{2}789\}, \{\underline{3}789\}, \{\underline{4}789\}, \{\underline{5}789\}, \{\underline{6}789\},$$
$$\{123\underline{7}, 456\underline{7}\},$$
$$\{127\underline{8}, 567\underline{8}, 347\underline{9}, 125\underline{9}\}\}.$$

Using a method similar to the method in 2), a unique list coloring can be found from the given list.

4) Consider the following 4-list assignment for $K_{1*6,20}$:

$$\{\{\underline{1}789\},\{\underline{2}789\},\{\underline{3}789\},\{\underline{4}789\},\{\underline{5}789\},\{\underline{6}789\},$$
$$\{123\underline{8},124\underline{8},125\underline{9},126\underline{9},134\underline{7},135\underline{7},136\underline{8},145\underline{9},146\underline{7},156\underline{7},$$
$$234\underline{7},235\underline{7},236\underline{9},245\underline{8},246\underline{7},256\underline{7},345\underline{9},346\underline{9},356\underline{8},456\underline{8}\}\}.$$

Obviously, the last part must take 3 colors, and $\{7,8,9\}$ is the only choice after checking. The colors of other vertices are determined sequentially.

5) We use the hexadecimal counting method here. Consider the following 4-list assignment for $K_{1*12,6}$:

$$\{\{\underline{1}DEF\},\{\underline{2}DEF\},\{\underline{3}DEF\},\{\underline{4}DEF\},\{\underline{5}DEF\},\{\underline{6}DEF\},$$
$$\{\underline{7}DEF\},\{\underline{8}DEF\},\{\underline{9}DEF\},\{\underline{A}DEF\},\{\underline{B}DEF\},\{\underline{C}DEF\},$$
$$\{123\underline{D},456\underline{D},123\underline{E},789\underline{E},789\underline{F},ABC\underline{F}\}\}.$$

The last part must take the colors D, E and F, then the colors of the other vertices are uniquely determined. □

Theorem 3.2. *Let G be a complete multipartite graph which has at least 7 parts. If G is not $K_{1*r,s}$ for $6 \le r \le 11$ and $6 \le s \le 19$, $K_{1*5,3,4}$ or $K_{1*5,4,4}$, then G is $U4LC$ if and only if it has one of the graphs in Theorem 3.1 as an induced subgraph.*

Proof. If G has one of the graphs in Theorem 3.1 as an induced subgraph, then it is $U4LC$ by Lemma 2.1. On the other hand, assume that G is not one of the graphs mentioned in the statement and it does not have any one of graphs in Theorem 3.1 as an induced subgraph. We will show that G is not $U4LC$. As G is a complete k-partite($k \ge 7$) graphs, there are two cases to be considered.

(i) G has at most one part whose size is greater than 1, namely $G = K_{1*r,s}(r \ge 6)$. Since G does not contain $K_{1*6,20}$ or $K_{1*12,6}$, and G is not $K_{1*r,s}$ for $6 \le r \le 11$ and $6 \le s \le 19$, we must have $s \le 5$. Thus G is not $U4LC$ from Proposition 2.2 and Lemma 2.1.

(ii) G has at least two parts whose sizes are greater than 1. Since G does not contain a $K_{1*4,2,2,2}$, G has exactly two parts whose size are greater than 1. Since it does not contain $K_{1*5,2,5}$ or $K_{1*6,2,4}$, then it must be one of $K_{1*5,3,4}$, $K_{1*5,4,4}$, $K_{1*5,2,4}$ and $K_{1*r,s,t}$ ($r \ge 5$, $2 \le s \le 3$ and $2 \le t \le 3$). The first two graphs are exempted and the last two graphs are not $U4LC$ from Proposition 2.5 and Proposition 2.4. □

4 On complete 6-partite graphs

In this section, the $U4LC$ complete 6-partite graphs are characterized except for finitely many of them.

Theorem 4.1. *$K_{1*3,2,2,3}$, $K_{1*4,2,7}$ and $K_{1*4,4,6}$ are $U4LC$.*

Proof. (i) Consider the following 4-list assignment for $K_{1*3,2,2,3}$:

$$\{\{123\underline{4}\},\{123\underline{5}\},\{234\underline{6}\},\{1\underline{2}34,\underline{2}356\},\{12\underline{3}4,1\underline{3}56\},\{\underline{1}234,\underline{1}256,\underline{1}456\}\}.$$

Note that $\{1,2,3,4\}$ are contained in all the lists of first part, the fourth part, the fifth part and the sixth part, so the second part and the third part have to take the colors 5 and 6, respectively. Since there are only 6 colors in total, each part can only take one color. Then the sixth part must take the color 1, the fifth part must take the color 3, the fourth part must take the color 2 and the first part must take the color 4. Thus $K_{1*3,2,2,3}$ is $U4LC$.

(ii) Consider the following 4-list assignment for $K_{1*4,2,7}$:

$$\begin{array}{l}\{\{\underline{1}567\},\{\underline{2}567\},\{\underline{3}567\},\{\underline{4}567\},\\ \{123\underline{5},4\underline{5}67\},\\ \{123\underline{6},125\underline{6},135\underline{6},145\underline{7},235\underline{7},245\underline{7},345\underline{6}\}\}.\end{array}$$

Note that there are only 7 colors in total and the sixth part must take two colors, then the fifth part has to take one color and the color 5 is the unique choice. Therefore the last part has to take the colors 6 and 7, and the colors of other vertices are uniquely determined afterwards.

(iii) For $K_{1*4,4,6}$, give a list as follows:

$$\begin{array}{l}\{\{\underline{1}567\},\{\underline{2}567\},\{\underline{3}567\},\{\underline{4}567\},\\ \{1\underline{5}78,2\underline{5}78,34\underline{6}7,12\underline{6}8\},\\ \{456\underline{7},236\underline{7},345\underline{8},136\underline{8},124\underline{8},125\underline{7}\}\}.\end{array}$$

Each of the last two parts must take two colors obviously. The last part has to take the colors 7 and 8, then the fifth part has to take the colors 5 and 6. And the colors of other vertices are uniquely determined afterwards. □

Theorem 4.2. *Let G be a complete 6-partite graph. If G is not $K_{1*4,2,6}$, $K_{1*4,3,5}$, $K_{1*4,3,6}$, $K_{1*4,4,4}$, $K_{1*4,4,5}$, $K_{1*4,5,5}$, $K_{1,1,2*4}$, $K_{1,2*5}$ and K_{2*6}, then G is $U4LC$ if and only if it has one of the graphs in Theorem 4.1 as an induced subgraph.*

Proof. If G has one of the graphs in Theorem 4.1 as an induced subgraph, then it is $U4LC$ by Lemma 2.1. On the other hand, assume that G is not one of the graphs mentioned in the statement and it does not have any one of graphs in Theorem 4.1 as an induced subgraph. We will show that G is not $U4LC$. As G is a complete 6-partite graphs, there are three cases to be considered.

(i) G has at most one part whose size is greater than 1, namely, $G = K_{1*5,r}$. From the Proposition 2.3, G has the property $M(4)$.

(ii) G has exactly two parts whose size are greater than 1. Since G does not contain $K_{1*4,3,7}$ or $K_{1*4,4,6}$, and G is not $K_{1*4,2,6}$, $K_{1*4,3,5}$, $K_{1*4,3,6}$, $K_{1*4,4,4}$, $K_{1*4,4,5}$, $K_{1*4,5,5}$, then it must be $K_{1*4,3,4}$, $K_{1*4,2,5}$ or their induced subgraphs. From Proposition 2.6, they have the property $M(4)$.

(iii) G has at least three parts whose size are greater than 1. Since G does not contain $K_{1*3,2,2,3}$, then it must be $K_{1*3,2,2,2}$, $K_{1,1,2*4}$, $K_{1,2*5}$ or K_{2*6}. From Proposition 2.6, $K_{1*3,2,2,2}$ is not $U4LC$. The last three graphs have been exempted. □

5 Some results about some complete k-partite graphs($k \leq 5$)

In this section, we state some results which will pave the way in characterization of $U4LC$ complete k-partite($k \leq 5$) graphs.

Theorem 5.1. *$K_{1,2,2,2,3}$ and $K_{1,1,2,3,4}$ are $U4LC$.*

Proof. For $K_{1,2,2,2,3}$, consider the following 4-list assignment:

$$\{\{\underline{1}234\}, \{1\underline{2}34, \underline{2}567\}, \{12\underline{3}4, \underline{3}567\}, \{123\underline{4}, \underline{4}567\}, \{123\underline{5}, 123\underline{6}, 123\underline{7}\}\}.$$

For $K_{1,1,2,3,4}$, consider the following 4-list assignment:

$$\{\{\underline{1}234\}, \{1\underline{2}34, \underline{2}345\}, \{12\underline{3}4, 1\underline{3}45, 2\underline{3}45\}, \{123\underline{4}, 13\underline{4}5, 12\underline{4}5, 23\underline{4}5\}, \{123\underline{5}\}\}.$$

Obviously two unique list colorings for $K_{1,2,2,2,3}$ and for $K_{1,1,2,3,4}$ can be found from the above given list assignments respectively. □

Theorem 5.2. *$K_{1,2*4}$, $K_{1,1,2,2,3}$, $K_{1,1,1,3,3}$, $K_{1,1,1,2,r}(r \geq 1)$ have the property $M(4)$. If $r \geq 3$, the m-number of all of them are equal to 4.*

Proof. From Lemma 2.4, it can be easily found that the first three graphs have the property $M(4)$. From Proposition 2.3 and Theorem 1.4, we know that $K_{1*5,r}$ has the property $M(4)$ and $K_{1,1,1,r}$ has the property $M(3)$. With a similar way with the way used in Proposition 2.5, we can obtain that $K_{1,1,1,2,r}$ are not $U4LC$.

If $r \geq 3$, all the graphs are $U3LC$ by Theorem 1.4, so their m-numbers are equal to 4. □

Theorem 5.3. *$K_{1,1,s,t}$, $K_{1,2,2,r}$ have property $M(4)$. If $r \geq 2$, then $m(K_{1,2,2,r}) = 4$.*

Proof. From the Lemma 2.3 and Proposition 2.1 it is obviously that $K_{1,1,s,t}$ has the property $M(4)$. From Proposition 2.3 and Theorem 1.4, we know that $K_{1*5,r}$ has the property $M(4)$ and $K_{1,2,r}$ has the property $M(3)$. With a similar way with the way used in proposition 2.5, we obtain that $K_{1,2,2,r}$ are not $U4LC$.

If $r \geq 2$, $K_{1,2,2,r}$ are $U3LC$ by Theorem 1.4, so its m-number is equal to 4. □

Theorem 5.4. *$K_{3,4,4,4}$, $K_{2,3,4,6}$, $K_{3,3,3,6}$, $K_{1,3,4,10}$ are $U4LC$.*

Proof. One can check that all the graphs have unique list coloring from the lists assignments as follows respectively:

For $K_{3,4,4,4}$: $\{\{135\underline{8}, 146\underline{7}, 235\underline{7}\}, \{12\underline{3}7, 1\underline{4}57, \underline{3}567, 12\underline{4}6\},$
$\{13\underline{6}7, 14\underline{5}7, 23\underline{5}7, 124\underline{6}\}, \{\underline{1}347, \underline{1}357, \underline{2}457, \underline{1}456\}\}.$

For $K_{2,3,4,6}$: $\{\{\underline{1}356, \underline{2}478\}, \{\underline{3}567, 12\underline{4}8, \underline{4}567\}, \{1\underline{5}78, 2\underline{6}78, 34\underline{6}7, 23\underline{5}8\},$
$\{456\underline{7}, 236\underline{7}, 345\underline{8}, 136\underline{8}, 124\underline{8}, 125\underline{7}\}\}.$

For $K_{3,3,3,6}$: $\{\{\underline{1}347, \underline{1}357, \underline{2}458\}, \{12\underline{3}7, 1\underline{4}57, \underline{3}568\}, \{13\underline{6}7, 14\underline{5}7, 23\underline{5}8\},$
$\{456\underline{7}, 236\underline{7}, 345\underline{8}, 136\underline{8}, 124\underline{8}, 125\underline{7}\}\}.$

For $K_{1,3,4,10}$: $\{\{\underline{1}467\}, \{1\underline{2}56, 1\underline{3}57, \underline{3}467\}, \{1\underline{4}67, 2\underline{4}67, 3\underline{5}67, 123\underline{5}\},$
$\{123\underline{6}, 124\underline{6}, 125\underline{7}, 134\underline{7}, 135\underline{6}, 145\underline{6}, 234\underline{7}, 235\underline{7}, 245\underline{7}, 345\underline{6}\}\}.$
□

6 Some open problems

The following problems arise naturally from the work.

Problem 1. *Determine whether the graphs $K_{1*r,s}$ for $6 \leq r \leq 11$ and $6 \leq s \leq 19$, $K_{1*5,3,4}$ and $K_{1*5,4,4}$ are $U4LC$ or not.*

Problem 2. *Determine whether the graphs $K_{1*4,2,6}$, $K_{1*4,3,5}$, $K_{1*4,3,6}$, $K_{1*4,4,4}$, $K_{1*4,4,5}$, $K_{1*4,5,5}$, $K_{1,1,2*4}$, $K_{1,2*5}$ and K_{2*6} are $U4LC$ or not.*

Problem 3. *Characterize the $U4LC$ complete k-partite graphs for $k \leq 5$.*

References

[1] B. Bollobás, Modern graph theory, Science Publishing Co., Bejing, China, 2001.

[2] J.H. Dinitz and W.J. Martin, The stipulation polynomial of a uniquely list-colorable graph, Austral. j. Combin., 11 (1995), pp. 105-115.

[3] P. Erdős, A.L. Rubin, and H. Taylor. Choosability in graphs. In Proceedings of west coast conference on combinatorics, graph theory, and computing, number 26 in Congr. Numer., pages 125-157, Arcata, CA, September 1979.

[4] Ch. Eslahchi, M. Ghebleh, H. Hajiabolhassan, Some concepts in list coloring, J. Combin. Math. Combin. Comput. 41 (2002) 151-160.

[5] M. Ghebleh, E.S. Mahmoodian, On uniquely list colorable graphs, Ars Combin. 59 (2001) 307-318.

[6] W. He, Y. Shen Y. Zhao, Y. Wang, X. Ma, On property $M(3)$ of some complete multipartite graphs, Australasian Journal of Combinatorics, to appear.

[7] W. He, Y. Shen, Y. Zhao, S. Sun, $K_{2*2,r}, r = 4,5,6,7,8$, have the property $M(3)$, submitted.

[8] A.D. Keedweel. Critical sets for Latin squares, graphs: a survey, Condr. Numer., 113-245, 1996

[9] M. Mahdian, E.S. Mahmoodian, A characterization of uniquely 2-list colorable graphs, Ars Combin. 51 (1999) 295-305.

[10] E.S. Mahmoodian and M. Mahdian. On the uniquely list colorable graphs. In Proceedings of the 28th Annual Iranian Mathematics Conference, Part 1, number 377 in Tabriz Univ. Ser., pages 319-326, Tabriz, 1997.

[11] E.S. Mahmoodian, R. Naseraser and M. Zaker: Defning sets of vertex coloring of graphsand Latin rectangles, Discrete Math., 167(168): 451-460, 1997.

[12] Y. Shen, Y. Wang, W. He,Y. Zhao, On Uniquely List Colorable Complete Multipartite Graphs. Ars Comb. 88: (2008)

[13] V.G. Vizing, Coloring the vertices of a graph in prescribed colors (in Russian), Diskret. Anal. 29 (1976) 3-10.

[14] D.R. Woodall, List coloourings of graphs, Survey in combinatorics, 2001, London Math. Soc. Lecture Note Series 288, Cambridge University Press, 2001, 269-301.

[15] Y. Zhao, W. He, Y. Shen, Y. Wang, Note on Characterization of Uniquely 3-List Colorable Complete Multipartite Graphs. CJCDGCGT 2005: 278-287.