Near hexagons with two possible orders for the quads

Bart De Bruyn*

Bart De Bruyn, Department of Pure Mathematics and Computer Algebra, Ghent University, Galglaan 2, B-9000 Gent, Belgium, email: bdb@cage.ugent.be

Abstract

We study near hexagons which satisfy the following properties: (i) every two points at distance 2 from each other are contained in a unique quad of order (s, r_1) or (s, r_2), $r_1 \neq r_2$; (ii) every line is contained in the same number of quads; (iii) every two opposite points are connected by the same number of geodesics. We show that there exists an association scheme on the point set of such a near hexagon and calculate the intersection numbers. We also show how the eigenvalues of the collinearity matrix and their corresponding multiplicities can be calculated. The fact that all multiplicities and intersection numbers are nonnegative integers gives restrictions on the parameters of the near hexagon. We apply this to the special case in which the near hexagon has big quads.

Keywords: near hexagon, generalized quadrangle, association scheme, eigenvalue
MSC-2000: 05B25, 05E30, 51E12

1 Introduction

A *near polygon* ([9]) is a partial linear space $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$, $\mathrm{I} \subseteq \mathcal{P} \times \mathcal{L}$, with the property that for every point p and every line L, there exists a unique point on L nearest to p. Distances $\mathrm{d}(\cdot,\cdot)$ will always be measured in the *point graph* or *collinearity graph* Γ of the geometry. If d is the diameter of Γ, then the near polygon is called a *near $2d$-gon*. A near 0-gon is just a point and a near 2-gon is a line. Near quadrangles are usually called *generalized*

*Postdoctoral Researcher of Science Foundation - Flanders *quadrangles* (GQ's, [7]). A generalized quadrangle is called *degenerate* if it consists of a number of lines through a given point.

If X_1 and X_2 are two nonempty sets of points, then $\mathrm{d}(X_1, X_2)$ denotes the minimal distance between a point of X_1 and a point of X_2. If X_1 is a singleton $\{x_1\}$, then we will also write $\mathrm{d}(x_1, X_2)$ instead of $\mathrm{d}(\{x_1\}, X_2)$. If X is a nonempty set of points and if $i \in \mathbb{N}$, then $\Gamma_i(X)$ denotes the set of all points y for which $\mathrm{d}(y, X) = i$. If X is a singleton $\{x\}$, then we will also write $\Gamma_i(x)$ instead of $\Gamma_i(\{x\})$.

Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ be a near polygon. A nonempty subset $\mathcal{P}'$ of $\mathcal{P}$ is called *geodetically closed* if all *geodesics* (i.e. shortest pats) between two points of $\mathcal{P}'$ are contained in $\mathcal{P}'$. Suppose $\mathcal{P}'$ is a geodetically closed subspace, let $\mathcal{L}'$ denote the set of lines of $\mathcal{S}$ which are completely contained in $\mathcal{P}'$ and put $\mathrm{I}' := \mathrm{I} \cap (\mathcal{P}' \times \mathcal{L}')$. Then $\mathcal{S}' := (\mathcal{P}', \mathcal{L}', \mathrm{I}')$ is a sub near polygon of $\mathcal{S}$. We will say that $\mathcal{S}'$ is a *geodetically closed sub near polygon* of $\mathcal{S}$. A nondegenerate geodetically closed subGQ is also called a *quad*. In Proposition 2.5 of [9] sufficient conditions were given for the existence of quads: if x and y are two points at distance 2 from each other and if c and d are two common neighbours of x and y such that at least one of the lines xc, xd, yc, yd contains at least three points, then x and y are contained in a unique quad.

A near polygon is called *dense* if every line is incident with at least three points and if every two points at distance 2 have at least two common neighbours. A dense near $2d$-gon $\mathcal{S}$ satisfies the following properties.

- By Lemma 19 of [5], every point of $\mathcal{S}$ is incident with the same number of lines.
- By Theorem 4 of [5], every two points at distance $\delta \in \{0, \dots, d\}$ from each other are contained in a unique geodetically closed sub near 2δ-gon. This result generalizes Proposition 2.5 of [9].
- By Theorem 1 of [6], there exist constants m_i, $i \in \{0, \dots, d\}$, such that $|\Gamma_i(x)| = m_i$ for every point x of $\mathcal{S}$.

A near polygon is said to have *order* (s, t) if every line is incident with precisely $s+1$ points and if every point is incident with precisely $t+1$ lines. A near polygon is called *regular* if it has an order (s, t) and if there exists constants t_i, $i \in \{0, \dots, d\}$, such that $|\Gamma_{i-1}(x) \cap \Gamma_1(y)| = t_i + 1$ for every two points x and y at distance i from each other. Obviously, $t_0 = -1$, $t_1 = 0$ and $t_d = t$. The point graph of a regular near polygon is a distance-regular graph whose associated parameters satisfy $a_i = (s-1)(t_i+1)$, $b_i = s(t - t_i)$ and $c_i = t_i + 1$, see [4].

Techniques from linear algebra are a very important tool for studying regular near polygons, see [4], [8] and [9]. We will show that these techniques can also be useful for studying "almost regular" near hexagons.

More concrete, let S be a near hexagon which satisfies the following properties: (i) every two points at distance 2 from each other are contained in a unique quad of order (s, r_1) or (s, r_2), $r_1 \neq r_2$, and both types of quads occur; (ii) every line is contained in the same number of quads; (iii) every two opposite points are connected by the same number of geodesics. Define the following relations on the point set $\mathcal{P}$ of S:

$$\begin{aligned}
R_0 &:= \{(x,x) \in \mathcal{P} \times \mathcal{P} \,|\, x \in \mathcal{P}\}; \\
R_1 &:= \{(x,y) \in \mathcal{P} \times \mathcal{P} \,|\, \mathrm{d}(x,y) = 1\}; \\
R_2 &:= \{(x,y) \in \mathcal{P} \times \mathcal{P} \,|\, \mathrm{d}(x,y) = 2 \text{ and } |\Gamma_1(x) \cap \Gamma_1(y)| = r_1 + 1\}; \\
R_3 &:= \{(x,y) \in \mathcal{P} \times \mathcal{P} \,|\, \mathrm{d}(x,y) = 2 \text{ and } |\Gamma_1(x) \cap \Gamma_1(y)| = r_2 + 1\}; \\
R_4 &:= \{(x,y) \in \mathcal{P} \times \mathcal{P} \,|\, \mathrm{d}(x,y) = 3\}.
\end{aligned}$$

In Section 3, we prove the following theorem.

Theorem 1.1 *There exist constants p^i_{jk} such that for every $(x,y) \in R_i$, there are p^i_{jk} points $z \in \mathcal{P}$ such that $(x,z) \in R_j$ and $(z,y) \in R_k$. As a consequence, $(\mathcal{P}, \{R_0, R_1, R_2, R_3, R_4\})$ is a symmetric association scheme.*

We also show in Section 3 how the eigenvalues of the collinearity matrix and their multiplicities can be calculated. In Section 4, we calculate the intersection numbers of the association scheme. The fact that all multiplicities and all intersection numbers are nonnegative integers gives restrictions on the parameters of the near hexagon. In the final section, we will apply the previous theory to the special case in which the near hexagon has big quads.

2 Somme easy lemmas

Let S be a near hexagon which satisfies the following properties: (i) every line is incident with precisely $s+1$ points; (ii) every two points at distance 2 are contained in a unique quad; (iii) every quad has order (s, r_1) or (s, r_2), $r_1 \neq r_2$, and both types of quads occur. For every point x of S, let $t_x + 1$ denote the total number of lines through x. If a line L is contained in $\alpha_i(L)$ quads of order (s, r_i), $i \in \{1, 2\}$, then

$$t_x = r_1 \cdot \alpha_1(L) + r_2 \cdot \alpha_2(L) \tag{1}$$

for every point x of L. Since S is connected, every point of S is incident with the same number of lines, say $t+1$. If $\alpha(L)$ denote the total number of quads through L, then

$$\alpha_1(L) + \alpha_2(L) = \alpha(L). \tag{2}$$

By equations (1) and (2), the following lemma immediately follows.

Lemma 2.1 *The following are equivalent:*

(a) *there exists a constant* α *such that every line is contained in* α *quads;*

(b) *there exists constants* α_1 *and* α_2 *such that every line of* $\mathcal{S}$ *is contained in* α_i *quads of order* (s, r_i), $i \in \{1, 2\}$.

If either (a) *or* (b) *holds, then* $\alpha_1 = \frac{t - r_2\alpha}{r_1 - r_2}$ *and* $\alpha_2 = \frac{t - r_1\alpha}{r_2 - r_1}$.

Let v denote the total number of points of $\mathcal{S}$. For every point x of $\mathcal{S}$, we have $|\Gamma_0(x)| = 1$, $|\Gamma_1(x)| = 2(t+1)$, $\sum_{i=0}^{3} |\Gamma_i(x)| = v$ and $\sum_{i=0}^{3}(-\frac{1}{s})^i |\Gamma_i(x)| = 0$ (see e.g. [6, Lemma 3]). As a consequence,

$$\begin{aligned} |\Gamma_2(x)| &= \frac{v}{s+1} - 1 + s^2 t - st, \\ |\Gamma_3(x)| &= \frac{sv}{s+1} - s - s^2 t, \end{aligned}$$

for every point x of $\mathcal{S}$. Let $N_i(x)$, $i \in \{1, 2\}$, denote the total number of quads of order (s, r_i) through x. Then $t(t+1) = (r_1 + 1)r_1 \cdot N_1(x) + (r_2 + 1)r_2 \cdot N_2(x)$ (every two intersecting lines are contained in a unique quad) and $|\Gamma_2(x)| = s^2 r_1 \cdot N_1(x) + s^2 r_2 \cdot N_2(x)$ (every two points at distance 2 are contained in a unique quad). It follows that the numbers $N_1(x)$ and $N_2(x)$ are independent from the point x.

Lemma 2.2 *The following are equivalent:*

(a) *there exists a constant* δ *such that every two points* x *and* y *at distance* 3 *from each other are connected by precisely* δ *geodesics;*

(b) *there exist constants* β_1 *and* β_2 *such that for every two points* x *and* y *at distance* 3 *from each other there are* β_i, $i \in \{1, 2\}$, *quads of order* (s, r_i) *through* x *containing a point of* $\Gamma_1(y)$.

If either (a) *or* (b) *occurs, then* $\beta_1 = \frac{\delta - (r_2+1)(t+1)}{r_1 - r_2}$ *and* $\beta_2 = \frac{\delta - (r_1+1)(t+1)}{r_2 - r_1}$.

Proof. Suppose property (b) holds, then the number of geodesics between x and y is equal to $(r_1+1)\beta_1 + (r_2+1)\beta_2$. This number is independent from x and y. Conversely, suppose that (a) holds. Let $\beta_i(x, y)$, $i \in \{1, 2\}$, denote the number of quads of order (s, r_i) through x containing a point of $\Gamma_1(y)$. By (a), $(r_1 + 1) \cdot \beta_1(x, y) + (r_2 + 1) \cdot \beta_2(x, y) = \delta$. Since any line through y contains a unique point at distance 2 from x, $\beta_1(x, y) + \beta_2(x, y) = t + 1$. The lemma now easily follows. □

3 The existence of an association scheme

Let $\mathcal{S}$ be a near hexagon which satisfies the following properties:

(1) every line is incident with precisely $s+1$ points;

(2) every two points at distance 2 are contained in a unique quad;

(3) every quad has order (s, r_1) or (s, r_2), $r_1 \neq r_2$, and both types of quads occur;

(4) every line of $\mathcal{S}$ is contained in precisely α quads;

(5) there exists a constant δ such that every two points at distance 3 from each other are connected by precisely δ geodesics.

Let $t+1$ denote the constant number of lines through a point. Put $\alpha_1 = \frac{t-r_2\alpha}{r_1-r_2}$, $\alpha_2 = \frac{t-r_1\alpha}{r_2-r_1}$, $\beta_1 = \frac{\delta-(r_2+1)(t+1)}{r_1-r_2}$ and $\beta_2 = \frac{\delta-(r_1+1)(t+1)}{r_2-r_1}$. By Lemma 2.1, every line of $\mathcal{S}$ is contained in α_i quads of order (s, r_i). If x and y are two points at distance 3 from each other, then by Lemma 2.2 there are β_i quads of order (s, r_i) through x containing a point collinear with y.

Let $\mathcal{P}$ denote the point set of $\mathcal{S}$ and put $\mathcal{P} = \{p_1, p_2, \ldots, p_v\}$ where $v := |\mathcal{P}|$. Let $\mathcal{M}$ denote the set of all $(v \times v)$-matrices with real entries. Then $\mathcal{M}$ is a v^2-dimensional vector space over the field $\mathbb{R}$. If M is an arbitrary element of $\mathcal{M}$, then we label the i-th row and i-th column of M by the point p_i. Let I denote the $v \times v$ identity matrix and let J denote the $(v \times v)$-matrix with all entries equal to 1. Let A denote the *collinearity matrix* of $\mathcal{S}$, i.e.

$$\begin{aligned} A_{xy} &= 1 \quad \text{if } \mathrm{d}(x,y) = 1, \\ &= 0 \quad \text{otherwise.} \end{aligned}$$

For every $i \in \{1, 2\}$, let B_i denote the following matrix of $\mathcal{M}$:

$$\begin{aligned} (B_i)_{xy} &= 1 \quad \text{if } \mathrm{d}(x,y) = 2 \text{ and } |\Gamma_1(x) \cap \Gamma_1(y)| = r_i + 1, \\ &= 0 \quad \text{otherwise.} \end{aligned}$$

Let C denote the following matrix of $\mathcal{M}$:

$$\begin{aligned} C_{xy} &= 1 \quad \text{if } \mathrm{d}(x,y) = 3, \\ &= 0 \quad \text{otherwise.} \end{aligned}$$

Obviously, I, A, B_1, B_2 and C are linearly independent elements of $\mathcal{M}$ and $J = I + A + B_1 + B_2 + C$.

Lemma 3.1

$$\begin{aligned}
A^2 &= s(t+1)\cdot I + (s-1)\cdot A + (r_1+1)\cdot B_1 + (r_2+1)\cdot B_2 \\
A\cdot B_1 &= sr_1\alpha_1\cdot A + (s-1)(r_1+1)\cdot B_1 + \beta_1\cdot C \\
A\cdot B_2 &= sr_2\alpha_2\cdot A + (s-1)(r_2+1)\cdot B_2 + \beta_2\cdot C \\
A\cdot C &= s(t-r_1)\cdot B_1 + s(t-r_2)\cdot B_2 + (s-1)(t+1)\cdot C
\end{aligned}$$

Proof. If M and N are two elements of $\mathcal{M}$, then $(MN)_{xy} = \sum_z M_{xz}N_{zx}$. The lemma now easily follows. □

By Lemma 3.1, there exist (unique) integers a_n, b_n, c_n, d_n and e_n such that

$$A^n = a_n\cdot I + b_n\cdot A + c_n\cdot B_1 + d_n\cdot B_2 + e_n\cdot C, \tag{3}$$

for every $n \in \mathbb{N}$. The initial values are

$$a_0 = 1, b_0 = 0, c_0 = 0, d_0 = 0, e_0 = 0,$$

and the recursion relations read as follows ($n \geq 0$):

$$\begin{aligned}
a_{n+1} &= s(t+1)\cdot b_n, \\
b_{n+1} &= a_n + (s-1)\cdot b_n + sr_1\alpha_1\cdot c_n + sr_2\alpha_2\cdot d_n, \\
c_{n+1} &= (r_1+1)\cdot b_n + (s-1)(r_1+1)\cdot c_n + s(t-r_1)\cdot e_n, \\
d_{n+1} &= (r_2+1)\cdot b_n + (s-1)(r_2+1)\cdot d_n + s(t-r_2)\cdot e_n, \\
e_{n+1} &= \beta_1\cdot c_n + \beta_2\cdot d_n + (s-1)(t+1)\cdot e_n.
\end{aligned}$$

Obviously,

$$a_1 = 0, b_1 = 1, c_1 = 0, d_1 = 0, e_1 = 0$$

and

$$a_2 = s(t+1), b_2 = s-1, c_2 = r_1+1, d_2 = r_2+1, e_2 = 0.$$

One calculates that

$$\begin{aligned}
a_3 &= s(t+1)(s-1), \\
b_3 &= s(t+1) + (s-1)^2 + sr_1\alpha_1(r_1+1) + sr_2\alpha_2(r_2+1), \\
c_3 &= (r_1+1)(s-1) + (s-1)(r_1+1)^2, \\
d_3 &= (r_2+1)(s-1) + (s-1)(r_2+1)^2, \\
e_3 &= \beta_1(r_1+1) + \beta_2(r_2+1) = \delta,
\end{aligned}$$

and that

$$a_4 = s(t+1)b_3,$$

$$
\begin{aligned}
b_4 &= a_3 + (s-1)b_3 + (sr_1\alpha_1)c_3 + (sr_2\alpha_2)d_3,\\
c_4 &= (r_1+1)b_3 + s(t-r_1)\delta + (s-1)^2(r_1+1)^2 + (s-1)^2(r_1+1)^3,\\
&= c_2b_3 + s(t+1)\delta - s(r_1+1)\delta + (s-1)[c_3-(s-1)c_2]\\
&\qquad\qquad\qquad\qquad +(s-1)^2(r_1+1)^3,\\
&= [b_3 - s\delta - (s-1)^2]c_2 + (s-1)c_3 + s(t+1)\delta + (s-1)^2(r_1+1)^3,\\
d_4 &= [b_3 - s\delta - (s-1)^2]d_2 + (s-1)d_3 + s(t+1)\delta + (s-1)^2(r_2+1)^3,\\
e_4 &= (s-1)(\beta_1c_2+\beta_2d_2) + (s-1)[\beta_1c_2^2+\beta_2d_2^2] + (s-1)(t+1)\delta.
\end{aligned}
$$

Put

$$
\begin{aligned}
\begin{bmatrix} c_3' \\ d_3' \\ e_3' \end{bmatrix} &:= \begin{bmatrix} c_3 \\ d_3 \\ e_3 \end{bmatrix} - (s-1)\cdot\begin{bmatrix} c_2 \\ d_2 \\ e_2 \end{bmatrix}\\
&= \begin{bmatrix} (s-1)(r_1+1)^2 \\ (s-1)(r_2+1)^2 \\ \delta \end{bmatrix},\\
\begin{bmatrix} c_4' \\ d_4' \\ e_4' \end{bmatrix} &:= \begin{bmatrix} c_4 \\ d_4 \\ e_4 \end{bmatrix} - (b_3 - s\delta - (s-1)^2)\cdot\begin{bmatrix} c_2 \\ d_2 \\ e_2 \end{bmatrix} - (s-1)\cdot\begin{bmatrix} c_3 \\ d_3 \\ e_3 \end{bmatrix}\\
&= \begin{bmatrix} (s-1)^2(r_1+1)^3 + s(t+1)\delta \\ (s-1)^2(r_2+1)^3 + s(t+1)\delta \\ (s-1)(\beta_1(r_1+1)(r_1+2) + \beta_2(r_2+1)(r_2+2) + t\delta) \end{bmatrix},
\end{aligned}
$$

and

$$
\Delta := \begin{vmatrix} c_2 & d_2 & e_2 \\ c_3 & d_3 & e_3 \\ c_4 & d_4 & e_4 \end{vmatrix}.
$$

Lemma 3.2 $\Delta \neq 0$.

Proof. We have

$$
\Delta = \begin{vmatrix} c_2 & d_2 & e_2 \\ c_3' & d_3' & e_3' \\ c_4' & d_4' & e_4' \end{vmatrix}.
$$

We find $\Delta = (r_2-r_1)\Delta'$ with $\Delta' = (r_1+1)(r_2+1)(s-1)^2[\beta_1(r_1+1)(r_1+2)+\beta_2(r_2+1)(r_2+2)+t\delta] - \delta\cdot[(r_1+1)(r_2+1)(s-1)^2(r_1+r_2+2)] + \delta^2 s(t+1)$. Putting $\Delta'' := \frac{\Delta'}{(r_1+1)(r_2+1)(s-1)^2}$, we find

$$
\begin{aligned}
\Delta'' &> \beta_1(r_1+1)(r_1+2) + \beta_2(r_2+1)(r_2+2) + t\delta - \delta(r_1+r_2+2)\\
&> (\beta_1(r_1+1)+\beta_2(r_2+1))\cdot\min\{r_1+2, r_2+2\} + \delta\cdot\max\{r_1,r_2\}\\
&\qquad -\delta(r_1+r_2+2),
\end{aligned}
$$

and this latter number is equal to 0. This proves the lemma. □

Corollary 3.3 *The matrices I, A, A^2, A^3 and A^4 are linearly independent.*

Proof. We have

$$\begin{cases} I &= 1\cdot I + 0\cdot A + 0\cdot B_1 + 0\cdot B_2 + 0\cdot C \\ A &= 0\cdot I + 1\cdot A + 0\cdot B_1 + 0\cdot B_2 + 0\cdot C \\ A^2 &= a_2\cdot I + b_2\cdot A + c_2\cdot B_1 + d_2\cdot B_2 + e_2\cdot C \\ A^3 &= a_3\cdot I + b_3\cdot A + c_3\cdot B_1 + d_3\cdot B_2 + e_3\cdot C \\ A^4 &= a_4\cdot I + b_4\cdot A + c_4\cdot B_1 + d_4\cdot B_2 + e_4\cdot C \end{cases}$$

The matrices I, A, B_1, B_2 and C are linearly independent, and since $\Delta \neq 0$, also the matrices I, A, A^2, A^3 and A^4 are linearly independent. □

Put

$$p_1(x) := \frac{1}{\Delta}\cdot\begin{vmatrix} x^2 - b_2x - a_2 & d_2 & e_2 \\ x^3 - b_3x - a_3 & d_3 & e_3 \\ x^4 - b_4x - a_4 & d_4 & e_4 \end{vmatrix},$$

$$p_2(x) := \frac{1}{\Delta}\cdot\begin{vmatrix} c_2 & x^2 - b_2x - a_2 & e_2 \\ c_3 & x^3 - b_3x - a_3 & e_3 \\ c_4 & x^4 - b_4x - a_4 & e_4 \end{vmatrix},$$

$$p_3(x) := \frac{1}{\Delta}\cdot\begin{vmatrix} c_2 & d_2 & x^2 - b_2x - a_2 \\ c_3 & d_3 & x^3 - b_3x - a_3 \\ c_4 & d_4 & x^4 - b_4x - a_4 \end{vmatrix},$$

$$p(x) := 1 + x + p_1(x) + p_2(x) + p_3(x),$$

$$m(x) := p(x)\cdot(x - s(t+1)).$$

Lemma 3.4 *(a)* $B_1 = p_1(A)$, $B_2 = p_2(A)$, $C = p_3(A)$, $p(A) = J$ *and* $m(A) = 0$.

(b) $\deg(p(x)) = 4$, $\deg(m(x)) = 5$ *and* $m(x)$ *is a minimal polynomial of* A*;*

(c) *A has 5 distinct eigenvalues λ_1, λ_2, λ_3, λ_4 and λ_5 with* $-s(t+1) \leq \lambda_1 < \lambda_2 < \lambda_3 < \lambda_4 < \lambda_5 = s(t+1)$.

Proof.

(a) By equation (3) and the definitions of $p_1(x)$, $p_2(x)$ and $p_3(x)$, $B_1 = p_1(A)$, $B_2 = p_2(A)$ and $C = p_3(A)$. We also have $p(A) = I + A + B_1 + B_2 + C = J$ and $m(A) = p(A)\cdot(A - s(t+1)I) = J\cdot A - s(t+1)J = 0$.

(b) By Corollary 3.3, a minimal polynomial of A has degree at least 5. So, either $m(x) = 0$ or $\deg(m(x)) \geq 5$. Since $p(A) = J$, $p(x) \neq 0$ and $m(x) \neq 0$. Since $\deg(p(x)) \leq 4$, $\deg(m(x)) \leq 5$. The property now easily follows.

(c) Since A is symmetric, it is diagonizable and the number of eigenvalues is equal to $\deg(m(x)) = 5$. Moreover, all eigenvalues are real. Obviously, $[1, \ldots, 1]^T$ is an eigenvector of A with eigenvalue $s(t+1)$. Since the sum of all entries on an arbitrary row of A is equal to $s(t+1)$, $|\lambda| \leq s(t+1)$ for every eigenvalue λ of A. □

Let f_i, $i \in \{1, \ldots, 5\}$, denote the multiplicity of the eigenvalue λ_i. The number v is an eigenvalue of J with multiplicity 1 and corresponding eigenvector $[1, \ldots, 1]^T$. It follows that $f_5 = 1$ and that $p(st + s) = v$. The other multiplicities can be obtained from the following nonsingular linear system of equations ($j \in \{0, 1, 2, 3\}$):

$$\sum_{i=1}^{4} f_i \lambda_i^j = \mathrm{Tr}(A^j) - s^j(t+1)^j = v \cdot a_j - s^j(t+1)^j.$$

The multiplicities f_1, f_2, f_3 and f_4 need to be strictly positive integers. So, we obtain a bunch of conditions that need to be satisfied by the parameters s, t, r_1, r_2, α_1, α_2, β_1 and β_2.

Let $\mathcal{M}'$ be the five-dimensional subspace of $\mathcal{M}$ generated by the matrices I, A, B_1, B_2 and C. Then $\mathcal{M}' = \langle I, A, A^2, A^3, A^4 \rangle$ and $N_1 \cdot N_2 \in \mathcal{M}'$ for all $N_1, N_2 \in \mathcal{M}'$. By Theorem 1.1, which we will prove now, the algebra defined on the set $\mathcal{M}'$ is a Bose-Mesner algebra ([2]) of an association scheme.

Proof of Theorem 1.1: Let R_0, R_1, R_2, R_3 and R_4 be the relations on $\mathcal{P}$ as defined in Section 1. Put $M_0 = I$, $M_1 = A$, $M_2 = B_1$, $M_3 = B_2$ and $M_4 = C$. We can write $M_j M_k \in \mathcal{M}'$ as a unique linear combination of the matrices $M_0, \ldots, M_4$. The number p_{jk}^i is the coefficient of M_i in this linear combination.

4 Calculation of the intersection numbers

In principle, it is possible to calculate the intersection numbers p_{jk}^i (see Theorem 1.1) with the method given in Section 3. Unfortunately, a lot of tedious calculations are necessary if one wants to proceed that way. In this section, we will calculate these numbers by counting. Note that the countings presented below already make use of the fact that the intersection numbers are constant (what we have shown in Theorem 1.1).

We will use the same notations as in Section 3. The numbers $p^i_{\delta j}$ are easily calculated if $\delta \in \{0,1\}$. If $i, j \in \{0,1,2,3,4\}$, then p^i_{0j} is equal to 1 if $i = j$ and equal to 0 otherwise. We list the numbers p^i_{1j} in the following table.

p^i_{1j}	$j=0$	$j=1$	$j=2$	$j=3$	$j=4$
$i=0$	0	$s(t+1)$	0	0	0
$i=1$	1	$s-1$	$s\alpha_1 r_1$	$s\alpha_2 r_2$	0
$i=2$	0	r_1+1	$(s-1)(r_1+1)$	0	$s(t-r_1)$
$i=3$	0	r_2+1	0	$(s-1)(r_2+1)$	$s(t-r_2)$
$i=4$	0	0	β_1	β_2	$(t+1)(s-1)$

In the following proposition, we calculate the numbers $n_i := p^0_{ii}$.

Proposition 4.1 *It holds* $n_0 = 1$, $n_1 = s(t+1)$, $n_2 = \frac{s^2 r_1(t+1)(t-r_2\alpha)}{(r_1-r_2)(r_1+1)}$, $n_3 = \frac{s^2 r_2(t+1)(t-r_1\alpha)}{(r_2-r_1)(r_2+1)}$ *and* $n_4 = \frac{s^3[(t+1)(t+\alpha r_1 r_2)-t(r_1+1)(r_2+1)]}{(r_1+1)(r_2+1)}$.

Proof. Obviously, $n_0 = 1$ and $n_1 = s(t+1)$. Since every line through a given point x is contained in precisely $\alpha_1 = \frac{t-r_2\alpha}{r_1-r_2}$ quads of order (s, r_1), there are precisely $\frac{(t+1)(t-r_2\alpha)}{(r_1-r_2)(r_1+1)}$ quads of order (s, r_1) through x. Hence, $n_2 = \{y \in \mathcal{P} \,|\, (x,y) \in R_2\} = \frac{s^2 r_1(t+1)(t-r_2\alpha)}{(r_1-r_2)(r_1+1)}$. In a similar way one shows that $n_3 = \frac{s^2 r_2(t+1)(t-r_1\alpha)}{(r_2-r_1)(r_2+1)}$. By the calculations following Lemma 2.1, $n_4 = |\Gamma_3(x)| = s \cdot (|\Gamma_2(x)| - s^2 t) = s \cdot (n_2 + n_3 - s^2 t)$. The value of n_4 readily follows. □

Proposition 4.2 *It holds* $\beta_1 = \frac{n_2 \cdot s(t-r_1)}{n_4}$ *and* $\beta_2 = \frac{n_3 \cdot s(t-r_2)}{n_4}$.

Proof. Let x denote an arbitrary point of $\mathcal{S}$. Counting in two different ways the number of pairs (y,z) satisfying $y \in \Gamma_2(x)$, $z \in \Gamma_3(x)$, $\mathrm{d}(y,z) = 1$ and $|\Gamma_1(x) \cap \Gamma_1(y)| = r_1 + 1$ gives $n_4\beta_1 = n_2 \cdot s(t-r_1)$. Hence, $\beta_1 = \frac{n_2 \cdot s(t-r_1)}{n_4}$. Similarly, $\beta_2 = \frac{n_3 \cdot s(t-r_2)}{n_4}$. □

Proposition 4.3 *It holds*

$$p^2_{24} = s(s-1)r_1\Big(\frac{(t+1)(t-r_2\alpha)}{(r_1-r_2)(r_1+1)} - 1\Big),$$

$$p^2_{34} = s(s-1)r_2\Big(\frac{(t+1)(t-r_1\alpha)}{(r_2-r_1)(r_2+1)}\Big).$$

Proof. Let x and y be two points of $\mathcal{S}$ such that $(x,y) \in R_2$ and let Q denote the unique quad of order (s, r_1) through x and y. Let Q' denote a quad of order (s, r_i) through x different from Q. Notice that there are $\frac{(t+1)(t-r_2\alpha)}{(r_1-r_2)(r_1+1)} - 1$ possibilities for Q' if $i = 1$ and $\frac{(t+1)(t-r_1\alpha)}{(r_2-r_1)(r_2+1)}$ if $i = 2$.

Suppose Q' intersects Q in a line L and let z denote the unique point of L collinear with y. There are precisely $s(s-1)r_i$ points $u \in Q' \cap \Gamma_2(x) \cap \Gamma_2(z)$ and these are precisely the points u of Q' satisfying $(x,u) \in R_{i+1}$ and $(u,y) \in R_4$.

Suppose Q' intersects Q in only the point x. Then $\Gamma_2(y) \cap Q'$ is an ovoid of Q' containing x. There are precisely $s^2 r_i - s r_i = s(s-1)r_i$ points $u \in Q' \cap \Gamma_2(x) \cap \Gamma_3(y)$ and these are precisely the points u of Q' satisfying $(x,u) \in R_{i+1}$ and $(u,y) \in R_4$.

The proposition now readily follows. □

In a similar way, one shows that

$$
\begin{aligned}
p^3_{34} &= s(s-1)r_2\Big(\frac{(t+1)(t-r_1\alpha)}{(r_2-r_1)(r_2+1)} - 1\Big), \\
p^3_{24} &= s(s-1)r_1\Big(\frac{(t+1)(t-r_2\alpha)}{(r_1-r_2)(r_1+1)}\Big).
\end{aligned}
$$

The above mentioned values of the intersection numbers together with the well-known equalities $(i,j,k \in \{0,1,2,3,4\})$

$$
\begin{aligned}
p^i_{jk} &= p^i_{kj}, \\
p^j_{ik} &= \frac{n_i p^i_{jk}}{n_j}, \\
n_j &= p^i_{j0} + p^i_{j1} + p^i_{j2} + p^i_{j3} + p^i_{j4},
\end{aligned}
$$

allows us to calculate all intersection numbers except for the numbers p^i_{jk} with $i,j,k \in \{2,3\}$. Knowing one of these values however, the remaining seven readily follow. We show how to calculate p^2_{22} and p^3_{22} in the following proposition.

Proposition 4.4 *The numbers p^2_{22} and p^3_{22} are the solution of the following non-singular linear system of equations:*

$$
\begin{cases}
n_2 p^2_{22} + n_3 p^3_{22} = (n_2)^2 - n_0 p^0_{22} - n_1 p^1_{22} - n_4 p^4_{22}, \\
\beta_1 p^2_{22} + \beta_2 p^3_{22} = (\sum_j p^4_{j2} p^j_{12}) - (t+1)(s-1)p^4_{22}.
\end{cases}
$$

Proof. (1) Let x be a given point of $\mathcal{S}$. Counting in two different ways the number of pairs (y,z) with $(x,y) \in R_2$ and $(y,z) \in R_2$ yields $\sum_i n_i p^i_{22} = (n_2)^2$, from which the first equation readily follows.

(2) Let x and y be two points such that $(x, y) \in R_4$. We will count in two different ways the number of pairs (u, v) satisfying $(x, u) \in R_1$, $(u, v) \in R_2$ and $(v, y) \in R_2$. We find $\sum_i p^4_{1i} p^i_{22} = \sum_j p^4_{j2} p^j_{12}$. The second equation follows.

(3) By Proposition 4.2, $n_2\beta_2 - n_3\beta_1 = \frac{n_2 n_3 s(r_1 - r_2)}{n_4} \neq 0$. So, the linear system of equations is indeed non-singular. □

Remark. The fact that all intersection numbers are nonnegative integers give rise to a number of (divisibility) conditions that need to be satisfied by the parameters.

5 Example: near hexagons with big quads

We will now consider a special case for which the previous applies.

Definition. Let Q be a quad of a near hexagon $\mathcal{S}$. The quad Q is called *big* in $\mathcal{S}$ if every point of $\mathcal{S}$ not contained in Q is collinear with a unique point of Q. If Q and Q' are quads of $\mathcal{S}$ such that Q is big, then precisely one of the following occurs: (i) $Q = Q'$, (ii) $Q \cap Q' = \emptyset$, (iii) $Q \cap Q'$ is a line.

In this section, we suppose that $\mathcal{S}$ is a near hexagon which satisfies the following properties:

(1) every point is incident with precisely $s + 1$ points;

(2) every two points at distance 2 are contained in a unique quad;

(3) every quad has order (s, r_1) or (s, r_2), $r_1 \neq r_2$, and both types of quads occur;

(4) every quad of order (s, r_1) is big in $\mathcal{S}$;

(5) if V_x is the set of quads of order (s, r_1) through a point x of $\mathcal{S}$, then $\bigcap_{Q \in V_x} Q = \{x\}$ for every point x of $\mathcal{S}$.

As before, let $t + 1$ denote the constant number of lines through a point of $\mathcal{S}$ and let v denote the total number of points of $\mathcal{S}$. If Q is a quad of order (s, r_1), then $|Q| = (s + 1)(sr_1 + 1)$ and $|\Gamma_1(Q)| = (s + 1)(sr_1 + 1)s(t - r_1)$. Hence,

$$v = (s + 1)(1 + sr_1)(1 + s(t - r_1)).$$

Put

$$\alpha_1 := \frac{t - r_2(r_1 + 1)}{r_1 - r_2}, \qquad \alpha_2 := \frac{r_1^2 + r_1 - t}{r_1 - r_2},$$
$$\beta_1 := \frac{(t+1)(t - r_2(r_1 + 1))}{(r_1 - r_2)(r_1 + 1)}, \qquad \beta_2 := \frac{(t+1)(r_1^2 + r_1 - t)}{(r_1 - r_2)(r_1 + 1)}.$$

Lemma 5.1 *(a) Every line is contained in* $\alpha = r_1 + 1$ *quads:* α_1 *of these quads have order* (s, r_1) *and* α_2 *have order* (s, r_2).

(b) Let x *and* y *be two points at distance 3 from each other. Then there are* β_i, $i \in \{1, 2\}$, *quads of order* (s, r_i) *through* x *containing a point of* $\Gamma_1(y)$.

(c) $r_1 > r_2$.

Proof.

(a) Let L be an arbitrary line of $\mathcal{S}$. Let x be a point of L and let Q be a quad of order (s, r_1) through x not containing L (such a quad exists by property (5)). Any quad through L intersects Q in a line. So, there are precisely $r_1 + 1$ quads through L. The rest of the statement follows from Lemma 2.1.

(b) There are $\frac{(t+1)\alpha_1}{r_1+1} = \beta_1$ quads of order (s, r_1) through x. Since all these quads are big, they all contain a point of $\Gamma_1(y)$. The number of quads of order (s, r_2) through x containing a point at distance 1 from y is equal to $t + 1 - \beta_1 = \beta_2$.

(c) The total number of quads of order (s, r_2) through x is equal to $\frac{(t+1)\alpha_2}{r_2+1}$. Since this number is at least β_2, we must have that $r_1 > r_2$. □

Lemma 5.2 *We have* $t + 1 = (r_1 + 1)(r_2 + 1)$.

Proof. Let x be an arbitrary point of $\mathcal{S}$. Since there are $\frac{(t+1)(t-r_2(r_1+1))}{(r_1-r_2)(r_1+1)}$ quads of order (s, r_1) and $\frac{(t+1)(r_1^2+r_1-t)}{(r_1-r_2)(r_2+1)}$ quads of order (s, r_2) through x, we have

$$|\Gamma_2(x)| = \frac{s^2(t+1)}{r_1 - r_2}\left(\frac{(t - r_2(r_1+1))r_1}{r_1+1} + \frac{(r_1^2 + r_1 - t)r_2}{r_2+1}\right).$$

On the other hand, we have

$$|\Gamma_2(x)| = \frac{v}{s+1} - 1 + s^2t - st = s^2(tr_1 - r_1^2 + t).$$

Equating both expressions for $|\Gamma_2(x)|$, we obtain a quadratic equation in t. The solutions are $t = r_1^2 + r_1$ and $t = r_1r_2 + r_1 + r_2$. If $t = r_1^2 + r_1$, then the total number of quads of order (s, r_2) through a given line is equal to 0, a contradiction. So, $t = r_1r_2 + r_1 + r_2$. □

Plugging $t = r_1r_2 + r_1 + r_2$ in the values of α_1, α_2, β_1 and β_2, we obtain:

$$\begin{array}{lll} \alpha_1 & := & \frac{r_1}{r_1-r_2}, \qquad \alpha_2 \;:=\; \frac{r_1^2-r_1r_2-r_2}{r_1-r_2}, \\ \beta_1 & := & \frac{r_1(r_2+1)}{r_1-r_2}, \qquad \beta_2 \;:=\; \frac{(r_2+1)(r_1^2-r_1r_2-r_2)}{r_1-r_2}. \end{array}$$

Lemma 5.3 *There exists a quad Q_1 of order (s, r_1) and a quad Q_2 of order (s, r_2) such that $Q_1 \cap Q_2 = \emptyset$.*

Proof. Let x denote an arbitrary point of $\mathcal{S}$ and let R_1, R_2 and R_3 be three quads of order (s, r_1) through x such that $R_1 \cap R_2 \cap R_3 = \{x\}$. Let y be a point of $\mathcal{S}$ at distance 3 from x and let Q_2 denote a quad of order (s, r_2) through y. Obviously, Q_2 is disjoint with at least one of the quads R_1, R_2, R_3. □

Proposition 5.4 $s \in \{1, 2\}$.

Proof. Suppose $s \geq 3$. Take quads Q_1 and Q_2 as in Lemma 5.3. Every point of Q_2 is collinear with a unique point of Q_1 and the set of points of Q_1 we obtain in this way form a subquadrangle of order (s, r_2) of Q_1. By Theorem 2.2.1 of [7], $sr_2 \leq r_1$. It follows that

$$\beta_1 = \frac{r_1}{r_1 - r_2} \leq \frac{r_1}{r_1 - \frac{r_1}{s}} = \frac{s}{s-1} \leq \frac{3}{2}.$$

So, every line is contained in a unique big quad of order (s, r_1). This is impossible since every two different quads of order (s, r_1) through a point meet in a line. □

We will now define two near hexagons.

(a) Let X be a set of size 8. Let $\mathbb{H}_3$ be the following incidence structure: (i) the points of $\mathbb{H}_3$ are the partitions of X in four sets of size 2; (ii) the lines of $\mathbb{H}_3$ are the partitions of X in two sets of size 2 and one set of size 4; (iii) a point is incident with a line if and only if the partition defined by the point is a refinement of the partition defined by the line. One easily verifies that $\mathbb{H}_3$ is a dense near hexagon of order $(2, 5)$, see also [3].

(b) Consider in $PG(6, 3)$ a nonsingular quadric $Q(6, 3)$ and a nontangent hyperplane π intersecting $Q(6, 3)$ in a nonsingular elliptic quadric $Q^-(5, 3)$. There is a polarity associated with $Q(6, 3)$ and we call two points orthogonal if one of them is contained in the polar hyperplane of the other. Let N denote the set of 126 internal points of $Q(6, 3)$ which are contained in π, i.e. the set of all 126 points in π for which the polar hyperplane intersects $Q(6, 3)$ in a nonsingular elliptic quadric. Let $\mathbb{E}_3$ be the following incidence structure: (i) the points of $\mathbb{E}_3$ are the 6-tuples of mutually orthogonal points; (ii) the lines of $\mathbb{E}_3$ are the pairs of mutually orthogonal points; (iii) incidence is reverse containment. By [5, Section (n)], $\mathbb{E}_3$ is a dense near hexagon of order $(2, 14)$. The near hexagon $\mathbb{E}_3$ was first constructed by Aschbacher [1].

Proposition 5.5 *If $s = 2$, then S is isomorphic to either $\mathbb{H}_3$ or $\mathbb{E}_3$.*

Proof. In this case S is a dense near hexagon of order $(2, t)$. All these near hexagons were classified in [3]. It follows that S is isomorphic to either $\mathbb{H}_3$ ($r_1 = 2$, $r_2 = 1$) or $\mathbb{E}_3$ ($r_1 = 4$, $r_2 = 2$). □

We will now take a look at the case $s = 1$. Then $v = 2(r_1+1)(r_1r_2+r_2+1)$. Since $\alpha_1 \in \mathbb{N}$, we have the following divisibility condition:

$$r_1 - r_2 \mid r_1.$$

The fact that all intersection numbers of the association scheme are nonnegative integers gives rise to no extra conditions. We will now calculate the eigenvalues of the collinearity matrix A and their corresponding multiplicities. This will give rise to a new divisibility condition. One calculates that

$$p(x) = \frac{x^4 + (r_1+1)(r_2+1)x^3 - r_1^2x^2 - r_1^2(r_1+1)(r_2+1)x}{(r_1+1)(r_2+1)^2(r_1r_2+r_2+2r_1+1)}.$$

So, the roots of $p(x)$ are

$$\lambda_1 = -(r_1+1)(r_2+1), \lambda_2 = -r_1, \lambda_3 = 0, \lambda_4 = r_1.$$

The fifth eigenvalue of A is equal to $\lambda_5 = (r_1+1)(r_2+1)$. We have $\mathrm{Tr}(A^0) = v = 2(r_1+1)(r_1r_2+r_2+1)$, $\mathrm{Tr}(A^1) = 0$, $\mathrm{Tr}(A^2) = v \cdot a_2 = 2(r_1+1)^2(r_1r_2+r_2+1)(r_2+1)$, $\mathrm{Tr}(A^3) = v \cdot a_3 = 0$. Solving the linear system ($j \in \{0, 1, 2, 3\}$)

$$\sum_{i=1}^{4} f_i\lambda_i^j = \mathrm{Tr}(A^j) - s^j(t+1)^j,$$

we find

$$f_1 = f_5 = 1,$$
$$f_2 = f_4 = \frac{r_2(r_2+1)(r_1+1)^2}{r_1},$$
$$f_3 = \frac{2(r_1^2 - r_1r_2 - r_2)(r_1r_2+r_2+1)}{r_1}.$$

Since these multiplicities are integral, we find the following divisibility condition:

$$r_1 \mid r_2(r_2+1).$$

References

[1] M. Aschbacher. Flag structures on Tits geometries. *Geom. Dedicata*, 14:21–32, 1983.

[2] R. C. Bose and D. M. Mesner. On linear associative algebras corresponding with association schemes of partially balanced designs. *Ann. Math. Statist.*, 30:21–38, 1959.

[3] A. E. Brouwer, A. M. Cohen, J. I. Hall and H. A. Wilbrink. Near polygons and Fischer spaces. *Geom. Dedicata*, 49:349–368, 1994.

[4] A. E. Brouwer, A. M. Cohen and A. Neumaier. *Distance-regular graphs.* Springer-Verlag, 1989.

[5] A. E. Brouwer and H. A. Wilbrink. The structure of near polygons with quads. *Geom. Dedicata*, 14:145–176, 1983.

[6] B. De Bruyn. Near polygons with two types of quads and three types of hexes. *J. Combin. Math. and Comb. Computing*, to appear.

[7] S. E. Payne and J. A. Thas. *Finite Generalized Quadrangles.* Vol. 110 of *Research Notes in Mathematics.* Pitman, Boston, 1984.

[8] S. Shad and E. E. Shult. The near n-gon geometries. preprint.

[9] E. E. Shult and A. Yanushka. Near n-gons and line systems. *Geom. Dedicata*, 9:1–72, 1980.