Perfect r-Codes in Lexicographic Products of Graphs

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Abstract. A perfect r-code in a graph is a subset of the graph's vertices with the property that each vertex in the graph is within distance r of exactly one vertex in the subset. We determine the relationship between perfect r-codes in the lexicographic product of two simple graphs and perfect r-codes in each of the factors.

1 Introduction

For a positive integer r, a *perfect r-code* in a simple graph $G = (V(G), E(G))$ is a subset C of $V(G)$ that has the property that each vertex in G is within distance r of exactly one vertex in C. The distance between vertices u and v in G, denoted by $d_G(u, v)$, is the number of edges in a shortest path from u to v. A vertex u in C is said to *r-dominate* a vertex v in G if $0 \leq d_G(u, v) \leq r$. Perfect r-codes were first introduced by Biggs [2] and generalize the notion of perfect codes. We note that a perfect code is simply a perfect 1-code. Perfect r-codes have numerous applications in efficient resource placement in networks and error correcting codes. The dark vertices in Figures 1a and 1b represent a perfect 2-code and perfect 3-code respectively.

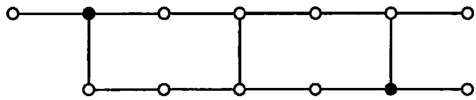

Figure 1a

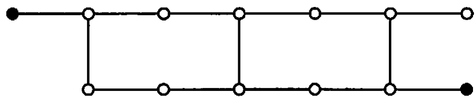

Figure 1b

Perfect r-codes have been studied in several standard product graphs; the Cartesian product [9, 3, 4], the direct product [6, 7, 8, 10] and the strong product [1]. In this paper we are interested in perfect r-codes in the final standard product, the lexicographic product. In particular, we will determine the relationship between perfect r-codes in the lexicographic product of two graphs and perfect r-codes in the two factors.

The *lexicographic product* of graphs H and G is the graph $H \circ G$ whose vertex set is the Cartesian product $V(H) \times V(G)$ and whose edges are the pairs $(h, g)(h', g')$ of distinct vertices where one of the following holds:

1. $hh' \in E(H)$ or
2. $h = h'$ and $gg' \in E(G)$.

We will refer to the graphs G and H as *factors* of the product. Figure 2 shows $P_4 \circ P_3$, where P_n denotes a path on n vertices.

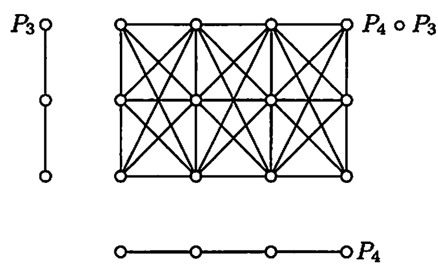

Figure 2

The lexicographic product is sometimes referred to as *composition* or *substitution*. The name composition comes from the familiar notion of composition of functions. We compose two graphs in much the same way that we compose two functions. Notice that $P_4 \circ P_3$ is obtained from P_4 by substituting a copy of P_3, denoted by $(P_3)_v$, for each vertex v in P_4 and by joining all vertices of $(P_3)_v$ to all vertices of $(P_3)_u$ whenever uv is an edge in P_4. Just as with function composition, the lexicographic product is associative, but in general is not commutative. In fact, $H \circ G \cong G \circ H$ only when one of the following is true [5, Theorem 6.9]:

1. Both H and G are complete
2. Both H and G are totally disconnected, or
3. Both H and G are powers of the same graph (with respect to the lexicographic product).

For nontrivial graphs H and G, the product $H \circ G$ is connected if and only if H is connected. We will denote by π_H (π_G respectively) the usual projection function from $V(H \circ G)$ to $V(H)$ defined by $\pi_H(h,g) = h$. For any $v \in V(H)$, the *fiber* in $H \circ G$ above v is the set $\pi_H^{-1}(v) = \{(v,g) | g \in V(G)\}$. The dark vertices in Figure 3 show the fiber in $P_2 \circ P_2$ above the vertex v.

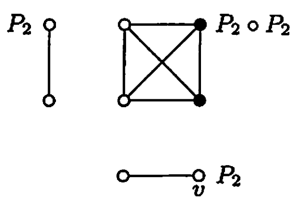

Figure 3

We make one important observation that follows from the definition of the lexicographic product. For distinct vertices (h,g) and (h',g') in the same connected component of $H \circ G$, we have $d_H(h,h') \leq d_{H\circ G}((h,g),(h',g'))$. If (h,g) and (h',g') are in the same fiber, then $h = h'$ and $d_H(h,h') = 0 < d_{H\circ G}((h,g),(h',g'))$. If they are not in the same fiber, then $d_{H\circ G}((h,g),(h',g')) = d_H(h,h')$. Thus every (h,g)-(h',g') path in $H \circ G$ projects to an h-h' path in H of length $d_H(h,h')$. In Figure 4 the path of dark edges in $H \circ G$ projects to the path of dark edges in H. For a survey of properties of the lexicographic product see [5].

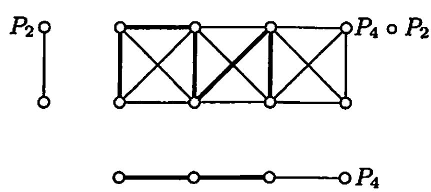

Figure 4

2 Results

In this section we examine the relationship between perfect r-codes in $H \circ G$ and perfect r-codes in the factors H and G. One might hope for a result that guarantees the existence of a perfect r-code in the product provided there are perfect r-codes in each of the two factors, and vice versa, as is true for the strong product [1]. As it turns out, it is a bit more subtle and requires some careful consideration. We begin by considering the case where $r \geq 2$.

Theorem 2.1 *Let G be any graph and let H be a graph with no isolated vertices. Then for $r \geq 2$, $H \circ G$ has a perfect r-code if and only if H has a perfect r-code.*

Proof. Suppose that $H \circ G$ has a perfect r-code C. We claim that $C_H = \pi_H(C) = \{h \in V(H) | (h, g) \in C\}$ is a perfect r-code in H. Let h be any vertex in H. Choose any vertex in the fiber above h, say (h, g). Then the vertex (h, g) must be within distance r of some vertex (c, c') in C. But this implies that $d_H(h, c) \leq r$. Hence h is r-dominated by c. Since $c \in C_H = \pi_H(C)$ we see that each vertex in H is r-dominated by a vertex in C_H.

Now suppose that h is r-dominated by two distinct vertices c and $c' \in C_H = \pi_H(C)$. This means that the fibers above c and c' each contain a vertex in C. If $h = c$ (or c'), then each vertex in the fiber above h is within distance r of each vertex in the fiber above c'. Thus two vertices in C are within distance r of each other. This is a contradiction since C is a perfect code in $H \circ G$. If $h \neq c$, then each vertex in the fiber above h is within distance r of each vertex in the fibers above c and c'. Thus every vertex in the fiber above h is r-dominated by two distinct vertices in C, again a contradiction. Thus h is r-dominated by exactly one vertex in C_H and C_H is a perfect r-code in H.

Conversely, suppose that H has a perfect r-code C. We claim that we can form a perfect r-code in $H \circ G$ in the following way. For each vertex $c \in C$ we choose exactly one vertex (c, g) in $H \circ G$ in the fiber above c. Let D denote this subset of vertices in $H \circ G$.

Let (h, g) be any vertex in $H \circ G$. Then either $h \in C$, and h dominates itself, or h is r-dominated by some $c \in C$. Suppose that $h \in C$. Since H has no isolated vertices, h must be adjacent to another vertex h' in H. This means that every vertex in the fiber above h is adjacent to every vertex in the fiber above h'. Since the fiber above h contains a vertex in D we see that (h, g) is at most distance two from a vertex in D. Since $r \geq 2$ we see that (h, g) is r-dominated by a vertex in D. Now suppose that h

is r-dominated by some $c \in C$. Then the fiber above c contains exactly one vertex in D and every vertex in the fiber above c is within distance r of every vertex in the fiber above h. Thus (h, g) is r-dominated by some vertex in D.

Now suppose that (h, g) is r-dominated by two distinct vertices (h', g') and (h'', g'') in D. This implies that h' and h'' are vertices in C. Thus $d_H(h, h') \leq r$ and $d_H(h, h'') \leq r$ and h is r-dominated by two vertices in C. This is a contradiction since C is a perfect r-code in H. Therefore every vertex in $H \circ G$ is r-dominated by exactly one vertex in D. Hence D is a perfect r-code in $H \circ G$.

Theorem 2.1 is illustrated in Figure 5 where the dark vertices indicate the vertices in a perfect 2-code. Observe that any one of the vertices in the fiber above each dark vertex in H could have been chosen to form the perfect 2-code in $H \circ G$. Thus we can imagine lifting a perfect 2-code in H to a perfect 2-code in $H \circ G$.

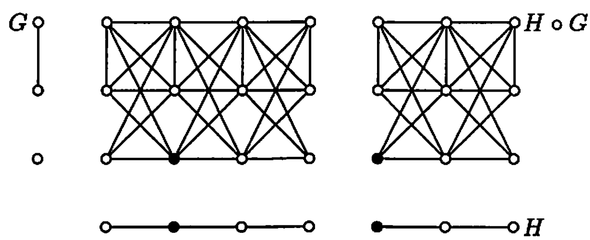

Figure 5

Notice in Theorem 2.1 that perfect r-codes in $H \circ G$ depend only on the graph H. In Figure 6 we see that $H \circ G$ has a perfect 2-code, take any vertex in the fiber above the dark vertex in H, but the factor G has no perfect 2-code. (For simplicity, we only draw the three copies of G appearing in $H \circ G$ and not all of the horizontal edges connecting the adjacent copies of G.)

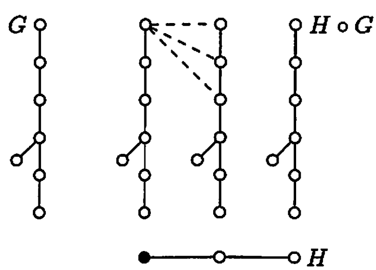

Figure 6

The fact that G may not have a perfect r-code is precisely why H is not allowed to have isolated vertices, for then we would get isolated copies of G appearing in $H \circ G$. If we allow H to have isolated vertices, then in order for $H \circ G$ to have a perfect r-code, it must also be the case that G has a perfect r-code. In this situation we get the following weaker result.

Theorem 2.2 *Let G and H be graphs. For $r \geq 2$, if H and G have perfect r-codes then $H \circ G$ has a perfect r-code.*

The proof of Theorem 2.2 follows by a similar argument from that of Theorem 2.1 where the perfect r-code for $H \circ G$ is D, together with the vertices that form a perfect r-code in each isolated copy of G. Theorem 2.2 is illustrated in Figure 7.

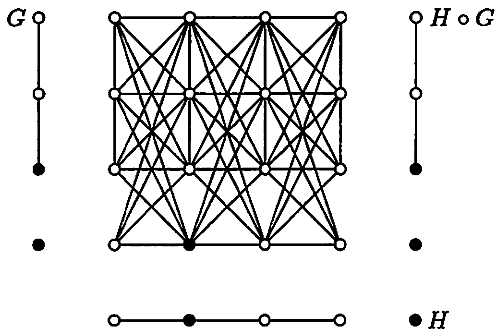

Figure 7

Unfortunately, Theorems 2.1 and 2.2 do not just carry over to the case where $r = 1$. For example, in Figure 8 we see that H and G have perfect codes, however, the product $H \circ G$ does not.

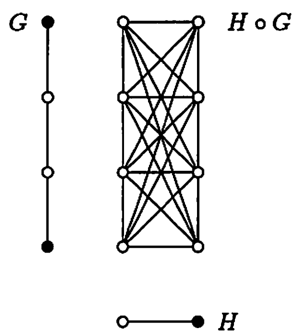

Figure 8

It is not hard to see that in order for $H \circ G$ to have a perfect code, we are going to have to put a restriction on the graph G. In particular, G

must have radius one (or zero). The following observation is simple, but nonetheless stated as a lemma.

Lemma 2.1 *Let G be a graph. Then G has a perfect code consisting of a single vertex if and only if G has radius one or zero.*

Proof. Clearly the radius of G is zero if and only if G is the trivial graph. Suppose that G is not the trivial graph and that G has perfect code $C = \{c\} \subseteq V(G)$. Then $d_G(c, g) \leq 1$ for all $g \in V(G)$. Thus the eccentricity of the vertex c is one and the radius of G is one. Conversely, suppose that G has radius one. Then G has a vertex v of eccentricity one. This means that $\max_{u \in V(G)} d(v, u) = 1$. Hence the vertex v dominates every vertex in G and $C = \{v\}$ is a perfect code in G. ■

Theorem 2.3 *Let G and H be graphs. Then $H \circ G$ has a perfect code if and only if H and G have perfect codes and the perfect code in G consists of a single vertex.*

Proof. Suppose that C is a perfect code in $H \circ G$. We claim that $C_H = \pi_H(C)$ is a perfect code in H. Let h be any vertex in H. Then the vertex (h, g) in $H \circ G$ must be dominated by some vertex $(c, c') \in C$. Since $d_{H \circ G}((h, g), (c, c')) \leq 1$, we have $d_H(h, c) \leq 1$. Thus h is dominated by c.

Suppose now that h is dominated by two distinct vertices c and $c' \in C_H$. Then the fibers above c and c' each contain a vertex in C. If $h = c$ (or c') then each vertex in the fiber above h is adjacent to each vertex in the fiber above c'. Thus we have two vertices in C adjacent to each other in $H \circ G$. This is a contradiction since C is a perfect code in $H \circ G$. If $h \neq c$, then each vertex in the fiber above h is adjacent to each vertex in the fibers above c and c'. Thus every vertex in the fiber above h is dominated by two distinct vertices in C. This is again a contradiction. Hence C_H is a perfect code in H.

Showing that the factor G has a perfect code requires a little more thought. If G is the trivial graph, then certainly $\pi_G(C)$ is a perfect code in G. Suppose that G is not the trivial graph. Let h be any vertex in H. Let K denote the connected component of $H \circ G$ containing the fiber above h. We claim that $C_G = \pi_G(C \cap V(K))$ is a perfect code in G. Let g be any vertex in G. Then the vertex (h, g) is either in C, in which case $g \in C_G$, or (h, g) is dominated by some $(c, c') \in C$. If the latter, then either $h = c$ and $gc' \in E(G)$, giving g adjacent to an element in C_G, or $hc \in E(H)$. Suppose that $hc \in E(H)$. Since two adjacent fibers in $H \circ G$ cannot both contain vertices in C, the fibers above h and c cannot both contain vertices in C. Thus $d_K((c, c'), (c, g')) \leq 1$ for all $g' \in V(G)$. This implies that

c' is a vertex of eccentricity one in G and that G has radius one. Hence $gc' \in E(G)$ and $C_G = \{c'\}$ is a perfect code in G.

Conversely, suppose that H and G have perfect codes C_H and C_G respectively. Suppose also that $C_G = \{c'\}$. We claim that the Cartesian product of C_H and C_G, denoted by $C_H \times C_G$, is a perfect code in $H \circ G$.

Let (h, g) be any vertex in $H \circ G$. Then h is dominated by some $c \in C_H$ and g is dominated by the vertex c'. Since $d_H(h, c) \leq 1$ and $d_G(g, c') \leq 1$, it follows that $d_{H \circ G}((h, g), (c, c')) \leq 1$. Hence every vertex in $H \circ G$ is dominated by some vertex in $C_H \times C_G$.

Suppose now that (h, g) is dominated by two distinct vertices $(c, c'), (\widetilde{c}, c') \in C_H \times C_G$. Note that these vertices must differ in the first coordinate as C_G consists of a single vertex. Since $d_{H \circ G}((h, g), (c, c')) \leq 1$ and $d_{H \circ G}((h, g), (\widetilde{c}, c')) \leq 1$, it follows that $d_H(h, c) \leq 1$ and $d_H(h, \widetilde{c}) \leq 1$. Thus the vertex h in H is dominated by c and $\widetilde{c}$. This contradicts the fact that C_H is a perfect code in H. Hence $C_H \times C_G$ is a perfect code in $H \circ G$. ■

Theorem 2.3 is illustrated in Figure 9. Notice that the perfect code in the product projects to a perfect code in H, but not in G. We need only look at the projection onto G of the perfect code from one connected component of $H \circ G$.

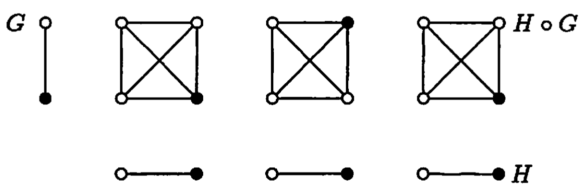

Figure 9

It is easy to show that any two perfect r-codes in a graph have the same cardinality, but in general it is not easy to determine the number of perfect r-codes in a graph. In the case of the lexicographic product however, we can determine the number of perfect r-codes in the product based on the number of perfect r-codes in the factors. First we consider the case where $r \geq 2$. If H has no isolated vertices, then for each perfect r-code C in H, we can form $|V(G)|^{|C|}$ perfect r-codes in $H \circ G$. Thus if H has n perfect r-codes, then $H \circ G$ will have a total of $n|V(G)|^{|C|}$ perfect r-codes. If H has x isolated vertices and G has m perfect r-codes, then $H \circ G$ will have $n|V(G)|^{(|C|-x)}m^x$ perfect r-codes. Finally, if $r = 1$ it is not possible to determine the number of perfect codes in the product based on the number of perfect codes in the factors (Figure 7 shows this), we must take Theorem 2.3 into account. Suppose that G has m perfect codes, each of cardinality one, and that H has n perfect codes. Then $H \circ G$ will have $n(m^{|C|})$, where

C is a perfect code in H. We can look back at Figure 8 to see this. Each perfect code in H has cardinality 3, and there are 8 such codes. The graph G has 2 perfect codes and $H \circ G$ has $8(2^3) = 64$ perfect codes.

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