Finding Hamiltonian Cycles in Four Subfamilies of Quasi-Claw-Free Graphs

Rao Li *
Dept. of mathematical sciences
University of South Carolina Aiken
Aiken, SC 29801
Email: raol@usca.edu

Abstract

A graph G is called *quasi-claw-free* if it satisfies the property: $d(x,y) = 2 \Rightarrow$ there exists $u \in N(x) \cap N(y)$ such that $N[u] \subseteq N[x] \cup N[y]$. It is shown that a Hamiltonian cycle can be found in polynomial time in four subfamilies of quasi-claw-free graphs.

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1. Introduction

We consider only finite undirected graphs without loops and multiple edges. For terminology, notation and concepts not defined here see [3]. If v is a vertex in a graph, the *closed neighbor* $N[v]$ of v is defined as $N(v) \cup \{v\}$. For any two distinct vertices x and y in a graph G, $d(x, y)$ denotes the distance in G from x and y. A graph G of order n is *pancyclic* if G has a cycle of length r for each r with $3 \leq r \leq n$. If C is a cycle in G, let $\overrightarrow{C}$ denote the cycle C with a given orientation. For $u, v \in C$, let $\overrightarrow{C}[u, v]$ denote the consecutive vertices on C from u to v in the direction specified by $\overrightarrow{C}$. The same vertices, in reverse order, are given by $\overleftarrow{C}[v, u]$. Both $\overrightarrow{C}[u, v]$ and $\overleftarrow{C}[v, u]$ are considered as paths and vertex sets. If u is on C, then the predecessor, successor, next predecessor and next successor of u along the orientation of C are denoted by u^-, u^+, u^{--} and u^{++} respectively. The symboles $\vee$ and $\wedge$ are used to denote " or " and " and ", respectively. The following concept was introduced in [5]. If H is a subgraph of a graph G and $u, v \in V(H)$, the H is said to satisfy property $\phi(u, v)$ if $(N(u) \cap N(v)) - V(H) \neq \emptyset$.

The graphs Z_i, $1 \leq i \leq 3$, P_7, P_7^+, and B are respectively defined as follows. $V(Z_1) = \{a, b_1, b_2, c\}, E(Z_1) = \{ac, b_1c, b_2c, b_1b_2\}$; $V(Z_2) = \{a_1, a_2, b_1, b_2, c\}, E(Z_2) = \{a_2a_1, a_1c, b_1c, b_2c, b_1b_2\}$; $V(Z_3) = \{a_1, a_2, a_3, b_1, b_2, c\}, E(Z_3) = \{a_3a_2, a_2a_1, a_1c, b_1c, b_2c, b_1b_2\}$; $V(P_7) = \{a, b_1, b_2, c_1, c_2, d_1, d_2\}, E(P_7) = \{b_1c_1, c_1d_1, d_1a, ad_2, d_2c_2, c_2b_2\}$; $V(P_7^+) = V(P_7)$, $E(P_7^+) = E(P_7) \cup \{d_1d_2\}$; and $V(B) = \{a_1, a_2, b_1, b_2, c\}, E(B) = \{a_1a_2, a_1c, a_2c, b_1b_2, b_1c, b_2c\}$.

If G and H are graphs, then G is called *H-free* if G contains no induced subgraph isomorphic to H. If $H = K_{1,3}$, the graph G is called *claw-free*. It is well known that every line graph is claw-free. The concept of quasi-claw-free graphs was introduced by Ainouche [1]. A graph G is called *quasi-claw-free* if it satisfies the property: $d(x, y) = 2 \Rightarrow$ there exists $u \in N(x) \cap N(y)$ such that $N[u] \subseteq N[x] \cup N[y]$. Obviously, every claw-free graph is quasi-claw-free.

Bertossi [2] proved that determining if a line graph has a Hamiltonian path is NP-complete. From Bertossi's result, one can easily see that determining if a line graph has a Hamiltonian cycle is NP-complete and therefore finding a Hamiltonian cycle in line graphs is a hard problem. Since every line graph is claw-free and every claw-free graph is quasi-claw-free, the problem for finding a Hamiltonian cycle in quasi-claw-free graphs is also hard. Hence we currently can only find polynomial time algorithms for Hamiltonian cycles in subfamilies of quasi-claw-free graphs.

Broersma and Veldman proved the following theorems for claw-free graphs.

Theorem 1. [5] *Let G be a 2-connected claw-free graph. If every induced Z_1 of G satisfies $\phi(a, b_1) \vee \phi(a, b_2)$, then G is pancyclic or G is a cycle.*

Theorem 2. [5] *Let G be a 2-connected claw-free graph. If every induced Z_2 of G satisfies $\phi(a_1, b_1) \wedge \phi(a_1, b_2)$, then G is pancyclic or G is a cycle.*

Theorem 3. [5] *Let G be a 2-connected claw-free graph. If every induced subgraph of G isomorphic to P_7 or P_7^+ satisfies $\phi(a, b_1) \vee \phi(a, b_2) \vee (\phi(a, c_1) \wedge \phi(a, c_2))$, then G is Hamiltonian.*

Theorem 4. [5] *Let G be a 2-connected graph that is claw-free and B-free. If every induced Z_3 of G satisfies $\phi(a_1, b_1) \vee \phi(a_1, b_2) \vee \phi(a_2, b_1) \vee \phi(a_2, b_2) \vee (\phi(a_3, b_1) \wedge \phi(a_3, b_2))$, then G is Hamiltonian.*

Motivated by Ainouche's work of extending a number of theorems for claw-free graphs to quasi-claw-free graphs in [1]. We will prove the following theorems in this paper.

Theorem 5. *Let G be a 2-connected quasi-claw-free graph. If every induced Z_1 of G satisfies $\phi(a, b_1) \vee \phi(a, b_2)$, then a Hamiltonian cycle can be found in polynomial time in G.*

Theorem 6. *Let G be a 2-connected quasi-claw-free graph. If every induced Z_2 of G satisfies $\phi(a_1, b_1) \wedge \phi(a_1, b_2)$, then a Hamiltonian cycle can be found in polynomial time in G.*

Theorem 7. *Let G be a 2-connected quasi-claw-free graph. If every induced subgraph of G isomorphic to P_7 or P_7^+ satisfies $\phi(a, b_1) \vee \phi(a, b_2) \vee (\phi(a, c_1) \wedge \phi(a, c_2))$, then a Hamiltonian cycle can be found in polynomial time in G.*

Theorem 8. *Let G be a 2-connected graph that is quasi-claw-free and B-free. If every induced Z_3 of G satisfies $\phi(a_1, b_1) \vee \phi(a_1, b_2) \vee \phi(a_2, b_1) \vee \phi(a_2, b_2) \vee (\phi(a_3, b_1) \wedge \phi(a_3, b_2))$, then a Hamiltonian cycle can be found in polynomial time in G.*

2. Lemmas

Based on Tarjan's depth-first search algorithm in [9], Köhler proved the following Lemma 1 in [7]. Lemma 1 was used by Brandstädt et al. in [4]. We will also use Lemma 1 in our proofs in Section 3.

Lemma 1. *Let G be a 2-connected graph, and let x, y be two different non-adjacent vertices of G. Then one can construct in linear time (of $|V(G)|$ and $|E(G)|$) two induced, internally disjoint paths, both joining x and y.*

The following Lemma 2 is a computational refomulation of Lemma 3 in [1].

Lemma 2. *Let G be a connected quasi-claw-free of order n. Suppose that G has a cycle C of length r, $4 \leq r \leq n-1$. Let u be a vertex on C such that it has a neighbor in $V(G) - V(C)$. Then either $u^-u^+ \in E$ or a cycle of length $r+1$ can be found in polynomial time in G.*

Proof of Lemma 2. Let G be a connected quasi-claw-free of order n. Suppose that G has a cycle C of length r with a fixed orientation, $4 \leq r \leq n-1$. Let u be a vertex on C such that it has a neighbor in $V(G) - V(C)$. If $u^-u^+ \in E$, then it is done. Now we assume that $u^-u^+ \notin E$. Choose a vertex $x \in N(u) \cap (V(G) - V(C))$. If $xu^- \in E$ or $xu^+ \in E$, we can easily construct a cycle of length $r+1$ in G. Now we assume that $xu^- \notin E$ and $xu^+ \notin E$.

Since $d(x, u^+) = 2$, there exists a vertex, say v, such that $v \in N(x) \cap N(u^+)$ and $N[v] \subseteq N[x] \cup N[u^+]$. Since $u^- \in N[u]$ but $u^- \notin N[x] \cup N[u^+]$, we have $v \neq u$. If $v \in V(C)$, then $v^+ \in N[v] \subseteq N[x] \cup N[u^+]$. Thus $v^+ \in N(x)$ or $v^+ \in N(u^+)$. In either case, we can easily construct a cycle of length $r+1$ in G. Now we assume $v \in V(G) - V(C)$. Symmetrically, we can prove that there exist a vertex, say w, in $V(G) - V(C)$ such that $w \in N(x) \cap N(u^-)$ and $N[w] \subseteq N[x] \cup N[u^-]$ otherwise we can construct a cycle of length $r+1$ in G. Obviously, $v \neq w$, otherwise $u^- \in N[w] = N[v] \subseteq N[x] \cup N[u^+]$, contradicting to our assumption.

If $vu^{++} \in E$, we can easily construct a cycle of length $r+1$ in G. We now assume that $vu^{++} \notin E$. Since $d(v, u^{++}) = 2$, there exists a vertex, say y, such that $y \in N(v) \cap N(u^{++})$ and $N[y] \subseteq N[v] \cup N[u^{++}]$. If $y = u^+$, then $u \in N[u^+] \subseteq N[v] \cup N[u^{++}]$. Thus $u \in N[v]$ or $u \in N[u^{++}]$. If $u \in N[v]$, we can easily construct a cycle of length $r+1$ in G. If $u \in N[u^{++}]$, we can construct a cycle $wxu\overrightarrow{C}[u^{++}, u^-]w$ of length $r+1$ in G. Moreover, we can assume that y is in $V(G) - V(C)$ otherwise using a construction which is similar to the one for the case $v \in V(G) - V(C)$ above we can construct a cycle of length $r+1$ in G.

If $y = w$, then $vw \in E$ and we can construct a cycle $wv\overrightarrow{C}[u^+, u^-]w$ of length $r+1$ in G. If $y = x$, then $u \in N[x] \subseteq N[v] \cup N[u^{++}]$. If $u \in N[v]$, we can easily construct a cycle of length $r+1$ in G. If $u \in N[u^{++}]$, we can also easily construct a cycle of length $r+1$ in G. Now we assume that $y \in V(G) - (V(C) \cup \{w, x\})$. Thus $y \in N[v] \subseteq N[x] \cup N[u^+]$. If $y \in N[u^+]$, we can easily construct a cycle of length $r+1$ in G. If $y \in N[x]$, we can construct a cycle $xy\overrightarrow{C}[u^{++}, u]x$ of lenght $r+1$ in G. Clearly, the above algorithmic procedure can be completed in polynomial time. QED

3. Proofs

Proof of Theorem 5. Let G be a graph satisfying the conditions in Theorem 5. Check if G has a pair of nonadjacent vertices. If we cannot find a pair of nonadjacent vertices in G, then G is a complete graph and we can easily find a Hamiltonian cycle in G. If we can find a pair of nonadjacent vertices in G, apply Lemma 1, we can find a cycle C of length $r \geq 4$ in G. This step can be completed in $O(|V(G)|+|V(E)|)$ time. If $r = |V(G)|$, then we are finished. we now assume that $r \leq |V(G)|-1$ and give an orientation on C.

Since G is 2-connected, we can find a vertex, say u, on C such that $N(u) \cap (V(G) - V(C)) \neq \emptyset$. Next we will show that a cycle C_1 of length $r+1$ can be found in polynomial time in G.

If $u^-u^+ \notin E$ we can apply Lemma 2 to construct a cycle C_1 of length $r+1$ in G. Now we assume that $u^-u^+ \in E$. Choose a vertex, say w, in $N(u) \cap (V(G) - V(C))$. If $wu^- \in E$ or $wu^+ \in E$, then we can easily construct a cycle C_1 of length $r+1$ in G. We now assume that $wu^- \notin E$ and $wu^+ \notin E$. Thus $H_1 := G[\{w, u, u^-, u^+\}] \cong Z_1$. Then $N(w) \cap N(u^+) - V(H_1) \neq \emptyset$ or $N(w) \cap N(u^-) - V(H_1) \neq \emptyset$.

When $N(w) \cap N(u^+) - V(H_1) \neq \emptyset$, choose a vertex $v \in N(w) \cap N(u^+) - V(H_1)$. If $v \in V(C)$, then by Lemma 2 we can construct a cycle C_1 of length $r+1$ in G if $v^-v^+ \notin E$. We now assume that $v^-v^+ \in E$. In this case, we can construct a cycle $C_1 := wv\overrightarrow{C}[u^+, v^-]\overrightarrow{C}[v^+, u]w$ of length $r+1$ in G. If $v \in V(G) - V(C)$, then by Lemma 2 we can construct a cycle C_1 of length $r+1$ in G if $uu^{++} \notin E$. We now assume that $uu^{++} \in E$.

If $vu^{++} \in E$, we can easily construct a cycle C_1 of length $r+1$ in G. If $vu^- \in E$, we can construct a cycle $C_1 := u^-vwu\overrightarrow{C}[u^{++}, u^-]$ of length $r+1$ in G. So now we assume that $vu^{++} \notin E$ and $vu^- \notin E$.

If $u^-u^{++} \in E$, then $H_2 := G[\{v, u^+, u^-, u^{++}\}] \cong Z_1$. Thus $N(v) \cap N(u^-) - V(H_2) \neq \emptyset$ or $N(v) \cap N(u^{++}) - V(H_2) \neq \emptyset$. When $N(v) \cap N(u^-) - V(H_2) \neq \emptyset$, choose a vertex $p \in N(v) \cap N(u^-) - V(H_2)$. If $p \in V(C)$, then by Lemma 2 we can construct a cycle C_1 of length $r+1$ in G if $p^-p^+ \notin E$ or if $p^-p^+ \in E$ we can construct a cycle $C_1 := vwu\overrightarrow{C}[u^{++}, p^-]\overrightarrow{C}[p^+, u^-]pv$ of length $r+1$ in G. If $p \in V(G) - V(C)$, then we can construct a cycle $C_1 := u^-pv\overrightarrow{C}[u^+, u^-]$ of length $r+1$ in G.

When $N(v) \cap N(u^{++}) - V(H_2) \neq \emptyset$, choose a vertex $q \in N(v) \cap N(u^{++}) - V(H_2)$. If $q \in V(C)$. Using a construction which is similar to the one for the case $v \in V(C)$ above, we can construct a cycle C_1 of length $r+1$ in G Now we assume that $q \in V(G) - V(C)$. Then by Lemma 2 we can construct a cycle C_1 of length $r+1$ in G if $u^+u^{+++} \notin E$ or if $u^+u^{+++} \in E$ we can construct a cycle $C_1 := vu^+\overrightarrow{C}[u^{+++}, u]wv$ of length $r+1$ in G.

Now we consider the case $u^-u^{++} \notin E$. Since $d(v, u^-) = 2$, there exists a vertex x such that $x \in N(v) \cap N(u^-)$ and $N[x] \subseteq N[v] \cup N[u^-]$. Since $u^-u^{++} \notin E$, $x \neq u^{++}$. If $x = u$, we can easily construct a cycle C_1 of length $r+1$ in G. So we can assume that $x \neq u$ and $x \neq u^{++}$. Since $u^{++} \in N[u^+]$ but $u^{++} \notin N[v] \cup N[u^-]$, we have $x \neq u^+$. Using a construction which is similar to the one for the case $N(v) \cap N(u^-) - V(H_2) \neq \emptyset$ before, we can construct a cycle C_1 of length $r+1$ in G.

Symmetrically, we can construct a cycle C_1 of length $r+1$ in G when $N(w) \cap N(u^-) - V(H_1) \neq \emptyset$

Since the algorithmic procedures in Lemma 1 and Lemma 2 can be completed in polynomial time, the step of enlarging the cycle C of length r in G to a cycle C_1 of length $r+1$ can be fulfilled in polynomial time time.

Apply the similar procedure as above to the cycle C_1 of length $r+1$ in G, we can construct a cycle of length $r+2$ in G. Repeat this process, we can construct a cycle of length s for each s with $r+1 \leq s \leq |V(G)|$. Notice that we can repeat the processes at most $|V(G)|$ times, therefore we can find a Hamiltonian cycle in the graph G in polynomial time. QED

Proof of Theorem 6. Let G be a graph satisfying the conditions in Theorem 6. Check if G has a pair of nonadjacent vertices. If we cannot find a pair of nonadjacent vertices in G, then G is a complete graph and we can easily find a Hamiltonian cycle in G. If we can find a pair of nonadjacent vertices in G, apply Lemma 1, we can find a cycle C of length $r \geq 4$ in G.

This step can be completed in $O(|V(G)|+|V(E)|)$ time. If $r = |V(G)|$, then we are finished. we now assume that $r \leq |V(G)|-1$ and give an orientation on C.

Using depth-first search algorithm in the graph $G[V(G) - V(C)]$, we can find a connected component, say H, in $G[V(G) - V(C)]$. More details on applying depth-first search algorithm to find a connected component in a graph can be found in Algorithm 8.3 on Page 330 in [8]. This step can be completed in $O(|V(G)| + |V(E)|)$ time. Since G is 2-connected, we can find two distinct vertices, say v_1 and v_2, on C such that $N(v_i) \cap V(H) \neq \emptyset$, where $1 \leq i \leq 2$. Next we will show that a cycle C_1 of length $r+1$ in G can be found in polynomial time.

If $v_1v_2 \in E$, define a graph $G_1 := G[V(H) \cup \{ v_1, v_2 \}] - v_1v_2$; otherwise define $G_1 := G[V(H) \cup \{ v_1, v_2 \}]$. Using breadth-first search algorithm in the graph G_1, we can find a shortest path $P := v_1u_1u_2...u_tv_2$ between v_1 and v_2 in G_1, where each vertex u_i with $1 \leq i \leq t$ is in $V(H)$. This step can completed in $O(|V(G)| + |E(G)|)$ time. More details on breadth-first search algorithm can be found in [6].

If $u_1v^- \in E$ or $u_1v^+ \in E$, we can easily construct a cycle C_1 of length $r+1$ in G. If $v_1^-v_1^+ \notin E$, we can apply Lemma 2 to construct a cycle C_1 of length $r+1$ in G. Now we assume that $u_1v^- \notin E$, $u_1v^+ \notin E$, and $v_1^-v_1^+ \in E$.

If $t \geq 2$, then $u_2v_1 \notin E$ since P is a shortest path between v_1 and v_2 in G_1. Now we consider the case $u_2v_1^+ \in E$. Then we can assume that $v_1v_1^{++} \in E$ otherwise we can apply Lemma 2 to construct a cycle of C_1 of length $r+1$ in G. Moreover, if $u_1v_1^{++} \in E$, $u_2v_1^{++} \in E$, or $u_2v_1^- \in E$, then we can construct a cycle $C_1 = u_2u_1\overrightarrow{C}[v_1^{++}, v_1^-]u_2$, $C_1 = u_2\overrightarrow{C}[v_1^{++}, v_1^+]v_1^+u_2$, or $C_1 = u_2u_1v_1\overrightarrow{C}[v_1^{++}, v_1^-]u_2$ of length $r+1$ in G, respectively. So we now assume that $u_1v_1^{++} \notin E$, $u_2v_1^{++} \notin E$, and $u_2v_1^- \notin E$.

If $v_1^-v_1^{++} \in E$, then $H_1 := G[\{u_2, u_1, v_1, v_1^-, v_1^{++}\}] \cong Z_2$. Thus $N(u_1) \cap N(v_1^-) - V(H_1) \neq \emptyset$. Choose a vertex, say x, in $N(u_1) \cap N(v_1^-) - V(H_1)$. Now if $x \in V(C)$, then if $x^-x^+ \notin E$ we can apply Lemma 2 to construct a cycle C_1 of length $r+1$ in G and if $x^-x^+ \in E$ we can construct a cycle $C_1 = u_1\overleftarrow{C}[v_1, x^+]\overrightarrow{C}[x^+, v_1^-]xu_1$ of length in G. If $x \in V(G) - V(C)$, then we can construct a cycle $C_1 = u_1v_1\overrightarrow{C}[v_1^{++}, v_1^-]xu_1$.

If $v_1^-v_1^{++} \notin E$, since $d(u_1, v_1^-) = 2$ and G is quasi-claw-free, there exists a vertex, say y, such that $y \in N(u_1) \cap N(v_1^-)$ and $N[y] \subseteq N[u_1] \cap N[v_1^-]$. Notice that $v_1^{++} \in N[v_1]$ but $v_1^{++} \notin N[u_1] \cap N[v_1^-]$. Thus $y \neq v_1$. Using a construction which is similar to the one for the case $N(u_1) \cap N(v_1^-) - V(H_1) \neq \emptyset$ just above, we can construct a cycle C_1 of length $r+1$ in G.

Symmetrically, if $u_2 v_1^- \in E$, we can construct a cycle C_1 of length $r+1$ in G.

We now assume that $u_2 v_1^+ \notin E$ and $u_2 v_1^- \notin E$. Thus $H_2 := G[\{u_2, u_1, v_1, v_1^-, v_1^+\}] \cong Z_2$. So $N(u_1) \cap N(v_1^-) - V(H_2) \neq \emptyset$. Using a construction which is similar to the one for the case $N(u_1) \cap N(v_1^-) - V(H_1) \neq \emptyset$ just above, we can construct a cycle C_1 of length $r+1$ in G.

Next we consider the case $t = 1$. Then $v_2 \neq v_1^{--}, v_1^-, v_1^+$, and v_1^{++} otherwise we can easily construct a cycle C_1 of length $r+1$ in G. We also have $v_2^- v_2^+ \in E$ otherwise we can apply Lemma 2 to construct a cycle C_1 of length $r+1$ in G.

If $v_2 v_1^- \in E$, we can construct a cycle $C_1 = u_1 \overrightarrow{C}[v_1, v_2^-] \overrightarrow{C}[v_2^+, v_1^-] v_2 u_1$ of length $r+1$ in G. Similarly, we can construct a cycle of length $r+1$ in G when $v_2 v_1^+ \in E$, . Now we assume that $v_2 v_1^- \notin E$ and $v_2 v_1^+ \notin E$.

If $v_1 v_2 \notin E$, then $H_3 := G[\{v_2, u_2, v_1, v_1^-, v_1^+\}] \cong Z_2$. Thus $N(v_1^-) \cap N(u_1) - V(H_3) \neq \emptyset$. Choose a vertex, say z, in $N(v_1^-) \cap N(u_1) - V(H_3)$. If $z \in V(C)$, then $z^- z^+ \in E$ otherwise we can apply Lemma 2 to construct a cycle of C_1 of length $r+1$ in G. Furthermore, if $z \in \overrightarrow{C}[v_1^{++}, v_2^-]$, then we can construct a cycle $C_1 = u_1 \overrightarrow{C}[v_1, z^-] \overrightarrow{C}[z^+, v_1^-] z u_1$ of length $r+1$ in G and if $z \in \overrightarrow{C}[v_2^+, v_1^{--}]$, then we can construct a cycle $C_1 = u_1 \overrightarrow{C}[v_1, z^-] \overrightarrow{C}[z^+, v_1^-] z u_1$ of length of $r+1$ in G. If $z \in V(G) - V(C)$, then we can construct a cycle $C_1 = u_1 \overrightarrow{C}[v_1, v_2^-] \overrightarrow{C}[v_2^+, v_1^-] z u_1$ of length $r+1$ in G.

Now we assume that $v_1 v_2 \in E$. If $v_2 v_1^- \in E$ or $v_2 v_1^{--} \in E$, then we can easily construct a cycle C_1 of length $r+1$ in G. So we assume that $v_2 v_1^- \notin E$ and $v_2 v_1^{--} \notin E$. Choose the first vertex w along $\overrightarrow{C}[v_2^+, v_1^-]$ such that $w \notin N(v_2)$. If $v_1 w^- \in E$, then we can construct a cycle $C_1 = u_1 v_1 \overrightarrow{C}[w^-, v_1^-] \overrightarrow{C}[v_1^+, v_2^-] \overrightarrow{C}[v_2^+, w^{--}] v_2 u_1$ of length $r+1$ in G. If $v_1 w \in E$, then we can construct a cycle $C_1 = u_1 v_1 \overrightarrow{C}[w, v_1^-] \overrightarrow{C}[v_1^+, v_2^-] \overrightarrow{C}[v_2^+, w^-] v_2 u_1$ of length $r+1$ in G. If $u_1 w^- \in E$, then we can construct a cycle $C_1 = u_1 \overrightarrow{C}[w^-, v_2^-] \overrightarrow{C}[v_2^+, w^{--}] v_2 u_1$ of length $r+1$ in G. If $u_1 w \in E$, then we can construct a cycle $C_1 = u_1 \overrightarrow{C}[w, v_2^-] \overrightarrow{C}[v_2^+, w^-] v_2 u_1$ of length $r+1$ in G. Now we can assume that $v_1 w^- \notin E$, $v_1 w \notin E$, $u_1 w^- \notin E$, and $u_1 w \notin E$. Therefore $H_4 := G[\{w, w^-, v_2, u_1, v_1\}] \cong Z_2$. Hence $N(u_1) \cap N(w^-) - V(H_4) \neq \emptyset$.

Choose a vertex $o \in N(u_1) \cap N(w^-) - V(H_4)$. If $o \in V(G) - V(C)$, then we can construct a cycle $C_1 = u_1 o\overrightarrow{C}[w^-, v_1^-]\ \overrightarrow{C}[v_1^+, v_2^-]\overrightarrow{C}[v_2^+, w^{--}]v_2 u_1$. If $o \in V(C)$, then $o^- o^+ \in E$ otherwise we can apply Lemma 2 to construct a cycle C_1 of length $r+1$ in G. If $o \in \overrightarrow{C}[v_2^+, w^{--}]$, then we can construct a cycle $C_1 = u_1\overrightarrow{C}[o, v_2^-]\overrightarrow{C}[v_2^+, o^-]v_2 u_1$ of length $r+1$ in G. If $o \in \overrightarrow{C}[w^+, v_1^-] \cup \overrightarrow{C}[v_1^+, v_2^-]$, then we can construct a cycle $C_1 = u_1 o\overrightarrow{C}[w^-, o^-]\overrightarrow{C}[o^+, v_2^-]\overrightarrow{C}[v_2^+, w^{--}]v_2 u_1$ of length $r+1$ in G.

Since the algorithmic procedures in Lemma 1 and Lemma 2, the depth-first search algorithm for finding a component H in $G[V(G) - V(C)]$, and the breadth-first search algorithm for finding the shortest path between v_1 and v_2 in G_1 all can be completed in polynomial time, the step of enlarging the cycle C of length r in G to a cycle C_1 of length $r+1$ can be fulfilled in polynomial time.

Apply the similar procedure as above to the cycle C_1 of length $r+1$ in G, we can construct a cycle of length $r+2$ in G. Repeat this process, we can construct a cycle of length s for each s with $r+1 \le s \le |V(G)|$. Notice that we can repeat the processes at most $|V(G)|$ times, therefore we can find a Hamiltonian cycle in the graph G in polynomial time. QED

Proof of Theorem 7. Let G be a graph satisfying the conditions in Theorem 7. Check if G has a pair of nonadjacent vertices. If we cannot find a pair of nonadjacent vertices in G, then G is a complete graph and we can easily find a Hamiltonian cycle in G. If we can find a pair of nonadjacent vertices in G, apply Lemma 1, we can find a cycle C of length $r \ge 4$ in G. This step can be completed in $O(|V(G)|+|V(E)|)$ time. If $r = |V(G)|$, then we are finished. we now assume that $r \le |V(G)| - 1$ and give an orientation on C.

Using depth-first search algorithm in the graph $G[V(G) - V(C)]$, we can find a connected component, say H, in $G[V(G) - V(C)]$. This step can be completed in $O(|V(G)| + |V(E)|)$ time. Since G is 2-connected, we can find two distinct vertices, say v_1 and v_2, on C such that $N(v_i) \cap V(H) \neq \emptyset$, where $1 \le i \le 2$. Next we will show that a cycle C_1 of length at least $r+1$ in G can be found in polynomial time.

If $v_1 v_2 \in E$, define a graph $G_1 := G[V(H) \cup \{v_1, v_2\}] - v_1 v_2$; otherwise define $G_1 := G[V(H) \cup \{v_1, v_2\}]$. Using breadth-first search algorithm in the graph G_1, we can find a shortest path $P := v_1 u_1 u_2 ... u_t v_2$ between v_1 and v_2 in G_1, where each vertex u_i with $1 \le i \le t$ is in $V(H)$. This step can be completed in $O(|V(G)| + |E(G)|$.

We can assume that $v_1^+v_1^-$, $v_2^+v_2^- \in E$, otherwise we can apply Lemma 2 to construct a cycle C_1 of length $r+1$ in G. We can further assume that $v_2 \neq v_1^{--}, v_1^-, v_1^+$, and v_1^{++}, otherwise we can easily construct a cycle C_1 of length at least $r+1$ in G. Also, $v_1 \neq v_2^{--}, v_2^-, v_2^+$, and v_2^{++}, otherwise we can easily construct a cycle C_1 of length at least $r+1$ in G. Moreover, we can assume that $v_1v_2^{--}$, $v_1v_2^-$, $v_1v_2^+$, $v_1v_2^{++}$, $v_2v_1^{--}$, $v_2v_1^-$, $v_2v_1^+$, and $v_2v_1^{++}$ are not in E, otherwise we can easily construct a cycle C_1 of length at least $r+1$ in G.

Find the first vertex, say w_1, along $\overrightarrow{C}[v_1^+, v_2^-]$ such that $v_1w_1 \notin E$ and the first vertex, say w_2, along $\overrightarrow{C}[v_2^+, v_1^-]$ such that $v_2w_2 \notin E$. We can assume that u_ip is not in E for each i with $1 \leq i \leq t$ and p is any vertex in $\overrightarrow{C}[v_1^+, w_1] \cup \overrightarrow{C}[v_2^+, w_2]$, otherwise we can construct a cycle C_1 of length at least $r+1$ in G. We can further assume that pq, v_1q, and v_2p are not in E, where p is any vertex in $\overrightarrow{C}[v_1^+, w_1]$ and q is any vertex in $\overrightarrow{C}[v_2^+, w_2]$ otherwise we can easily construct a cycle of length at least $r+1$ in G.

If $t = 1$, then $H_1 := G[\{w_1, w_1^-, v_1, u_1, v_2, w_2^-, w_2\}]$ is isomorphic to P_7 if $v_1v_2 \in E$ or P_7^+ otherwise. Thus $N(u_1) \cap N(w_1) - V(H_1) \neq \emptyset$ or $N(u_1) \cap N(w_2) - V(H_1) \neq \emptyset$ or $(N(u_1) \cap N(w_1^-) - V(H_1) \neq \emptyset$ and $N(u_1) \cap N(w_2^-) - V(H_1) \neq \emptyset)$.

When $N(u_1) \cap N(w_1) - V(H_1) \neq \emptyset$, choose a vertex, say o, in $N(u_1) \cap N(w_1) - V(H_1) \neq \emptyset$. If $o \in V(G) - V(C)$, then we can construct a cycle $C_1 = u_1o\overrightarrow{C}[w_1, v_1^-]\overrightarrow{C}[v_1^+, w_1^-]v_1u_1$ of length at least $r+1$ in G. If $o \in V(C)$, then we can assume that $o^-o^+ \in E$ otherwise we can apply Lemma 2 to construct a cycle C_1 of length $r+1$ in G. Thus we can construct a cycle of $C_1 = u_1o\overrightarrow{C}[w_1, o^-]\overrightarrow{C}[o^+, v_1^-]\overrightarrow{C}[v_1^+, w_1^-]v_1u_1$ of length at least $r+1$ in G. Similarly, we can construct a cycle C_1 of length at least $r+1$ in G when $N(u_1) \cap N(w_2) - V(H_1) \neq \emptyset$.

When $N(u_1) \cap N(w_1^-) - V(H_1) \neq \emptyset$ and $N(u_1) \cap N(w_2^-) - V(H_1) \neq \emptyset$, choose a vertex, say o, in $N(u_1) \cap N(w_1^-) - V(H_1) \neq \emptyset$. If $o \in V(G) - V(C)$, then we can construct a cycle $C_1 = u_1o\overrightarrow{C}[w_1^-, v_1^-]\overrightarrow{C}[v_1^+, w_1^{--}]v_1u_1$ of length at least $r+1$ in G. If $o \in V(C)$, then we can assume that $o^-o^+ \in E$ otherwise we can apply Lemma 2 to construct a cycle C_1 of length $r+1$ in G. Thus we can construct a cycle of $C_1 = u_1o\overrightarrow{C}[w_1^-, o^-]\overrightarrow{C}[o^+, v_1^-]\overrightarrow{C}[v_1^+, w_1^{--}]v_1u_1$ of length at least $r+1$ in G.

If $t \geq 2$, since $P = v_1u_1u_2...u_tv_2$ is a shortest path between v_1 and v_2 in G_1, $u_1v_2 \notin E$. If $v_1v_2 \in E$, since $d(u_1, v_1^-) = 2$ and G is quasi-claw-free, there exists a vertex, say y, such that $y \in N(u_1) \cap N(v_1^-)$ and

$N[y] \subseteq N[u_1] \cup N[v_1^-]$. Notice that $v_2 \in N[v_1]$ but $v_2 \notin N[u_1] \cup N[v_1^-]$, we have $y \neq v_1$. Thus if $y \in V(G) - V(C)$, we can construct a cycle $C_1 = u_1\overrightarrow{C}[v_1, v_1^-]yu_1$ of length at least $r+1$ in G. If $y \in V(C)$, then we can assume that $y^-y^+ \in E$ otherwise we can apply Lemma 2 to construct a cycle C_1 of length $r+1$ in G. Thus we can construct a cycle $C_1 = u_1\overrightarrow{C}[v_1, y^-]\overrightarrow{C}[y^+, v_1^-]yu_1$ of length at least $r+1$ in G. Next we assume that $v_1v_2 \notin E$.

If $t = 2$, then $H_2 = G[\{w_1, w_1^-, v_1, u_1, u_2, v_2, w_2^-\}] \cong P_7$. Thus $N(u_1) \cap N(w_1) - V(H_2) \neq \emptyset$ or $N(u_1) \cap N(w_2^-) - V(H_2) \neq \emptyset$ or $(N(u_1) \cap N(w_1^-) - V(H_2) \neq \emptyset$ and $N(u_1) \cap N(v_2) - V(H_2) \neq \emptyset)$.

When $N(u_1) \cap N(w_1) - V(H_2) \neq \emptyset$ or $N(u_1) \cap N(w_2^-) - V(H_2) \neq \emptyset$ or $N(u_1) \cap N(w_1^-) - V(H_2) \neq \emptyset$, using a construction which is similar to the one for the cases $t = 1$ and $N(u_1) \cap N(w_1) - V(H_1) \neq \emptyset$ or $t = 1$ $N(u_1) \cap N(w_1^-) - V(H_1) \neq \emptyset$, we can construct a cycle C_1 of length at least $r+1$ in G

If $t = 3$, then $H_3 = G[\{w_1, w_1^-, v_1, u_1, u_2, u_3, v_2\}] \cong P_7$. Thus $N(u_1) \cap N(v_2) - V(H_3) \neq \emptyset$ or $N(u_1) \cap N(w_1) - V(H_3) \neq \emptyset$ or $(N(u_1) \cap N(w_1^-) - V(H_3) \neq \emptyset$ and $N(u_1) \cap N(u_3) - V(H_3) \neq \emptyset)$.

When $N(u_1) \cap N(v_2) - V(H_3) \neq \emptyset$, choose a vertex $p \in N(u_1) \cap N(v_2) - V(H_3) \neq \emptyset$. Since $P = v_1u_1u_2u_3v_2$ is a shortest path between v_1 and v_2 in G_1, we have $u_1v_2 \notin E$, $p \in V(C)$, and $d(u_1, v_2) = 2$. Since G is quasi-claw-free, there exists a vertex, say o, such that $o \in N(u_1) \cap N(v_2)$ and $N[o] \subseteq N[u_1] \cup N[v_2]$. Again the fact that $P = v_1u_1u_2u_3v_2$ is a shortest path between v_1 and v_2 in G_1 implies that $o \in V(C)$. We can assume that $o^-o^+ \in E$ otherwise we can apply Lemma 2 to construct a cycle C_1 of length $r+1$ in G. We further have $o \in \overrightarrow{C}[w_1^+, v_2^{--}] \cup \overrightarrow{C}[w_2^+, v_1^{--}]$, otherwise we can construct a cycle C_1 of length at least $r+1$ in G.

If $o \in \overrightarrow{C}[w_1^+, v_2^{--}]$, since $o^- \in N[o]$, we have $o^- \in N[u_1] \cup N[v_2]$. If $o^- \in N[u_1]$, we can easily construct a cycle C_1 of length at least $r+1$ in G. If $o^- \in N[v_2]$, then we can construct a cycle $C_1 = u_1\overrightarrow{C}[o, v_2^-]\overrightarrow{C}[v_2^+, o^-]v_2u_3u_2u_1$ of length at least $r+1$ in G. If $o \in \overrightarrow{C}[w_2^+, v_1^{--}]$, since $o^+ \in N[o]$, we have $o^+ \in N[u_1] \cup N[v_2]$. If $o^+ \in N[u_1]$, we can easily construct a cycle C_1 of length at least $r+1$ in G. If $o^+ \in N[v_2]$, then we can construct a cycle $C_1 = u_1u_2u_3v_2\overrightarrow{C}[o^+, v_2^-]\overrightarrow{C}[v_2^+, o]u_1$ of length at least $r+1$ in G.

When $N(u_1) \cap N(w_1) - V(H_3) \neq \emptyset$ or $N(u_1) \cap N(w_1^-) - V(H_3) \neq \emptyset$, using a construction which is similar to the one for the cases $t = 1$ and

$N(u_1) \cap N(w_1) - V(H_1) \neq \emptyset$ or $t = 1$ $N(u_1) \cap N(w_1^-) - V(H_1) \neq \emptyset$, we can construct a cycle C_1 of length at least $r + 1$ in G

If $t \geq 4$, then $H_4 = G[\{w_1, w_1^-, v_1, u_1, u_2, u_3, u_4\}] \cong P_7$. Thus $N(u_1) \cap N(u_4) - V(H_4) \neq \emptyset$ or $N(u_1) \cap N(w_1) - V(H_4) \neq \emptyset$ or $(N(u_1) \cap N(w_1^-) - V(H_3) \neq \emptyset$ and $N(u_1) \cap N(u_3) - V(H_4) \neq \emptyset)$.

When $N(u_1) \cap N(u_4) - V(H_4) \neq \emptyset$, choose a vertex $p \in N(u_1) \cap N(u_4) - V(H_4) \neq \emptyset$. Since $P = v_1 u_1 u_2 ... u_t v_2$ is a shortest path between v_1 and v_2 in G_1, we have $u_1 u_4 \notin E$, $p \in V(C)$, and $d(u_1, u_4) = 2$. Since G is quasi-claw-free, there exists a vertex, say o, such that $o \in N(u_1) \cap N(u_4)$ and $N[o] \subseteq N[u_1] \cup N[u_4]$. Again the fact that $P = v_1 u_1 u_2 ... u_t v_2$ is a shortest path between v_1 and v_2 in G_1 implies that $o \in V(C)$. Since $o^- \in N[u_1] \cup N[u_4]$, we can easily construct a cycle C_1 of length $r+1$ in G.

When $N(u_1) \cap N(w_1) - V(H_4) \neq \emptyset$ or $N(u_1) \cap N(w_1^-) - V(H_4) \neq \emptyset$, using a construction which is similar to the one for the cases $t = 1$ and $N(u_1) \cap N(w_1) - V(H_1) \neq \emptyset$ or $t = 1$ $N(u_1) \cap N(w_1^-) - V(H_1) \neq \emptyset$, we can construct a cycle C_1 of length at least $r + 1$ in G

Since the algorithmic procedures in Lemma 1 and Lemma 2, the depth-first search algorithm for finding a component H in $G[V(G) - V(C)]$, and the breadth-first search algorithm for finding the shortest path between v_1 and v_2 in G_1 all can be completed in polynomial time, the step of enlarging the cycle C of length r in G to a cycle C_1 of length at least $r + 1$ can be fulfilled in polynomial time.

Apply the similar procedure as above to the cycle C_1 of length at least $r + 1$ in G, we can construct a cycle longer than C_1 in G. Repeat this process, we can construct a Hamiltonian cycle in G. Notice that we can repeat the processes at most $|V(G)|$ times, therefore we can find a Hamiltonian cycle in the graph G in polynomial time. QED

Proof of Theorem 8. Let G be a graph satisfying the conditions in Theorem 8. Check if G has a pair of nonadjacent vertices. If we cannot find a pair of nonadjacent vertices in G, then G is a complete graph and we can easily find a Hamiltonian cycle in G. If we can find a pair of nonadjacent vertices in G, apply Lemma 1, we can find a cycle C of length $r \geq 4$ in G. This step can be completed in $O(|V(G)| + |V(E)|)$ time. If $r = |V(G)|$, then we are finished. we now assume that $r \leq |V(G)| - 1$ and give an orientation on C.

Using depth-first search algorithm in the graph $G[V(G) - V(C)]$, we can find a connected component, say H, in $G[V(G)-V(C)]$. This step can be completed in $O(|V(G)|+|V(E)|)$ time. Since G is 2-connected, we can find two distinct vertices, say v_1 and v_2, on C such that $N(v_i)\cap V(H)\neq\emptyset$, where $1\leq i\leq 2$. Next we will show that a cycle C_1 of length at least $r+1$ in G can be found in polynomial time.

If $v_1v_2\in E$, define a graph $G_1:=G[V(H)\cup\{v_1,v_2\}]-v_1v_2$; otherwise define $G_1:=G[V(H)\cup\{v_1,v_2\}]$. Using breadth-first search algorithm in the graph G_1, we can find a shortest path $P:=v_1u_1u_2...u_tv_2$ between v_1 and v_2 in G_1, where each vertex u_i with $1\leq i\leq t$ is in $V(H)$. This step can be completed in $O(|V(G)|+|E(G)|)$ time.

We can assume that $v_1^+v_1^-$, $v_2^+v_2^-\in E$, otherwise we can apply Lemma 2 to construct a cycle C_1 of length $r+1$ in G. We can further assume that $v_2\neq v_1^{--}, v_1^-, v_1^+$, and v_1^{++}, otherwise we can easily construct a cycle C_1 of length at least $r+1$ in G. Also, $v_1\neq v_2^{--}, v_2^-, v_2^+$, and v_2^{++}, otherwise we can easily construct a cycle C_1 of length at least $r+1$ in G. Moreover, we can assume that $v_1v_2^{--}$, $v_1v_2^-$, $v_1v_2^+$, $v_1v_2^{++}$, $v_2v_1^{--}$, $v_2v_1^-$, $v_2v_1^+$, $v_1^-v_2^-$, $v_1^+v_2^+$, and $v_2v_1^{++}$ are not in E, otherwise we can easily construct a cycle C_1 of length at least $r+1$ in G.

We first prove the claim that if $N(v_1^-)\cap N(v_2^-)\neq\emptyset$, then we can construct a cycle C_1 of length at least $r+1$ in G.

Since $v_1^-v_2^-\notin E$ and $N(v_1^-)\cap N(v_2^-)\neq\emptyset$, we have $d(v_1^-,v_2^-)=2$. Since G is quasi-claw-free, there exists a vertex, say o, such that $o\in N(v_1^-)\cap N(v_2^-)$ and $N[o]\subseteq N[v_1^-]\cup N[v_2^-]$. If $o\in V(G)-V(C)$, we can easily construct a cycle C_1 of length at least $r+1$ in G. If $o\in V(C)$, then $\{o^-,o^+\}\subseteq N[o]\subseteq N[v_1^-]\cup N[v_2^-]$. When $o\in\overrightarrow{C}[v_2^+,v_1^{--}]$, if $o^-\in N(v_1^-)$ or $o^+\in N(v_2^-)$, we can construct a cycle $C_1=u_1\overrightarrow{C}[v_1,v_2^-]\overrightarrow{C}[o,v_1^-]\overleftarrow{C}[o^-,v_2]u_t...u_2u_1$ or $C_1=u_1\overrightarrow{C}[v_1,v_2^-]\overrightarrow{C}[o^+,v_1^-]\overleftarrow{C}[o,v_2]\ u_t...u_2u_1$ of length at least $r+1$ in G. Now we can assume that $o^-\notin N(v_1^-)$ and $o^+\notin N(v_2^-)$. Thus $o^-\in N(v_2^-)$ and $o^+\in N(v_1^-)$. Since $G[\{v_2^-,o^-,o,o^+,v_1^-\}]$ is not isomorphic to B, we have $o^-o^+\in E$ and so we can construct a cycle $C_1=u_1\overrightarrow{C}[v_1,v_2^-]o\overleftarrow{C}[v_1^-,o^+]\overleftarrow{C}[o^-,v_2]u_t...u_2u_1$ of length at least $r+1$ in G. Similarly, we can construct a cycle C_1 of length at least $r+1$ in G when $o\in\overrightarrow{C}[v_1^+,v_2^{--}]$.

Symmetrically, we can prove the claim that if $N(v_1^+)\cap N(v_2^+)\neq\emptyset$, then we can construct a cycle C_1 of length at least $r+1$ in G.

If $t=1$, since $G[\{v_1^-,v_1^+,v_1,u_1,v_2\}]$ is not isomorphic to B, we have

$v_1v_2 \notin E$. If $v_1^+v_2^- \in E$, then $N(v_1^+) \cap N(v_2^+) \neq \emptyset$. By the claim we just proved we can construct a cycle of C_1 of length $r+1$ in G.

We now consider the case $v_1^+v_2^- \notin E$. Then $H_1 := G[\{v_1^-, v_1^+, v_1, u_1, v_2, v_2^-\}] \cong Z_3$. Thus $N(u_1) \cap N(v_1^-) - V(H_1) \neq \emptyset$ or $N(u_1) \cap N(v_1^+) - V(H_1) \neq \emptyset$ or $N(v_2) \cap N(v_1^-) - V(H_1) \neq \emptyset$ or $N(v_2) \cap N(v_1^+) - V(H_1) \neq \emptyset$ or $(N(v_2^-) \cap N(v_1^-) - V(H_1) \neq \emptyset$ and $N(v_2^-) \cap N(v_1^+) - V(H_1) \neq \emptyset)$.

When $N(u_1) \cap N(v_1^-) - V(H_1) \neq \emptyset$, choose a vertex $p \in N(u_1) \cap N(v_1^-) - V(H_1)$. If $p \in V(G) - V(C)$, then we easily construct a cycle C_1 of length at least $r+1$ in G. If $p \in V(C)$, then we can assume that $p^-p^+ \in E$ otherwise we can apply Lemma 2 to construct a cycle C_1 of length $r+1$ in G. Now we can construct a cycle $C_1 = u_1\overrightarrow{C}[v_1, p^-]\overleftarrow{C}[p^+, v_1^-]pu_1$ of length at least $r+1$ in G. Similarly, we can construct a cycle C_1 of length at least $r+1$ in G when $N(u_1) \cap N(v_1^+) - V(H_1) \neq \emptyset$.

When $N(v_2) \cap N(v_1^-) - V(H_1) \neq \emptyset$, choose a vertex $q \in N(v_2) \cap N(v_1^-) - V(H_1)$. If $q \in V(G) - V(C)$, then we can construct a cycle $C_1 = u_1\overleftarrow{C}[v_1, v_2^-]\ \overleftarrow{C}[v_2^+, v_1^-]qv_2u_1$ of length at least $r+1$ in G. Now we assume that $q \in V(C)$. We can assume that $q \notin N(v_2^-)$ otherwise by the claim we proved above we can construct a cycle of length at least $r+1$ in G. We can further assume that $q \notin N(u_1)$ otherwise we can apply Lemma 2 to construct a cycle C_1 of length $r+1$ in G when $q^-q^+ \notin E$ or $C_1 = u_1\overrightarrow{C}[v_1, q^-]\overleftarrow{C}[q^+, v_1^-]pu_1$ of length at least $r+1$ in G when $q^-q^+ \in E$. Notice that $d(u_1, v_2^-) = 2$. Since G is quasi-claw-free, there exists a vertex, say w, such that $w \in N(u_1) \cap N(v_2^-)$ and $N[w] \subseteq N[u_1] \cup N[v_2^-]$. Since $q \in N[v_2]$ but $q \notin N[u_1] \cup N[v_2^-]$, we have $w \neq v_2$. Using a construction which is similar to the one for the case $N(u_1) \cap N(v_1^-) - V(H_1) \neq \emptyset$ above, we can construct a cycle C_1 of length at least $r+1$ in G. Similarly, we can construct a cycle C_1 of length at least $r+1$ in G when $N(v_2) \cap N(v_1^+) - V(H_1) \neq \emptyset$

When $N(v_2^-) \cap N(v_1^-) - V(H_1) \neq \emptyset$, using the claim we proved above, we can construct a cycle C_1 of length at least $r+1$ in G.

If $t \geq 2$, since $P = v_1u_1u_2...u_tv_2$ is a shortest path between v_1 and v_2 in G_1, we have $u_1v_2 \notin G$.

If $t = 2$ and $v_1v_2 \in E$, since $d(u_1, v_1^-) = 2$ and G is quasi-claw-free, there exists a vertex, say w, such that $w \in N(u_1) \cap N(v_1^-)$ and $N[w] \subseteq N[u_1] \cup N[v_1^-]$. Notice that $v_2 \in N[v_1]$ but $v_2 \notin N[u_1] \cup N[v_1^-]$, we have $w \neq v_1$. Using a construction which is similar to the one for the case

$t = 1$ and $N(u_1) \cap N(v_1^-) - V(H_1) \neq \emptyset$ above, we can construct a cycle C_1 of length at least $r+1$ in G.

If $t = 2$ and $v_1 v_2 \notin E$, then $H_2 := G[\{v_1^-, v_1^+, v_1, u_1, u_2, v_2\}] \cong Z_3$. Thus $N(u_1) \cap N(v_1^-) - V(H_2) \neq \emptyset$ or $N(u_1) \cap N(v_1^+) - V(H_2) \neq \emptyset$ or $N(u_2) \cap N(v_1^-) - V(H_2) \neq \emptyset$ or $N(u_2) \cap N(v_1^+) - V(H_2) \neq \emptyset$ or $(N(v_2) \cap N(v_1^-) - V(H_2) \neq \emptyset$ and $N(v_2) \cap N(v_1^+) - V(H_2) \neq \emptyset)$.

When $N(u_1) \cap N(v_1^-) - V(H_2) \neq \emptyset$ or $N(u_1) \cap N(v_1^+) - V(H_2) \neq \emptyset$ or $N(u_2) \cap N(v_1^-) - V(H_2) \neq \emptyset$ or $N(u_2) \cap N(v_1^+) - V(H_2) \neq \emptyset$, we can easily construct a cycle C_1 of length at least $r+1$ in G. Using a construction which is similar to the one for the case when $t = 1$ and $N(v_2) \cap N(v_1^-) - V(H_1) \neq \emptyset$, we can construct a cycle C_1 of length at least $r+1$ in G when $N(v_2) \cap N(v_1^-) - V(H_2) \neq \emptyset$ and $N(v_2) \cap N(v_1^+) - V(H_2) \neq \emptyset$.

If $t \geq 3$, then $H_3 := G[\{v_1^-, v_1^+, v_1, u_1, u_2, u_3\}] \cong Z_3$. Thus $N(u_1) \cap N(v_1^-) - V(H_3) \neq \emptyset$ or $N(u_1) \cap N(v_1^+) - V(H_3) \neq \emptyset$ or $N(u_2) \cap N(v_1^-) - V(H_3) \neq \emptyset$ or $N(u_2) \cap N(v_1^+) - V(H_3) \neq \emptyset$ or $(N(u_3) \cap N(v_1^-) - V(H_3) \neq \emptyset$ and $N(u_3) \cap N(v_1^+) - V(H_3) \neq \emptyset)$. Using or slightly modifying the constructions in the cases $t = 1$ and $t = 2$, we can construct a cycle C_1 of length at least $r+1$ in G .

Since the algorithmic procedures in Lemma 1 and Lemma 2, the depth-first search algorithm for finding a component H in $G[V(G) - V(C)]$, and the breadth-first search algorithm for finding the shortest path between v_1 and v_2 in G_1 all can be completed in polynomial time, the step of enlarging the cycle C of length r in G to a cycle C_1 of length at least $r+1$ can be fulfilled in polynomial time.

Apply the similar procedure as above to the cycle C_1 of length at least $r+1$ in G, we can construct a cycle longer than C_1 in G. Repeat this process, we can construct a Hamiltonian cycle in G. Notice that we can repeat the processes at most $|V(G)|$ times, therefore we can find a Hamiltonian cycle in the graph G in polynomial time. QED

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