A Neighborhood Condition for Graphs to Have (g, f)-Factors *

Hongxia Liu[a,b]

[a]School of Mathematics, Shandong University

Jinan, Shandong 250100, P. R. China

[b]School of Mathematics and Informational Science, Yantai University

Yantai, Shandong 264005, P. R. China

e-mail: mqy7174@sina.com

Guizhen Liu[†]

School of Mathematics, Shandong University

Jinan, Shandong 250100, P. R. China

e-mail: gzliu@sdu.edu.cn

Abstract

Let G be a graph of order n. Let a and b be integers with $1 \leq a < b$, and let $k \geq 2$ be a positive integer not larger than the independence number of G. Let $g(x)$ and $f(x)$ be two non-negative integer-valued functions defined on $V(G)$ such that $a \leq g(x) < f(x) \leq b$ for each $x \in V(G)$. Then G has a (g, f)-factor if the minimum degree $\delta(G) \geq \frac{b(b-1)(k-1)}{a+1}$, $n > \frac{(a+b)(k(a+b)-2)}{a+1}$ and $|N_G(x_1) \cup N_G(x_2) \cup \cdots \cup N_G(x_k)| \geq \frac{(b-1)n}{a+b}$ for any independent subset $\{x_1, x_2, ..., x_k\}$ of $V(G)$. Furthermore, we show that the result is best possible in some sense.

Keywords: graph, factor, (g, f)-factor, neighborhood union

Mathematics Subject Classification 2000: 05C70

*This work is supported by NSFC(10871119), SRFDP(200804220001) of China and Yantai University Youth Fundation(sx07211).

†Corresponding author: Guizhen Liu, e-mail: gzliu@sdu.edu.cn

1. Introduction

The graphs considered in this paper will be finite and undirected graphs without loops and multiple edges. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. For $x \in V(G)$, we denote by $d_G(x)$ the degree of x in G and by $N_G(x)$ the set of vertices adjacent to x in G. we use $N_G[x]$ to denote $N_G(x) \cup \{x\}$ and $\delta(G)$ to denote the minimum degree of G. For a subset $S \subseteq V(G)$, we denote by $N_G(S)$ the union of $N_G(x)$ for every $x \in S$, by $G[S]$ the subgraph of G induced by S, by $G - S$ the subgraph obtained from G by deleting the vertices in S together with the edges incident to the vertices in S. We write $d(S) = \sum_{v \in S} d(v)$.

Let $g(x)$ and $f(x)$ be two nonnegative integer-valued functions defined on $V(G)$ with $g(x) \leq f(x)$ for any $x \in V(G)$. Then a spanning subgraph F of G is called a (g, f)-factor if $g(x) \leq d_F(x) \leq f(x)$ holds for any $x \in V(G)$. A (g, f)-factor is called an $[a, b]$-factor if $g(x) \equiv a$ and $f(x) \equiv b$. An $[a, b]$-factor is called a k-factor if $a = b = k$. The other terminologies and notations may be found in [1].

Many authors have investigated graph factors[4,7] and (g, f)-factors[5,9,10] In [8], T. Nishimura gave the following result.

Theorem 1 [8]. *Let $k \geq 3$ be an integer and G be a connected graph of order n with $n \geq 4k - 3$, kn even, and $\delta(G) \geq k$. If for each pair of nonadjacent vertices x, y of $V(G)$*

$$\max\{d_G(x), d_G(y)\} \geq \frac{n}{2},$$

then G has a k-factor.

In [3], Yanjun Li and Maocheng Cai extended Theorem 1 to $[a, b]$- factors.

Theorem 2 [3]. *Let G be a graph of order n, and let a and b be integers such that $1 \leq a < b$. Then G has an $[a, b]$-factor if $\delta(G) \geq a$, $n \geq 2a + b + \frac{a^2 - a}{b}$ and*

$$\max\{d_G(x), d_G(y)\} \geq \frac{an}{a + b}$$

for any two nonadjacent vertices x and y in G.

In [7], Haruhide Matsuda gave a sufficient condition in terms of neighborhood union for the existence of $[a, b]$-factors.

Theorem 3 [7]. *Let a and b be integers such that $1 \leq a < b$, and let*

G be a graph of order n with $n \geq \frac{2(a+b)(a+b-1)}{b}$, *and* $\delta(G) \geq a$. *If*

$$|N_G(x) \cup N_G(y)| \geq \frac{an}{a+b}$$

for any two nonadjacent vertices x and y of G , then G has an $[a, b]$*-factor.*

In [2], Jianxiang Li proved the following theorem, which is an extension of Theorem 3.

Theorem 4 [2]. *Let a and b be integers such that* $1 \leq a < b$, *and let G be a graph of order n with* $n \geq \frac{(a+b)(k(a+b)-2)}{b}$. *If* $\delta(G) \geq (k-1)a$, *and*

$$|N_G(x_1) \cup N_G(x_2) \cup \cdots \cup N_G(x_k)| \geq \frac{an}{a+b}$$

for any independent subset $\{x_1, x_2, ...x_k\}$ *of* $V(G)$, *where* $k \geq 2$, *then G has an* $[a, b]$*-factor.*

It is easy to see that Theorem 3 is a special case of Theorem 4 for $k = 2$.

We extend Theorem 4 to (g, f)-factors and obtain the following result.

Theorem 5. *Let G be a graph of order n, and let a and b be integers with* $1 \leq a < b$. *Let* $g(x)$ *and* $f(x)$ *be two nonnegative integer-valued functions defined on* $V(G)$ *such that* $a \leq g(x) < f(x) \leq b$ *for each* $x \in V(G)$. *Let* $k \geq 2$ *be a positive integer not larger than the independence number of G. Then G has a* (g, f)*-factor if* $\delta(G) \geq \frac{b(b-1)(k-1)}{a+1}$, $n > \frac{(a+b)(k(a+b)-2)}{a+1}$ *and*

$$|N_G(x_1) \cup N_G(x_2) \cup \cdots \cup N_G(x_k)| \geq \frac{(b-1)n}{a+b} \tag{1}$$

for any independent subset $\{x_1, x_2, ...x_k\}$ *of* $V(G)$.

2. Proof of Theorem 5

We use the following lemma in our proof, which is a special case of Lovász's (g, f)-factor theorem.

lemma 1 [6]. *Let G be a graph, and let* $g(x)$ *and* $f(x)$ *be two non-negative integer-valued functions defined on* $V(G)$ *such that* $g(x) < f(x)$ *for each* $x \in V(G)$. *Then G has a* (g, f)*-factor if and only if*

$$\delta_G(S, T) = f(S) + d_{G-S}(T) - g(T) \geq 0$$

for all disjoint subsets S and T of $V(G)$.

Proof of Theorem 5. We use the similar method in [7] to prove Theorem 5. Suppose that G satisfies the condition of Theorem 5, but has no (g, f)-factor. Then, by Lemma 1, there exist two disjoint subsets S and T of $V(G)$ such that

$$\delta_G(S,T) = f(S) + d_{G-S}(T) - g(T) \leq -1. \quad (2)$$

We choose such subsets S and T which satisfy $|T|$ is minimum.

We first prove the following claims.

Claim 1. $d_{G-S}(x) < g(x) \leq b - 1$ for all $x \in T$.

Proof. If $d_{G-S}(x) \geq g(x)$ for some $x \in T$, then the subsets S and $T \setminus \{x\}$ satisfy (2). This contradicts the choice of S and T. Therefore,

$$d_{G-S}(x) < g(x) \leq b - 1$$

for all $x \in T$ holds. □

Claim 2. $|T| \geq a + 2$.

Proof. Assume $|T| \leq a + 1$. By (2) and $|S| + d_{G-S}(x) \geq d_G(x) \geq \delta(G) \geq \frac{b(b-1)(k-1)}{a+1} \geq b - 1$ (since $k \geq 2$ and $b \geq a + 1$) for all $x \in T$, we obtain

$$\begin{aligned} -1 &\geq \delta_G(S,T) = f(S) + d_{G-S}(T) - g(T) \\ &\geq (a+1)|S| + d_{G-S}(T) - (b-1)|T| \\ &\geq |T||S| + d_{G-S}(T) - (b-1)|T| \\ &= \textstyle\sum_{x\in T}(|S| + d_{G-S}(x) - (b-1)) \geq 0, \end{aligned}$$

which is a contradiction. Thus $|T| \geq a + 2$. □

Since $T \neq \phi$, in the following we shall construct a sequence $x_1, x_2, ...x_\pi$ of vertices of T. Let

$$h_1 = \min\{d_{G-S}(x) | x \in T\}$$

and choose $x_1 \in T$ to be a vertex such that $d_{G-S}(x_1) = h_1$. By Claim 1, we have $h_1 < g(x) \leq b - 1$.

If $j \geq 2$ and $T \setminus (\bigcup_{i=1}^{j-1} N_T[x_i]) \neq \phi$, let

$$h_j = \min\left\{d_{G-S}(x) | x \in T \setminus \left(\bigcup_{i=1}^{j-1} N_T[x_i]\right)\right\}$$

and choose $x_j \in T \setminus (\bigcup_{i=1}^{j-1} N_T[x_i])$ to be a vertex such that $d_{G-S}(x_j) = h_j$. This defines a nondecreasing sequence of integers $0 \leq h_1 \leq h_2 \leq \cdots \leq$

$h_\pi < g(x) \le b-1$ and a sequence of independent vertices $x_1, x_2, ..., x_\pi$ in T with $d_{G-S}(x_i) = h_i$ $(1 \le i \le \pi)$ and $T \setminus (\bigcup_{i=1}^{\pi} N_T[x_i]) = \phi$.

Claim 3. $|T| \ge (k-1)b+1$.

Proof. Suppose that $|T| \le (k-1)b$. Since $|S| + h_1 \ge d_G(x_1) \ge \delta(G) \ge \frac{b(b-1)(k-1)}{a+1}$, by (2) and $1 \le h_1 < b-1$, it follows that

$$\begin{aligned}
-1 &\ge f(S) + d_{G-S}(T) - g(T) \\
&\ge (a+1)|S| + h_1|T| - (b-1)|T| \\
&= (a+1)|S| + (h_1 - b + 1)|T| \\
&\ge (a+1)(\tfrac{b(b-1)(k-1)}{a+1} - h_1) + (h_1 - b + 1)(k-1)b \\
&= b(b-1)(k-1) - h_1(a+1) + h_1(k-1)b + b(1-b)(k-1) \\
&= b(b-1)(k-1) + b(1-b)(k-1) + h_1(b(k-1) - a - 1) \\
&\ge b(b-1)(k-1) + b(1-b)(k-1) \\
&= 0.
\end{aligned}$$

This is a contradiction. Thus we have $|T| \ge (k-1)b+1$. □

Since $|N_T[x_i]| = h_i + 1 \le d_{G-S}(x_i) + 1 \le b-1$(by Claim 1) and $|T| \ge (k-1)b+1 > (k-1)(b-1)+1$, we have $\pi \ge k$ and we can take an independent subset $\{x_1, x_2, ..., x_k\} \subseteq T$.

By the assumption of the theorem we can get the following inequalities:

$$\frac{(b-1)n}{a+b} \le |N_G(x_1) \cup N_G(x_2) \cup \cdots \cup N_G(x_k)| \le |S| + \sum_{i=1}^{k} h_i.$$

It follows that

$$|S| \ge \frac{(b-1)n}{a+b} - \sum_{i=1}^{k} h_i. \tag{3}$$

Since $n-|S|-|T| \ge 0$ and $b-1-h_k \ge 1$, we have $(n-|S|-|T|)(b-1-h_k) \ge 0$. Note that

$$N_T[x_i] \setminus \bigcup_{j=1}^{i-1} N_T[x_j] \ne \emptyset, i = 2, 3, ..., k-1$$

and

$$\left|\bigcup_{j=1}^{i} N_T[x_j]\right| \le \sum_{j=1}^{i} |N_T[x_j]| \le \sum_{j=1}^{i}(d_{G-S}(x_j)+1) = \sum_{j=1}^{i}(h_j+1), i = 1, 2, ..., k.$$

Therefore we have

$$(n-|S|-|T|)(b-1-h_k)$$

$$
\begin{aligned}
&\geq 0 \geq -1 \\
&\geq f(S) + d_{G-S}(T) - g(T) + 1 \\
&\geq (a+1)|S| + d_{G-S}(T) - (b-1)|T| + 1 \\
&= (a+1)|S| + h_1|N_T[x_1]| + h_2(|N_T[x_2] \setminus N_T[x_1]|) + \cdots \\
&\quad + h_{k-1}\left(|N_T[x_{k-1}] \setminus \bigcup_{i=1}^{k-2} N_T[x_i]|\right) \\
&\quad + h_k\left(|T| - \left|\bigcup_{i=1}^{k-1} N_T[x_i]\right|\right) - (b-1)|T| + 1 \\
&\geq (a+1)|S| + (h_1 - h_k)|N_T[x_1]| + \sum_{i=2}^{k-1} h_i + (h_k - b + 1)|T| \\
&\quad - h_k \sum_{i=2}^{k-1} |N_T[x_i]| + 1 \\
&= (a+1)|S| + (h_1 - h_k)(h_1 + 1) + \sum_{i=2}^{k-1} h_i + (h_k - b + 1)|T| \\
&\quad - h_k \sum_{i=2}^{k-1} (h_i + 1) + 1 \\
&= (a+1)|S| + h_1^2 + \sum_{i=1}^{k-1} h_i + (h_k - b + 1)|T| - h_k \sum_{i=1}^{k-1} (h_i + 1) + 1.
\end{aligned}
$$

Thus it follows that

$$
0 \leq n(b-1-h_k) - (a+b-h_k)|S| + h_k \sum_{i=1}^{k-1} h_i - \sum_{i=1}^{k-1} h_i + h_k(k-1) - h_1^2 - 1. \quad (4)
$$

By (3) and (4), $h_1 \leq h_2 \leq \cdots \leq h_k \leq b-2$, and $n > \frac{(a+b)(k(a+b)-2))}{a+1}$, we have

$$
\begin{aligned}
0 \leq\ & n(b-1-h_k) - (a+b-h_k)\left(\frac{(b-1)n}{a+b} - \sum_{i=1}^{k} h_i\right) + h_k \sum_{i=1}^{k-1} h_i \\
& - \sum_{i=1}^{k-1} h_i + h_k(k-1) - h_1^2 - 1
\end{aligned}
$$

$$
\begin{aligned}
&= -\frac{(a+1)n}{a+b}h_k + (a+b)\sum_{i=1}^{k} h_i - h_k\sum_{i=1}^{k} h_i + h_k\sum_{i=1}^{k-1} h_i - \sum_{i=1}^{k-1} h_i \\
&\quad + h_k(k-1) - h_1^2 - 1 \\
&= -\frac{(a+1)n}{a+b}h_k + ((a+b-1)h_1 - h_1^2) + (a+b-1)\sum_{i=2}^{k-1} h_i \\
&\quad + h_k(a+b+k-1) - h_k^2 - 1 \\
&\le -\frac{(a+1)n}{a+b}h_k + (a+b-1)h_k + (a+b-1)\sum_{i=2}^{k-1} h_k \\
&\quad + h_k(a+b+k-1) - h_k^2 - 1 \\
&= -\frac{(a+1)n}{a+b}h_k + k(a+b)h_k - h_k^2 - 1.
\end{aligned}
$$

If $h_k > 0$, then $0 < 2h_k - h_k^2 - 1 \le 0$ (since $n > \frac{(a+b)(k(a+b)-2)}{a+1}$), that is a contradiction. If $h_k = 0$, then $0 \le -1$, a contradiction. So we conclude that G has a (g, f)-factor.

Remark 1. By the following example we can show that the condition (1) is best possible in the sense that it can not be replaced by $|N_G(x_1) \cup N_G(x_2) \cup \cdots \cup N_G(x_k)| \ge \frac{(b-1)n}{a+b} - 1$.

We let $G_1 = K_{(b-1)t}$ be a complete graph and $G_2 = ((a+1)t+1)K_1$ be $(a+1)t+1$ independent vertices. Then let $G = G_1 + G_2$ be the join of G_1 and G_2(that is, $V(G) = V(G_1) \cup V(G_2)$, $E(G) = E(G_1) \cup E(G_2) \cup \{uv : u \in V(G_1), v \in V(G_2)\}$. Thus we have $|V(G_1)| = (b-1)t$ and $|V(G_2)| = (a+1)t+1$, where t is a sufficiently large positive integer(For some k, we choose $t \ge \frac{k(a+b)-2}{a+1} + \frac{a+b-1}{a+b}$, thus the conditions $\delta(G) \ge \frac{b(b-1)(k-1)}{a+1}$ and $n > \frac{(a+b)(k(a+b)-2)}{a+1}$ suffice). Then it follows that $n = |V(G_1)| + |V(G_2)| = (a+b)t + 1$ and

$$\frac{(b-1)n}{a+b} > |N_G(x_1) \cup N_G(x_2) \cup \cdots \cup N_G(x_k)| = (b-1)t > \frac{(b-1)n}{a+b} - 1$$

for any independent subset $\{x_1, x_2, ... x_k\}$ of $V(G_2)$. We take $S = V(G_1)$, $g(x) = a$ and $f(x) = a+1$ for $x \in V(G_1)$; $T = V(G_2)$, $g(x) = b-1$ and $f(x) = b$ for $x \in V(G_2)$. It is easy to see that G has no (g, f)-factors because $\delta_G(S,T) = f(S) - g(T) = (a+1)|V(G_1)| - (b-1)|V(G_2)| = (a+1)(b-1)t - (b-1)((a+1)t+1) = -(b-1) < 0$.

Remark 2. We can see that the minimum degree bound in Theorem 5 $\delta(G) \ge \frac{b(b-1)(k-1)}{a+1}$ is best possible when $b = a+1$ and $k = 2$.

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Hongxia Liu is now doing Ph.D. on operational research under the direction of Professor Guizhen Liu.
Guizhen Liu is Professor and Ph.D. Advisor at Shandong University of China. More than 180 research papers have published in national and international leading journals. Her current research interests are graph theory, matroid theory and operational research.