A note on the characterization of potentially $K_{1,1,s}$–graphic sequences *

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Abstract: For given a graph H, a graphic sequence $\pi = (d_1, d_2, \ldots, d_n)$ is said to be potentially H-graphic if there is a realization of π containing H as a subgraph. In this paper, we characterize potentially $K_{1,1,6}$-positive graphic sequences. This characterization implies the value of $\sigma(K_{1,1,6}, n)$. Moreover, we also give a simple sufficient condition for a positive graphic sequence $\pi = (d_1, d_2, \ldots, d_n)$ to be potentially $K_{1,1,s}$–graphic for $n \geq s+2$ and $s \geq 2$.
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1. Introduction

The set of all sequences $\pi = (d_1, d_2, \ldots, d_n)$ of non-negative, non-increasing integers with $d_1 \leq n-1$ is denoted by NS_n. A sequence $\pi \in NS_n$ is said to be *graphic* if it is the degree sequence of a simple graph G on n vertices, and such a graph G is called a *realization* of π. The set of all graphic sequences in NS_n is denoted by GS_n. If each term of a graphic sequence π is nonzero, then π is said to be *positive graphic.* For a sequence $\pi = (d_1, d_2, \ldots, d_n) \in NS_n$, denote $\sigma(\pi) = d_1 + d_2 + \cdots + d_n$. For given a graph H, a sequence $\pi \in GS_n$ is said to be *potentially H-graphic*, if there is a realization of π containing H as a subgraph. Yin and Chen [9] characterized potentially $K_{2,3}$-graphic sequences and potentially $K_{2,4}$-graphic sequences, where $K_{r,s}$ is the $r \times s$ complete bipartite graph. Hu and Lai [6] characterized potentially $K_{3,3}$-graphic sequences. Yin and Li [11] gave

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two sufficient conditions for $\pi \in GS_n$ to be potentially $K_{1,1,\ldots,1,2}$-graphic, where $K_{1,1,\ldots,1,2}$ is the $r_1 \times r_2 \times \cdots \times r_t$ complete t-partite graph with $r_1 = r_2 = \cdots = r_{t-1} = 1$ and $r_t = 2$. Eschen and Niu [3] characterized potentially $K_{1,1,2}$-graphic sequences. Hu and Lai [5] characterized potentially $K_{1,1,3}$-graphic sequences.In the following, the symbol x^y in a sequence stands for y consecutive terms, each equal to x.

Theorem 1.1 [3] Let $n \geq 4$ and $\pi = (d_1, d_2, \ldots, d_n) \in GS_n$ be a positive sequence. Then π is potentially $K_{1,1,2}$-graphic if and only if the conditions hold (1) $d_1 \geq d_2 \geq 3$ and $d_4 \geq 2$, (2) $\pi \neq (3^6), (3^2, 2^4), (3^2, 2^3)$.

Theorem 1.2 [5] Let $\pi = (d_1, d_2, \ldots, d_n)$ be a graphic sequence with $n \geq 5$. Then π is potentially $K_{1,1,3}$-graphic if and only if the conditions hold (1) $d_2 \geq 4$ and $d_5 \geq 2$, (2) $\pi \neq (4^2, 2^4), (4^2, 2^5), (4^3, 2^3)$ and (4^6).

Recently, Yin et al. [12] characterized potentially $K_{1,1,4}-$graphic sequences and potentially $K_{1,1,5}-$graphic sequences.

In [4], Gould et al. posed an extremal problem on potentially H-graphic sequences as follows: determine the smallest even integer $\sigma(H, n)$ such that every positive sequence $\pi \in GS_n$ with $\sigma(\pi) \geq \sigma(H, n)$ is potentially H-graphic.

In this paper, we first characterize potentially $K_{1,1,6}$-positive graphic sequences. That is the following.

Theorem 1.3 Let $n \geq 6$ and $\pi = (d_1, d_2, \ldots, d_n) \in GS_n$ be a positive sequence. Then π is potentially $K_{1,1,6}$-graphic if and only if π satisfies the following conditions: (1) $d_2 \geq 7$ and $d_8 \geq 2$, (2) π is not one of the following sequences

$n = 9 : (7^2, 2^7), (7^2, 4, 2^6), (7^2, 5, 3, 2^5), (7^2, 6, 3^2, 2^4), (7^2, 6, 4^2, 2^4), (7^3, 6, 3^5), (7^4, 4, 3^4), (7^3, 5^3, 2^3), (7^3, 5, 4^2, 2^3)$,

$n = 10 : (7^2, 2^8), (8, 7, 5, 2^7), (7^2, 6, 2^7), (7^3, 3, 2^6), (8, 7, 6, 3, 2^6), (7^2, 5, 2^6, 1), (7^2, 6, 3, 2^5, 1), (7^2, 6, 4, 2^6), (7^3, 4, 3, 2^5), (8, 7^2, 3^2, 2^5), (8, 7^2, 4^2, 2^5), (7^3, 3^2, 2^4, 1), (7^3, 4^2, 2^4, 1), (7^4, 3^5, 1), (7^3, 5, 4, 2^5)$,

$n = 11 : (8^2, 6, 2^8), (8, 7, 6, 2^7, 1), (7^2, 6, 2^8), (9, 7, 6, 2^8), (8^2, 7, 3, 2^7), (8, 7^2, 2^8), (8, 7^2, 4, 2^7), (8, 7^2, 3, 2^6, 1), (9, 7^2, 3, 2^7), (7^2, 6, 2^6, 1^2), (7^3, 2^7, 1), (7^3, 3, 2^5, 1^2), (7^3, 4, 2^6, 1), (7^3, 3, 2^7), (7^3, 5, 2^7), (8^3, 3^2, 2^6), (8^3, 4^2, 2^6)$,

$n = 12 : (8^3, 2^9), (9, 8, 7, 2^9), (9, 7^2, 2^8, 1), (8^2, 7, 2^8, 1), (8, 7^2, 2^7, 1^2), (8, 7^2, 2^9), (10, 7^2, 2^9), (9, 8^2, 3, 2^8), (7^3, 2^8, 1), (8^3, 3, 2^7, 1), (7^3, 2^6, 1^3), (8^3, 4, 2^8)$,

$n = 13 : (9^2, 8, 2^{10}), (9, 8^2, 2^9, 1), (10, 8^2, 2^{10}), (8^3, 2^8, 1^2), (8^3, 2^{10}), (9^3, 3, 2^9)$

$n = 14 : (10, 9^2, 2^{11}), (9^3, 2^{10}, 1)$,

$n = 15 : (10^3, 2^{12})$.

As an application of the characterization, it is straightforward to find the value of $\sigma(K_{1,1,6}, n)$. Moreover, we also give a sufficient condition for a positive graphic sequence π to be potentially $K_{1,1,s}-$graphic.

Theorem 1.4 Let $\pi = (d_1, d_2, \ldots, d_n) \in GS_n$ be a positive sequence with $n \geq s + 2$ and $s \geq 2$. If $\pi \neq (3^6)$ for $s = 2$, $\pi \neq ((s+1)^{s+3})$(s is odd) and $d_{s+2} \geq s + 1$, then π is potentially $K_{1,1,s}$-graphic.

2. Proofs of Theorem 1.3 and Theorem 1.4

In order to prove Theorem 1.3 and Theorem 1.4, we also need the following known results.

Let $\pi = (d_1, d_2, \ldots, d_n) \in NS_n$ and $1 \le k \le n$. Let

$$\pi''_k = \begin{cases} (d_1 - 1, \ldots, d_{k-1} - 1, d_{k+1} - 1, \ldots, d_{d_k+1} - 1, d_{d_k+2}, \ldots, d_n), \\ \text{if } d_k \ge k, \\ (d_1 - 1, \ldots, d_{d_k} - 1, d_{d_k+1}, \ldots, d_{k-1}, d_{k+1}, \ldots, d_n), \\ \text{if } d_k < k. \end{cases}$$

Let $\pi'_k = (d'_1, d'_2, \ldots, d'_{n-1})$, where $d'_1 \ge \cdots \ge d'_{n-1}$ is the rearrangement in non-increasing order of the $n-1$ terms of π''_k. π'_k is called the *residual sequence* obtained by laying off d_k from π.

Theorem 2.1 [7] Let $\pi = (d_1, d_2, \ldots, d_n) \in NS_n$ and $1 \le k \le n$. Then $\pi \in GS_n$ if and only if $\pi'_k \in GS_{n-1}$.

Theorem 2.2 [2] Let $\pi = (d_1, d_2, \ldots, d_n)$ be a non-increasing sequence of nonnegative integer with even $\sigma(\pi)$. Then $\pi \in GS_n$ if and only if for any $t, 1 \le t \le n-1$,

$$\sum_{i=1}^{t} d_i \le t(t-1) + \sum_{i=t+1}^{n} min\{t, d_i\}$$

Theorem 2.3 [10] Let $\pi = (d_1, d_2, \ldots, d_n) \in NS_n$, $d_1 = m$ and $\sigma(\pi)$ be even. If there exists an integer $n_1, n_1 \le n$ such that $d_{n_1} \ge h \ge 1$ and $n_1 \ge \frac{1}{h}\left\lfloor \frac{(m+h+1)^2}{4} \right\rfloor$, then $\pi \in GS_n$.

Theorem 2.4 [8] Let $\pi = (d_1, d_2, \ldots, d_n) \in NS_n$ and $\sigma(\pi)$ be even. If $d_1 - d_n \le 1$ and $d_1 \le n-1$, then $\pi \in GS_n$.

Theorem 2.5 [4] If $\pi = (d_1, d_2, \ldots, d_n) \in GS_n$ has a realization G containing H as a subgraph, then there exists a realization G' of π containing H as a subgraph so that the vertices of H have the largest degrees of π.

Theorem 2.6 [10] Let $\pi = (d_1, \ldots, d_r, d_{r+1}, \ldots, d_{r+s}, d_{r+s+1}, \ldots, d_n) \in GS_n$, where $d_r \ge r+s-1$ and $d_n \ge r$. If $n \ge (r+2)(s-1)$, then π is potentially $K_{r,s}-$graphic.

Theorem 2.7 [1] Let $\pi = (d_1, d_2, \ldots, d_n) \in GS_n$. If π is potentially $K_{2,s+2}-$graphic, then π is potentially $K_{1,1,s}-$graphic.

Theorem 2.8 [11] Let $\pi = (d_1, d_2, \ldots, d_n) \in NS_n$, where $d_1 = r$ and $\sigma(\pi)$ is even. If $d_{r+1} \ge r-1$, then π is graphic.

Theorem 2.9 [10] Let $\pi = (d_1, \ldots, d_r, d_{r+1}, \ldots, d_{r+s}, d_{r+s+1}, \ldots, d_n) \in GS_n$, where $d_{r+s} \ge r+s-1$ and $d_n \ge r$. Then π is potentially $K_{r,s}-$graphic.

In order to prove our main result, we need the following definition.

Let $n \ge s+2$ and $\pi = (d_1, d_2, \ldots, d_n) \in NS_n$ with $d_2 \ge s+1, d_{s+2} \ge 2$ and $d_n \ge 1$. We define sequences π_0, π_1 and π_2 as follows. Let $\pi_0 = \pi$. Let

$$\pi_1 = (d_2 - 1, \ldots, d_{s+2} - 1, d^{(1)}_{s+3}, \ldots, d^{(1)}_n),$$

where $d^{(1)}_{s+3} \ge \cdots \ge d^{(1)}_n$ is the rearrangement in non-increasing order of

$d_{s+3}-1,\ldots,d_{d_1+1}-1,d_{d_1+2},\ldots,d_n$.

$$\pi_2=(d_3-2,\ldots,d_{s+2}-2,d^{(2)}_{s+3},\ldots,d^{(2)}_n),$$

where $d^{(2)}_{s+3},\ldots,d^{(2)}_n$ is the rearrangement in non-increasing order of $d^{(1)}_{s+3}-1,\ldots,d^{(1)}_{d_2+1}-1,d^{(1)}_{d_2+2},\ldots,d^{(1)}_n$.

The following lemma is obvious from the definition of π_2.

Lemma 2.1 Let $n\ge s+2$ and $\pi=(d_1,d_2,\ldots,d_n)\in GS_n$ with $d_2\ge s+1,d_{s+2}\ge 2$ and $d_n\ge 1$. If π_2 is graphic, then π is potentially $K_{1,1,s}$-graphic.

Lemma 2.2 Let $n\ge s+2$ and $\pi=(d_1,d_2,\ldots,d_n)\in GS_n$ with $d_2\ge s+1,d_{s+2}\ge 2$ and $d_n\ge 1$. If $d_1=n-1$ or there exists an integer $t,s+2\le t\le d_1+1$ such that $d_t>d_{t+1}$, then π_2 is graphic. Hence, π is potentially $K_{1,1,s}$-graphic.

Proof. Since there exists an integer $t,s+2\le t\le d_1+1$ such that $d_t>d_{t+1}$, the residual sequence $\pi'_1=(d'_1,d'_2,\ldots,d'_{n-1})$ obtained by laying off d_1 from π satisfies $d'_i=d_{i+1}-1$ for $i=1,2,\ldots,t-1$. If $d_1=n-1$, then the residual sequence $\pi'_1=(d'_1,d'_2,\ldots,d'_{n-1})$ satisfies $d'_i=d_{i+1}-1$ for $i=1,2,\ldots,n-1$. Since π is graphic, π'_1 is graphic by Theorem 2.1. Thereby, $\pi''_{11}=(d'_2-1,\ldots,d'_{d'_1+1}-1,d'_{d'_1+2},\ldots,d'_{n-1})$ is graphic. It is easy to see that π_2 is graphic. □

Lemma 2.3 Let $\pi=(3^x,2^y,1^z)$, where $x+y+z=n\ge 1$ and $\sigma(\pi)$ is even. Then $\pi\in GS_n$ if and only if $\pi\notin S$, where $S=\{(2),(2^2),(3,1),(3^2),(3,2,1),(3^2,2),(3^3,1),(3^2,1^2)\}$.

Proof. For $n=1$, since $\sigma(\pi)$ is even, π must be (2), which belongs to S. For $n\ge 2$, we consider the following cases.

Case 1. $n=2$. Then π is one of the sequences $(3,1),(2^2),(3^2),(1^2)$. It is easy to check that only one sequence (1^2) is graphic.

Case 2. $n=3$. Since $\sigma(\pi)$ is even, π must be one of the sequences $(3,2,1),(3^2,2),(2^3),(2,1^2)$. The sequences (2^3) and $(2,1^2)$ are graphic.

Case 3. $n=4$. Then π is one of the sequences $(3^3,1),(3,1^3),(3^4),(2^4),(3,2^2,1),(2^2,1^2),(3^2,2^2),(1^4),(3^2,1^2)$. Only $(3^2,1^2)$ and $(3^3,1)$ are excepted.

Case 4. $n=5$. Then π must be one of the graphic sequences $(2,1^4),(3,2,1^3),(3^2,2,1^2),(3^3,2,1),(3,2^3,1),(2^5),(3^2,2^3),(2^3,1^2),(3^4,2)$.

Case 5. $n\ge 6$. If $x>0$ and $z>0$, then $6=\lfloor\frac{(3+1+1)^2}{4}\rfloor\le n$. Hence, π is graphic from Theorem 2.3. Otherwise, π is graphic by Theorem 2.4. □

Lemma 2.4 Let $\pi=(4^x,3^y,2^z,1^m)$ with even $\sigma(\pi),x+y+z+m=n\ge 1$ and $x\ge 1$. Then $\pi\in GS_n$ if and only if $\pi\notin A$, where $A=\{(4),(4,2),(4^2),(4,1^2),(4,3,1),(4,3^2),(4,2^2),(4^2,2),(4^3),(4,2,1^2),(4,2^3),(4,3,2,1),(4,3^2,2),(4^2,1^2),(4^2,2^2),(4^2,3,1),(4^2,3^2),(4^3,2),(4^4),(4,3^2,1^2),(4,3,1^3),(4^2,2,1^2),(4^2,3,2,1),(4^3,1^2),(4^3,2^2),(4^3,3,1),(4^4,2),(4^2,3,1^3),(4^2,1^4),(4^3,2,1^2),(4^4,1^2),(4^3,1^4)\}$.

Proof. It is easy to see that the sequences of the set A are not graphic. Now we verify the sufficient condition. If $n\le 4$ and $x\ge 1$, then π is not graphic. In other words, the following sequences are not graphic $(4),(4,2)$, $(4^2),(4,1^2),(4,3,1),(4,3^2),(4,2^2),(4^2,2),(4^3),(4,2,1^2),(4,2^3),(4,3,2,1)$, $(4,3^2,2),(4^2,1^2),(4^2,2^2),(4^2,3,1),(4^2,3^2),(4^3,2),(4^4)$. It is enough to consider the following cases. For convenience, denote $\pi=(d_1,d_2,\ldots,d_n)$.

Case 1. $n=5$. Consider the residual sequence π_1' obtained by laying off d_1 from π. If $x=1$ and $\pi_1'\neq(2),(2^2)$, then π_1' is graphic by Lemma 2.3 and so is π. If π_1' is (2) or (2^2), then π is $(4,3^2,1^2)$ or $(4,3,1^3)$, a contradiction. If $x\geq 2$ and $\pi_1'\neq(3,1),(3^2),(3,2,1),(3^2,2),(3^3,1)$ and $(3^2,1^2)$, then π_1' is graphic by Lemma 2.3. If π_1' is one of the sequences $(3,1),(3^2),(3,2,1),(3^2,2),(3^3,1),(3^2,1^2)$, then π is one of the excepted sequences $(4^2,2,1^2),(4^3,1^2),(4^2,3,2,1),(4^3,3,1),(4^4,2),(4^3,2^2)$.

Case 2. $n=6$. If $d_6\geq 2$, then π_1' is graphic by Lemma 2.3. For $d_6=1$ and $x=1$, π_1' is also graphic. If $d_6=1,x\geq 2$ and $\pi_1'\neq(3,1),(3,2,1),(3^3,1),(3^2,1^2)$, then π_1' is graphic by Lemma 2.3. If π_1' is one of the sequences $(3,1),(3,2,1),(3^3,1),(3^2,1^2)$, then π is one of the sequences $(4^2,1^4),(4^2,3,1^3),(4^4,1^2),(4^3,2,1^2)$, which is contradict.

Case 3. $n=7$. If $d_7\geq 2$, then π_1' is graphic by Lemma 2.3. For $d_7=1$ and $x=1$, π_1' is also graphic. If $d_7=1,x\geq 2$ and $d_6\geq 2$, then π_1' is graphic by Lemma 2.3. If $d_7=d_6=1,x\geq 2$ and $\pi_1'\neq(3^2,1^2)$, then π_1' is graphic. If π_1' is $(3^2,1^2)$, then π is $(4^3,1^4)$, a contradiction.

Case 4. $n=8$. By Lemma 2.3, it is easy to check that π_1' is graphic. Hence, π is graphic.

Case 5. $n\geq 9$. For $n\geq 9\geq max\{\frac{1}{2}\lfloor\frac{(4+2+1)^2}{4}\rfloor,\lfloor\frac{(4+1+1)^2}{4}\rfloor\}$, π is graphic by Theorem 2.3 and 2.4. □

Lemma 2.5 Let $n\geq 6$ and $\pi=(d_1,d_2,\ldots,d_n)\in GS_n$ with $d_2\geq 7$. If $d_n\geq 3$ and $\pi\neq(7^3,6,3^5),(7^4,4,3^4)$, then π is potentially $K_{1,1,6}$-graphic.

Proof. We use induction on n. For $n=8$, $d_1=7$ since $d_2\geq 7$. So π is potentially $K_{1,1,6}$-graphic by Lemma 2.2. Assume this Lemma holds for $n-1(n\geq 9)$. It is enough to consider the following two cases.

Case 1. $d_2\geq 8$. If $d_3\geq 4$, then the residual sequence $\pi_n'=(d_1',\ldots,d_{n-1}')$ obtained by laying off d_n from π satisfies $d_{n-1}'\geq 3$ and $d_2'\geq 7$. By the induction hypothesis, π_n' is potentially $K_{1,1,6}$-graphic, and so is π. If $d_3=3$, then $\pi=(d_1,d_2,3^{n-2})$. It is easy to compute the corresponding sequence $\pi_2=(3^x,2^y,1^z)$(reordering the terms in π_2 to make them non-increasing), where $z\geq 6$. According to Lemma 2.3, π_2 is graphic. So π is potentially $K_{1,1,6}$-graphic by Lemma 2.1.

Case 2. $d_2=7$. By Lemma 2.2, we suppose that π satisfies $n-2\geq d_1\geq\cdots\geq d_8=d_9=\cdots=d_{d_1+2}\geq d_{d_1+3}\geq\ldots\geq d_n$. It is enough to show that π_2 is graphic by Lemma 2.1.

If $d_8=3$, then $m(\pi_2)\leq 5,h(\pi_2)=1$ and $|\pi_2|\geq 7$, where $m(\pi),h(\pi)$ and $|\pi|$ mean the largest positive term, the smallest positive term and the number of the positive terms of π, respectively. If $d_3\leq 6$, then $m(\pi_2)\leq$

$4, h(\pi_2) = 1$ and $|\pi_2| \geq 7$. According to Lemma 2.3 and 2.4, π_2 is graphic. If $d_3 = 7$, then $\pi_2 = (5, d_4 - 2, d_5 - 2, d_6 - 2, d_7 - 2, 1, 3^{n-d_1-1}, 2^{d_1-7})$. If $d_7 \geq 5$ and $n - d_1 - 1 \geq 2$, then $5 + (n - d_1 - 1) \geq 7 \geq \frac{1}{3}\lfloor\frac{(5+3+1)^2}{4}\rfloor$ and π_2 is graphic by Theorem 2.3. If $d_7 \geq 5$ and $n - d_1 - 1 = 1$, then $d_1 = n - 2$ and $\pi_2 = (5, d_4 - 2, d_5 - 2, d_6 - 2, d_7 - 2, 1, 3, 2^{n-9})$. If $n - 9 \geq 2$, then $6 + (n - 9) \geq 8 \geq \frac{1}{2}\lfloor\frac{(5+2+1)^2}{4}\rfloor$ and π_2 is graphic. If $n - 9 \leq 1$, then we consider the residual sequence π'_{21} obtained by laying off the largest term of π_2(reordering the terms of π_2 to make them non-increasing). It is easy to check that π'_{21} is graphic by Lemma 2.3 and 2.4. Thereby, π_2 is graphic by Theorem 2.1. If $d_7 = 4$ and $n - d_1 - 1 + (d_1 - 7) \geq 3$, i.e. $n \geq 11$, then $5 + n - d_1 - 1 + (d_1 - 7) \geq 8 \geq \frac{1}{2}\lfloor\frac{(5+2+1)^2}{4}\rfloor$ and π_2 is graphic by Theorem 2.1. If $d_7 = 4$ and $n = 10$, then d_1 is 7 or 8 and the residual sequence π'_{21} is graphic and so is π_2. If $d_7 = 4, n = 9$ and $d_4 \leq 6$, then π'_{21} is graphic. If $d_7 = 4, n = 9$ and $d_4 = 7$, then π'_{21} is graphic by Lemma 2.4 and $\pi \in GS_n$. If $d_7 = 3, d_4 \leq 6$ and $d_6 \geq 4$, then π'_{21} is graphic by Lemma 2.3 and so is π_2. If $d_7 = 3, d_4 \leq 6, d_6 = 3$ and $d_5 \geq 4$, then π'_{21} is also graphic. If $d_7 = 3, d_4 \leq 5$ and $d_6 = d_5 = 3$, then $m(\pi'_{21}) \leq 3, |\pi'_{21}| \geq 2$ and there must exist one term which is equal to two and one term equal to one in π'_{21}. it is easy to check that π'_{21} is graphic. If $d_7 = 3, d_4 = 6$ and $d_6 = d_5 = 3$, then there must exist one term which is equal to two and one term equal to one in π'_{21}. If $\pi'_{21} \neq (3, 2, 1)$, then π'_{21} is graphic by Lemma 2.3. If $\pi'_{21} = (3, 2, 1)$, then $\pi = (7^3, 6, 3^5)$, a contradiction. If $d_7 = 3, d_4 = 7$ and $d_6 \geq 4$, then there must exist one term which is equal to one and one term which is equal to two in π'_{21}. Thus, π'_{21} is graphic by Lemma 2.4 and $\pi \in GS_n$. If $d_7 = 3, d_4 = 7, d_6 = 3$ and $d_5 \geq 4$, then there are at most two terms equal to four in π'_{21}. If there are exactly two terms equal to four in π'_{21}, then $\pi_2 = (5^3, 1^3, 3^{n-d_1-1}, 2^{d_1-7})$. Since $\sigma(\pi_2)$ is even, $n - d_1 - 1$ must be even. If $n - d_1 - 1 \geq 4$, then $3 + (n - d_1 - 1) \geq \frac{1}{3}\lfloor\frac{(5+3+1)^2}{4}\rfloor$ and π_2 is graphic by Theorem 2.3. If $n - d_1 - 1 = 2$, then π_2 is the graphic sequence $(5^3, 1^3, 3^2, 2^{n-10})$. If there is only one term which is equal to four in π'_{21} and $\pi'_{21} \neq (4, 2, 1^2)$, then π'_{21} is graphic by Lemma 2.4. If $\pi'_{21} = (4, 2, 1^2)$, then $\pi = (7^4, 4, 3^4)$, a contradiction. If $d_7 = 3, d_4 = 7$ and $d_5 = d_6 = 3$, then $\pi_2 = (5^2, 1^4, 3^{n-d_1-1}, 2^{d_1-7})$. If $|\pi_2| \geq 12 \geq \lfloor\frac{(5+1+1)^2}{4}\rfloor$, then π_2 is graphic. If $|\pi_2| \leq 11$, then π_2 is one of the graphic sequences $(5^2, 1^4, 3^2), (5^2, 1^4, 3^2, 2), (5^2, 1^4, 3^2, 2^2), (5^2, 1^4, 3^4), (5^2, 1^4, 3^2, 2^3), (5^2, 1^4, 3^4, 2)$.

If $d_8 = 4$ and $d_3 \leq 6$, then $|\pi_2| \geq 7 \geq \frac{1}{2}\lfloor\frac{(4+2+1)^2}{4}\rfloor$ and π_2 is graphic. If $d_8 = 4, d_3 = 7$ and $d_4 \leq 6$, then π'_{21} is graphic by Lemma 2.3 and 2.4. If $d_8 = 4, d_3 = d_4 = 7$ and $d_7 \geq 5$, then π'_{21} is also graphic. If $d_8 = 4, d_3 = d_4 = 7, d_7 = 4$ and $|\pi_2| \geq 8$, then π_2 is graphic by Theorem 2.3. If $|\pi_2| = 7$, then π_2 is one of the graphic sequences $(5^2, 2^4, 4), (5^2, 3^2, 2^2, 4), (5^2, 4, 2^3, 4), (5^2, 4^2, 2^2, 4), (5^3, 3, 2^2, 4), (5^4, 2^2, 4)$.

If $d_8 = 5$, then π_2 is graphic by Theorem 2.3, Lemma 2.3 and 2.4.

If $d_8 = 6$, then $m(\pi_2) = 6$ and $h(\pi_2) \geq 3$. If $|\pi_2| \geq 9 \geq max\{\frac{1}{4}\lfloor\frac{(6+4+1)^2}{4}\rfloor, \frac{1}{3}\lfloor\frac{(6+3+1)^2}{4}\rfloor\}$, then π_2 is graphic. If $|\pi_2| = 7$, then π_{21}' is graphic. If $|\pi_2| = 8$ and $d_{10} \geq 4$, then $|\pi_2| = 8 \geq \frac{1}{4}\lfloor\frac{(6+4+1)^2}{4}\rfloor$. If $|\pi_2| = 8$ and $d_{10} = 3$, then π_2 is one of the graphic sequences $(5, 4^5, 6, 3), (5^3, 4^3, 6, 3), (5^5, 4, 6, 3)$.

If $d_8 = 7$, then $\pi = (d_1, 7^{d_1+1+x}, 6^y, 5^z, 4^m, 3^{n-(d_1+2+x+y+z+m)})(x, y, z, m \geq 0)$ and $\pi_2 = (5^6, 7^{x+1}, 6^{y+d_1-7}, 5^z, 4^m, 3^{n-(d_1+2+x+y+z+m)})$. If $6+x+1+y+d_1-7+z \geq 9 \geq \frac{1}{5}\lfloor\frac{(7+5+1)^2}{4}\rfloor$, then π_2 is graphic by Theorem 2.3. If $6+x+1+y+d_1-7+z = 7$, then π_{21}' is graphic by Lemma 2.4. If $6+x+1+y+d_1-7+z = 8$, then π_2 is one of the graphic sequences $(5^6, 7, 6, 4^m, 3^{n-m-10}), (5^6, 7, 5, 4^m, 3^{n-m-10}), (5^6, 7^2, 4^m, 3^{n-m-10})$. □

Proof of Theorem 1.3 Assume that π is potentially $K_{1,1,6}$-graphic. (1) is obvious. It is easy to compute the corresponding π_2(re-ordering the terms in π_2 to make them non-increasing and zero omitted) of the excepted sequences is one of the sequences $(2), (2^2), (3, 2, 1), (4, 2, 1^2), (4, 2^2), (4, 2^3), (5, 2^2, 1), (5, 2, 1^3), (6, 2^2, 1^2), (6, 2, 1^4), (5^2, 3, 1^5), (5, 4, 3, 1^4), (5^2, 3, 2, 1^3), (5, 2^3, 1), (6, 2^3, 1^2), (5, 3, 2^3), (5, 3^3, 2), (7, 2, 1^5), (8, 2, 1^6)$, which are not graphic.

To prove the sufficiency, we use induction on n. It is enough to show that π_2 is graphic by Lemma 2.1. Assume that $n = 8$ and $\pi = (d_1, d_2, \ldots, d_n) \in GS_n$ satisfies (1) and (2). Then $d_1 = 7 = n-1$ and π is potentially $K_{1,1,6}$-graphic by Lemma 2.2. Now suppose that the sufficiency holds for $n-1 (n \geq 9)$, and let $\pi = (d_1, d_2, \ldots, d_n) \in GS_n$ satisfy (1) and (2). According to Lemma 2.2 and 2.5, we can assume that π satisfies $n-2 \geq d_1 \geq \cdots \geq d_8 = d_9 = \cdots = d_{d_1+2} \geq d_{d_1+3} \geq \cdots \geq d_n$ and $d_n \leq 2$. In the following, we will use Theorem 2.5, repeatedly. We now prove that π is potentially $K_{1,1,6}$-graphic in terms of the following two cases.

Case 1. $d_n = 2$. Consider $\pi_n' = (d_1', d_2', \ldots, d_{n-1}')$, where $d_2' \geq 6$ and $d_{n-1}' \geq 2$. If π_n' satisfies (1) and (2), then by the induction hypothesis, π_n' is potentially $K_{1,1,6}$-graphic, and so is π.

If π_n' does not satisfy (1), i.e., $d_2' = 6$, then $d_2 = 7$.

If $d_1 = 7$, then $d_4 \leq 6$. If $d_4 = 2$, then $\pi_2 = (d_3 - 2, 0^5, 2^{n-8})$. If π_2 (zero omitted) is not $(2), (2^2), (4, 2^2)$ or $(4, 2^3)$, then π_2 is graphic by Lemma 2.3 and 2.4. If π_2 is $(2), (2^2), (4, 2^2)$ or $(4, 2^3)$, then π is one of the sequences $(7^2, 2^7), (7^2, 2^8), (7^2, 4, 2^6), (7^2, 6, 2^7), (7^2, 6, 2^8)$, a contradiction. If $d_4 = 3$ and $d_3 \leq 4$, then π_2 is graphic. If $d_4 = 3, d_3 = 5$ and $d_8 = 3$, then π_2 is graphic. If $d_4 = 3, d_3 = 5, d_8 = 2$ and $\pi_2 \neq (3, 2, 1)$, then π_2 is graphic. If $\pi_2 = (3, 2, 1)$, then $\pi = (7^2, 5, 3, 2^5)$, a contradiction. If $d_4 = 3, d_3 = 6$ and $d_8 = 3$, then π_2 is graphic. If $d_4 = 3, d_3 = 6$ and $d_8 = 2$, then $|\pi_2| \geq 3$. If $|\pi_2| \geq 7$, then π_2 is graphic. Since $\sigma(\pi_2)$ is even, $|\pi_2| \neq 3$. If $|\pi_2| = 4$, then $\pi = (7^2, 6, 3^2, 2^4)$, a contradiction. If $|\pi_2| \leq 6$, then π_2 is graphic sequences $(4, 2^2, 1^2), (4, 2, 1^4)$ or $(4, 2^3, 1^2)$. If $d_4 = 3, d_3 = 7$ and $d_8 = 3$, then $m(\pi_2) = 5$ and $|\pi_2| \geq 8$. If $|\pi_2| \geq$

$12 \geq \lfloor \frac{(5+1+1)^2}{4} \rfloor$, then π_2 is graphic. If $|\pi_2| \leq 11$, then π_2 is one of the graphic sequences $(5,3^2,2,1^5),(5,3^2,2^2,1^5),(5,3^2,2^3,1^5),(5,3^4,2,1^5)$. If $d_4 = 3, d_3 = 7$ and $d_8 = 2$, then π_2 is $(5,2^{n-8},1)$ or $(5,2^{n-8},1^3)$. If $\pi_2 = (5,2^{n-8},1)$ and $n \geq 16$, then $1 + (n-8) \geq \frac{1}{2}\lfloor \frac{(5+2+1)^2}{4} \rfloor$ and π_2 is graphic. If $12 \leq n \leq 15$, then π_2 is one of the graphic sequences $(5,2^4,1),(5,2^5,1),(5,2^6,1),(5,2^7,1)$. Since π is graphic, $n \neq 9$. If n is 10 or 11, then $\pi = (7^3,3,2^6)$ or $(7^3,3,2^7)$, a contradiction. If $\pi_2 = (5,2^{n-8},1^3)$ and $n \geq 16$, then $1 + (n-8) \geq \frac{1}{2}\lfloor \frac{(5+2+1)^2}{4} \rfloor$ and π_2 is graphic. Since π is graphic, $n \neq 9$. If $10 \leq n \leq 15$, then π_2 is one of the graphic sequences $(5,2^2,1^3),(5,2^3,1^3),(5,2^4,1^3),(5,2^5,1^3),(5,2^6,1^3),(5,2^7,1^3)$. If $d_4 = 4$ and $d_3 \leq 5$, then π_2 is graphic. If $d_4 = 4, d_3 = 6$ and $d_8 = 2$, then there must exist at least two terms equal to two in π_2. If $|\pi_2| \geq 7$, then π_2 is graphic. Since π is graphic, $|\pi_2| \neq 3$. If $|\pi_2| = 4$, then π is $(7^2,6,4^2,2^4)$ or $(7^2,6,4,2^6)$, a contradiction. If $5 \leq |\pi_2| \leq 6$, then π_2 is one of the graphic sequences $(4,2^2,1^2),(4,2^4),(4,2^3,1^2),(4,2^5)$. If $d_4 = 4, d_3 = 6$ and $d_8 = 3$, then π_2 is graphic. If $d_4 = 4, d_3 = 6$ and $d_8 = 4$, then π_2 is also graphic. If $d_4 = 4, d_3 = 7$ and $d_8 = 4$, then π_2 is graphic by Theorem 2.3. If $d_4 = 4, d_3 = 7$ and $d_8 = 3$, then $|\pi_2| \geq 8$. If $|\pi_2| \geq 12 \geq \lfloor \frac{(5+1+1)^2}{4} \rfloor$, then π_2 is graphic. If $|\pi_2| \leq 11$, then the residual sequence π_{21}' is graphic and so is π_2. If $d_4 = 4, d_3 = 7$ and $d_8 = 2$, then π_2 is one of the sequences $(5,2^{n-7},1),(5,2^{n-5},1),(5,2^{n-7},1^3)$. It is easy to check that the sequences $(5,2^{n-5},1)$ and $(5,2^{n-7},1^3)$ are graphic. If $\pi_2 = (5,2^{n-7},1)$ and $n \geq 14$, then $1 + (n-7) \geq \frac{1}{2}\lfloor \frac{(5+2+1)^2}{4} \rfloor$ and π_2 is graphic. If $11 \leq n \leq 13$, then π_2 is one of the graphic sequences $(5,2^4,1),(5,2^5,1),(5,2^6,1)$. Since π is graphic, $n \neq 9$. If $n = 10$, then $\pi = (7^3,4,3,2^5)$, which is contradict. If $d_4 = 5, d_3 = 5$ and $d_8 \geq 3$, then π_2 is graphic by Theorem 2.3, Lemma 2.3 and 2.4. If $d_4 = 5, d_3 = 5$ and $d_8 = 2$, then $|\pi_2| \geq 3$. If $|\pi_2| \geq 4$, then π_2 is graphic. Since π is graphic, $|\pi_2| \neq 3$. If $d_4 = 5, d_3 = 6$ and $d_8 \geq 3$, then π_2 is graphic by Theorem 2.3, Lemma 2.3 and 2.4. If $d_4 = 5, d_3 = 6$ and $d_8 = 2$, then $|\pi_2| \geq 3$. Since $\sigma(\pi_2)$ is even, $|\pi_2| \neq 3, 4$. If $|\pi_2| \geq 5$, then π_2 is graphic. If $d_4 = 5, d_3 = 7$ and $d_8 \geq 4$, then π_2 is graphic by Theorem 2.3. If $d_4 = 5, d_3 = 7$ and $d_8 = 3$, then π_{21}' is graphic by Lemma 2.3. If $d_4 = 5, d_3 = 7$ and $d_8 = 2$, then π_2 is one of the sequences $(5,3,2^{n-8}),(5,3,2^{n-8},1^2),(5,3,2^{n-7}),(5,3,2^{n-7},1^2),(5,3,2^{n-5}),,(5,3,2^{n-6}),(5,3^2,2^{n-8},1),(5,3^2,2^{n-7},1),(5,3^3,2^{n-8}),(5,3^3,2^{n-7})$. If $\pi_2 = (5,3,2^{n-8})$ and $n \geq 12$, then it is easy to see that π_2 is graphic. Since π is graphic, $n \geq 11$. If $n = 11$, then $\pi = (7^3,5,2^7)$, a contradiction. If $\pi_2 = (5,3,2^{n-7})$, then $n \geq 10$ since π is graphic. It is easy to check that π_2 is graphic for $n \geq 11$. If $n = 10$, then $\pi = (7^3,5,4,2^5)$, which is contradict. If $\pi_2 = (5,3,2^{n-6})$ and $n \geq 10$, then π_2 is graphic. If $n = 9$, then $\pi = (7^3,5,4^2,2^3)$, a contradiction. If $\pi_2 = (5,3^3,2^{n-8})$ and $n \geq 10$, then π_2 is graphic. If $n = 9$, then $\pi = (7^3,5^3,2^3)$, a contradiction. For π_2 is one of the sequences $(5,3,2^{n-8},1^2),(5,3,2^{n-7},1^2),(5,3,2^{n-5}),(5,3^2,2^{n-8},1),$ $(5,3^2,2^{n-7},1),(5,3^3,2^{n-7})$, it is easy to check that π_2 is graphic by $\pi \in GS_n$. If $d_4 = 6$ and $d_3 = d_8 = 6$, then $m(\pi_2) = 6$ and $h(\pi_2) = 2$. If $|\pi_2| \geq 10 \geq \frac{1}{2}\lfloor\frac{(6+2+1)^2}{4}\rfloor$, then π_2 is graphic. If $|\pi_2| \leq 9$, then π_2 is one of the graphic sequences $(6,4^6,2),(6,4^7,2),(6,4^6,2^2),(6^2,4^6,2)$. If $d_4 = 6, d_3 = 6$ and $2 \leq d_8 \leq 5$, then π_2 is graphic by Theorem 2.3, Lemma 2.4 and $\pi \in GS_n$. If $d_4 = 6, d_3 = 7, d_8 = 6$ and $d_{10} \geq 4$, then $8 \geq \frac{1}{4}\lfloor\frac{(6+4+1)^2}{4}\rfloor$ and π_2 is graphic. If $d_4 = 6, d_3 = 7, d_8 = 6$ and $d_{10} \leq 3$, then π'_{21} is also graphic. If $d_4 = 6, d_3 = 7$ and $d_8 = 3$, then π'_{21} is graphic. If $d_4 = 6, d_3 = 7$ and $4 \leq d_8 \leq 5$, then π_2 is graphic. If $d_4 = 6, d_3 = 7, d_7 \geq 4$ and $d_8 = 2$, then π'_{21} is also graphic. If $d_4 = 6, d_3 = 7, d_7 \leq 3$ and $d_8 = 2$, then π_2 is one of the graphic sequences $(5,4,2^{n-8},1^3),(5,4,2^{n-6},1),(5,4,3,2^{n-8},1^2),(5,4,3^2,2^{n-8},1),(5,4^3,2^{n-8},1),(5,4,2^{n-8},1),(5,4,2^{n-7},1),(5,4,3,2^{n-8}),(5,4,3,2^{n-7}),(5,4^2,2^{n-8},1),$ $(5,4^2,3,2^{n-8}),(5,4^2,2^{n-7},1)$.

If $d_1 \geq 8$, then $d_3 \leq 6$. If $d_3 \leq 4$, then π_2 is graphic. If $d_3 = 5$ and $d_8 = 2$, then $m(\pi_2) = 3$ and $|\pi_2| \geq 3$. If $\pi_2 \neq (3,2,1)$, then π_2 is graphic. If $\pi_2 = (3,2,1)$, then π is $(8,7,5,2^7)$, a contradiction. If $d_3 = 5$ and $3 \leq d_8 \leq 4$, then π_2 is graphic. If $d_3 = 5$ and $d_8 = 5$, then $9 \geq \frac{1}{3}\lfloor\frac{(5+3+1)^2}{4}\rfloor$ and π_2 is graphic. If $d_3 = 6$ and $d_8 = 2$, then $|\pi_2| \geq 3$. If $|\pi_2| \geq 7$, then π_2 is graphic by Lemma 2.4. Since $\sigma(\pi_2)$ is even, $|\pi_2| \neq 3$. Since π is graphic, $|\pi_2| \neq 5$ and 6. If $|\pi_2| = 4$, then π is $(8,7,6,3,2^6),(9,7,6,2^8)$ or $(8,7,6,5,2^6)$. The sequences $(8,7,6,3,2^6)$ and $(9,7,6,2^8)$ are excepted and the sequence $(8,7,6,5,2^6)$ is not graphic. If $d_3 = 6$ and $3 \leq d_8 \leq 4$, then π_2 is graphic. If $d_3 = 6$ and $d_8 = 5$, then $|\pi_2| \geq 9 \geq \frac{1}{2}\lfloor\frac{(5+2+1)^2}{4}\rfloor$ and π_2 is graphic. If $d_3 = 6$ and $d_8 = 6$, then $|\pi_2| \geq 9$. If $|\pi_2| \geq 10 \geq \frac{1}{2}\lfloor\frac{(6+2+1)^2}{4}\rfloor$, then π_2 is graphic by Theorem 2.3. Since $\sigma(\pi_2)$ is even, $|\pi_2| \neq 9$.

If π'_n does not satisfy (2), then π'_n is one of the sequences $(7^2,2^7),(7^2,4,2^6),(7^2,5,3,2^5),(7^2,2^8),(7^2,6,4^2,2^4),(7^3,6,3^5),(7^4,4,3^4),(8,7,5,2^7),$ $(10,9^2,2^{11})(8^2,6,2^8),(7^2,6,2^7),(7^3,3,2^6),(8,7,6,3,2^6),(7^2,6,4,2^6),(7^3,4,3,2^5),(8,7^2,3^2,2^5),(7^2,6,3^2,2^4),(7^3,3,2^7),(7^2,6,2^8),(9,7,6,2^8),(10^3,2^{12}),$ $(8^2,7,3,2^7),(8,7^2,2^8),(8,7^2,4,2^7),(8^3,2^9),(8^3,2^{10}),(9,7^2,3,2^7),(7^3,5,2^7),$ $(8,7^2,4^2,2^5),(7^3,5,4,2^5),(7^3,5^3,2^3),(8^3,3^2,2^6),(8^3,4^2,2^6),(8^3,4,2^8),(9,8,7,2^9),(8,7^2,2^9),(10,7^2,2^9),(9,8^2,3,2^8),(9^2,8,2^{10}),(7^3,5,4^2,2^3),(10,8^2,2^{10}),(9^3,3,2^9)$. Since π satisfies the condition (2), π is one of the sequences $(9,8,7,4^2,2^6),(9^2,6,2^9),(10,8,6,2^9),(9^2,7,3,2^8),(8^2,7,3,2^8),(8^2,7,5,2^8),$ $(8^2,6,2^9),(9^2,8,2^{11}),(8^2,7,5,4,2^6),(8^2,7,5^3,2^4),(11,9,8,2^{11}),(9^2,8,4^2,2^7),$ $(10,8,7,3,2^8),(9,8,7,4,2^8),(10^2,8,2^{11}),(10,9,7,2^{10}),(9^2,8,4,2^9),(9,8,7,2^{10}),(11,8,7,2^{10}),(10,9,8,3,2^9),(8^2,7,5,4^2,2^4),(10^2,9,3,2^{10}),(8^2,2^8),(8^2,4,2^7),(8^2,5,3,2^6),(8^2,6,3^2,2^5),(8^2,6,4^2,2^5),(8^2,7,6,3^5,2),(9,8,7,3^2,2^6),$ $(8^2,2^9),(8^2,7^2,4,3^4,2),(9,8,5,2^8),(9,8,6,3,2^7),(8^2,6,4,2^7),(8^2,7,4,3,2^6),$

$(8,7^3,3^5,2),(9^2,8,3^2,2^7),(11,10,9,2^{12}),(11^2,10,2^{13})$. It is easy to compute the corresponding π_2 is one of the graphic sequences $(1^2),(2,1^2),(3,1^3),(4,1^4),(4,2^2,1^2),(5,4,2,1^5),(5^2,2^2,1^4),(5,2^2,1^3),(5,1^5),(5,3,2^2,1^2),(5,3^3,1^2),(6,1^6),(6,2^2,1^4),(7,1^7),(8,1^8)$. So π is potentially $K_{1,1,6}$-graphic.

Case 2. $d_n = 1$. Then $\pi'_n = (d'_1, d'_2, \ldots, d'_{n-1})$ satisfies $d'_2 \geq 6$ and $d'_8 \geq 2$. If π'_n satisfies (1) and (2), then by the induction hypothesis, π'_n is potentially $K_{1,1,6}$-graphic, and hence so is π.

If π'_n does not satisfy (1), i.e. $d'_2 = 6$, then $d_1 = d_2 = 7, d_3 \leq 6$ and $n \geq 10$. Thus, $\pi_2 = (d_3-2, d_4-2, d_5-2, d_6-2, d_7-2, d_8-2, d_9, \ldots, d_n)$. If $d_3 \leq 4$, then π_2 is graphic. If $d_3 = 5$ and $d_8 \geq 3$, then π_2 is also graphic. If $d_3 = 5, d_8 = 2$ and $\pi_2 \neq (3,2,1)$, then π_2 is graphic. If $\pi_2 = (3,2,1)$, then $\pi = (7^2,5,2^6,1)$, a contradiction. If $d_3 = 6$ and $d_8 = 2$, then $|\pi_2| \geq 3$. Since $\sigma(\pi_2)$ is even, $|\pi_2| \neq 3$. If $|\pi_2| \geq 7$, then π_2 is graphic. If $|\pi_2| = 4$, then π is one of the sequences $(7^2,6,3,2^5,1),(7^2,6,2^6,1^2),(7^2,6,5,2^5,1)$. The sequences $(7^2,6,3,2^5,1)$ and $(7^2,6,2^6,1^2)$ are excepted and the sequence $(7^2,6,5,2^5,1)$ is not graphic. Since π is graphic, $|\pi_2| \neq 5$ and 6. If $d_3 = 6$ and $3 \leq d_8 \leq 5$, then π_2 is graphic. If $d_3 = 6, d_8 = 6$ and $d_{10} \geq 4$, then $8 \geq \frac{1}{4}\lfloor \frac{(6+4+1)^2}{4} \rfloor$ and π_2 is graphic. If $d_3 = 6, d_8 = 6$ and $d_{10} \leq 3$, then the residual sequence π'_{21} is graphic.

If π'_n does not satisfy (2), then π'_n is one of the sequences $(7^2,2^7),(7^2,4,2^6),(7^2,6,3^2,2^4),(7^4,3^5,1),(7^2,6,4^2,2^4),(7^3,6,3^5),(7^4,4,3^4),(7^2,6,2^7),(7^3,3,2^6),(7^2,5,3,2^5),(7^2,6,3,2^5,1),(8,7,6,3,2^6),(7^2,5,2^6,1),(7^3,4^2,2^4,1),(8^2,6,2^8),(7^3,5,2^7),(7^3,3,2^7),(7^2,6,4,2^6),(7^3,4,3,2^5),(9,7,6,2^8),(8,7,5,2^7),(8,7^2,4,2^7),(7^3,3^2,2^4,1),(8,7,6,2^7,1),(7^2,6,2^8),(8,7^2,3,2^6,1),(8,7^2,3^2,2^5),(7^2,2^8),(7^3,3,2^5,1^2),(8,7^2,4^2,2^5),(9,8,7,2^9),(10,7^2,2^9),(7^3,5^3,2^3),(8^3,3^2,2^6),(8^3,4^2,2^6),(7^3,5,4,2^5),(7^3,2^6,1^3),(8^3,2^9),(8,7^2,2^7,1^2),(8,7^2,2^9),(8^3,2^{10}),(7^3,2^8,1),(9,7^2,3,2^7),(7^2,6,2^6,1^2),(7^3,4,2^6,1),(8^2,7,3,2^7),(8,7^2,2^8),(7^3,2^7,1),(8^2,7,2^8,1),(9,7^2,2^8,1),(8^3,3,2^7,1),(8^3,4,2^8),(9^2,8,2^{10}),(9,8^2,2^9,1),(7^3,5,4^2,2^3),(10,8^2,2^{10}),(8^3,2^8,1^2),(9,8^2,3,2^8),(9^3,3,2^9),(10,9^2,2^{11}),(9^3,2^{10},1),(10^3,2^{12})$. Since π satisfies (2), π is one of the sequences $(8^2,7,3^2,2^5,1),(8,7,6,3,2^5,1^2),(8,7,6,3^2,2^4,1),(9,7,6,3,2^6,1),(9,7^2,3^2,2^5,1),(8,7^2,6,3^5,1),(9,7^2,4^2,2^5,1),(8^2,7,4^2,2^5,1),(9,7,5,2^7,1),(8,7^2,3^2,2^4,1^2),(8,7^2,4^2,2^4,1^2),(8,7,6,4^2,2^4,1),(9,7^2,4,2^7,1),(8,7^3,3^5,1^2),(9,7,6,2^7,1^2),(8^2,7,2^9,1),(8,7,6,2^8,1),(10,7,6,2^8,1),(9^2,7,2^9,1),(8,7^2,2^8,1^2),(9,8^2,2^8,1^3),(9,8,6,2^8,1),(8,7^2,3,2^7,1),(9^2,8,2^9,1^2),(9,7^2,2^7,1^3),(8^2,7,2^7,1^3),(8^2,7,4,2^7,1),(8,7^2,5,4,2^5,1),(9,7^2,3,2^6,1^2),(8^2,7,3,2^6,1^2),(10,7^2,3,2^7,1),(9,8,7,3,2^7,1),(9,8^2,2^{10},1),(8^2,6,2^7,1^2),(9,7^2,2^9,1),(8^2,6,3,2^6,1),(8^2,5,2^7,1),(8,7,2^8,1),(8,7,5,3,2^5,1),(8,7,5,2^6,1^2),(8,7^3,4,3^4,1),(8,7,6,4,2^6,1),(8,7^2,5,4^2,2^3,1),(8,7^2,4,3,2^5,1),(8,7,4,2^6,1),(10,9^2,3,2^9,1),(9,8^2,3,2^7,1^2),(11,10^2,2^{12},1),(9,8,7,2^8,1^2),(8,7^2,4,2^6,1^2),(8,7^2,5^3,2^3,1),(8,7^2,3,2^5,1^3),(10,8,7,2^9,1),(9,8^2,3^2,2^6,1),(9,8^2,4^2,2^6,1),(8,7,6,2^6,1^3),(10,9,8,2^{10},1),(11,7^2,2^9,1),(10,8^2,2^9,1^2),(11,$

$8^2, 2^{10}, 1), (8, 7^2, 5, 2^7, 1), (10, 9^2, 2^{10}, 1^2), (11, 9^2, 2^{11}, 1), (8, 7^2, 2^6, 1^4), (8, 7, 2^7, 1), (10, 8^2, 3, 2^8, 1), (9^2, 8, 3, 2^8, 1), (10, 7^2, 2^8, 1^2), (9, 8^2, 4, 2^8, 1), (10^2, 9, 2^{11}, 1), (10, 7^2, 2^8, 1^2)$. It is easy to compute the corresponding π_2 is one of the graphic sequences $(1^2), (2, 1^2), (3, 1^3), (4, 1^4), (4, 2^2, 1^2), (5, 4, 2, 1^5), (5^2, 2^2, 1^4), (5, 2^2, 1^3), (5, 1^5), (5, 3, 2^2, 1^2), (5^2, 2, 1^6), (5, 3^3, 1^2), (6, 1^6), (6, 2^2, 1^4), (7, 1^7), (8, 1^8)$. So π is potentially $K_{1,1,6}$-graphic. □

We now give an application of Theorem 1.3.

Corollary For $n \geq 8$,

$$\sigma(K_{1,1,6}, n) = \begin{cases} 7n-5, & \text{if } n \text{ is odd,} \\ 7n-6, & \text{if } n \text{ is even.} \end{cases}$$

Proof. Take $\pi = (n-1, 6^{n-1})$ if n is odd and $\pi = (n-1, 6^{n-2}, 5)$ if n is even. Obviously, π is graphic. If π is potentially $K_{1,1,6}$-graphic, then there are at least two terms in π which are greater or equal to seven, a contradiction. Hence, π is not potentially $K_{1,1,6}$-graphic. In other words,

$$\sigma(K_{1,1,6}, n) \geq \sigma(\pi) + 2 = \begin{cases} 7n-5, & \text{if } n \text{ is odd,} \\ 7n-6, & \text{if } n \text{ is even.} \end{cases}$$

Let $n \geq 8$ and $\pi = (d_1, d_2, \ldots, d_n) \in GS_n$ be a positive sequence with $\sigma(\pi) \geq 7n-6$. We show that π is potentially $K_{1,1,6}$-graphic.

(1) By $n \geq 8$ and $\sigma(\pi) \geq 7n-6$, it is easy to check that π is not one of the sequences $(7^2, 2^7), (7^2, 4, 2^6), (7^2, 5, 3, 2^5), (7^2, 6, 3^2, 2^4), (7^2, 6, 4^2, 2^4), (7^3, 6, 3^5), (7^4, 4, 3^4), (7^2, 6, 3, 2^5, 1), (7^2, 6, 2^8), (9, 7, 6, 2^8), (8^2, 7, 3, 2^7), (8, 7^2, 2^8), (8, 7^2, 4, 2^7), (8, 7^2, 3, 2^6, 1), (8^2, 6, 2^8), (7^3, 2^7, 1), (7^2, 6, 2^6, 1^2), (7^3, 3, 2^5, 1^2), (7^3, 4, 2^6, 1), (7^3, 5, 2^7), (7^3, 5, 4, 2^5), (7^3, 5^3, 2^3), (8^3, 3^2, 2^6), (8^3, 2^9), (7^2, 6, 2^7), (7^3, 3, 2^6), (8, 7, 5, 2^7), (8, 7, 6, 3, 2^6), (7^2, 5, 2^6, 1), (7^2, 6, 4, 2^6), (7^3, 4, 3, 2^5), (7^2, 2^8), (8^3, 4^2, 2^6), (7^3, 3, 2^7), (9, 8, 7, 2^9), (9, 7^2, 2^8, 1), (8^2, 7, 2^8, 1), (8, 7^2, 2^7, 1^2), (8, 7^2, 2^9), (7^3, 2^8, 1), (10, 7^2, 2^9), (9, 8^2, 3, 2^8), (8^3, 3, 2^7, 1), (7^3, 2^6, 1^3), (8^3, 4, 2^8), (9^2, 8, 2^{10}), (8^3, 2^{10}), (9, 8^2, 2^9, 1), (8, 7^2, 3^2, 2^5), (8, 7^2, 4^2, 2^5), (7^3, 3^2, 2^4, 1), (7^3, 4^2, 2^4, 1), (7^4, 3^5, 1), (8, 7, 6, 2^7, 1), (9, 7^2, 3, 2^7), (8^3, 2^8, 1^2), (10, 8^2, 2^{10}), (9^3, 3, 2^9), (10, 9^2, 2^{11}), (9^3, 2^{10}, 1), (10^3, 2^{12})$.

(2) We claim that $d_2 \geq 7$. Otherwise, $d_2 \leq 6$. Then $\sigma(\pi) = d_1 + d_2 + \cdots + d_n \leq n-1+6(n-1) < 7n-6$, a contradiction.

(3) We claim that $d_8 \geq 2$. Otherwise, $d_8 \leq 1$. Then $\sigma(\pi) = \sum_{i=1}^{7} d_i + \sum_{i=8}^{n} d_i \leq 42 + \sum_{i=8}^{n} min\{7, d_i\} + \sum_{i=8}^{n} d_i =< 7n-6$, a contradiction.

Thus, π is potentially $K_{1,1,6}$-graphic by Theorem 1.3 and $\sigma(K_{1,1,6}, n)$ being even. □

Proof of Theorem 1.4 Use induction on s. By Theorem 1.1 and Theorem 1.2, Theorem 1.4 holds for $s=2$ and $s=3$. Now assume Theorem 1.4 holds for $s-1(\geq 2)$. We will prove by using induction on n that Theorem 1.4 holds for s. If $n = s+2$, then $d_1 = s+1 = n-1$. By Lemma 2.2, π is potentially $K_{1,1,s}-$graphic. By Lemma 2.2, we can assume that π satisfies $n-2 \geq d_1 \geq \cdots \geq d_{s+1} \geq d_{s+2} = d_{s+3} = \cdots = d_{d_1+2} \geq d_{d_1+3} \geq \cdots \geq d_n$. If $n = s+3$, then $d_1 = s+1$ and $\pi = ((s+1)^{s+3})$, a contradiction. If

$n = s+4$, then $d_1 = s+1$ or $s+2$. If $d_1 = s+1$, then $\pi = ((s+1)^{s+3}, d_{s+4})$ and $\pi_2 = ((s-1)^s, s+1, d_{s+4})$. It is easy to check that π_2 is graphic by Theorem 2.4 and Theorem 2.8. If $d_1 = s+2$ and $d_2 = s+1$, then $\pi = (s+2, (s+1)^{s+3})$ and $\pi_2 = ((s-1)^s, s, s+1)$, which is graphic. If $d_1 = s+2 = d_2$ and $d_{s+2} = s+2$, then $\pi = ((s+2)^{s+4})$ and $\pi_2 = (s^s, (s+1)^2)$. If $d_1 = s+2 = d_2$ and $d_{s+2} = s+1$, then $\pi = ((s+2)^x, (s+1)^{s+4-x})(x \geq 2)$ and $\pi_2 = (s^y, (s-1)^{s+2-y})(y \geq 2, s+2-y \geq 1)$, which is graphic by Theorem 2.8. In the following, we assume that $n \geq s+5$ and $s \geq 3$.

If there exists an integer $t, 2 \leq t \leq s+1$, such that $d_t > d_{t+1}$, then the residual sequence $\pi_{s+2} = (d'_1, \ldots, d'_{n-1})$ obtained by laying off d_{s+2} from π satisfies $d'_1 = d_1 - 1, \ldots, d'_t = d_t - 1$. If $d_{s+2} \geq s+2$, then $d'_{s+2} \geq s+1$. By the induction hypothesis, π_{s+2} is potentially $K_{1,1,s}$−graphic and so is π. If $d_{s+2} = s+1$, then $d'_{s+2} \geq s$. By the induction hypothesis, π_{s+2} is potentially $K_{1,1,s-1}$−graphic. Thus, π is potentially $K_{1,1,s}$−graphic from $d'_1 = d_1 - 1, \ldots, d'_t = d_t - 1 (2 \leq t \leq s+1)$.

Now assume that $n-2 \geq d_1 \geq d_2 = d_3 = \cdots = d_{s+1} = d_{s+2} = \cdots = d_{d_1+2} \geq d_{d_1+3} \geq \cdots \geq d_n$. We consider the following two cases.

Case 1. $d_n = 1$. Then the residual sequence $\pi'_n = (d'_1, \ldots, d'_{n-1})$ obtained by laying off d_n from π satisfies $d'_{s+2} \geq s+1$. By the induction hypothesis, π'_n is potentially $K_{1,1,s}$−graphic and so is π.

Case 2. $d_n \geq 2$. If $d_{s+4} \geq s+3$, then π is potentially $K_{2,s+2}$−graphic by Theorem 2.9. According to Theorem 2.7, π is potentially $K_{1,1,s}$−graphic. Suppose that $d_{s+4} \leq s+2$, we consider the following two subcases.

Subcase 2.1. $d_1 \geq s+2$. Then $d_{s+4} = s+1$ or $d_{s+4} = s+2$. If $d_{s+4} = s+2$, then the residual sequence $\pi'_1 = (d'_1, \ldots, d'_{n-1})$ obtained by laying off d_1 from π satisfies $d'_{s+2} \geq s+1$. By the induction hypothesis, π'_1 is potentially $K_{1,1,s}$−graphic. If $d_{s+4} = s+1$, then $\pi = (d_1, (s+1)^{d_1+1}, d_{d_1+3}, \ldots, d_n)$. Denote $l = max\{i | d_{d_1+1+i} = d_2\}$. Obviously $l \geq 1$. Thereby, $\pi = (d_1, (s+1)^{d_1+l}, d_{d_1+1+(l+1)}, \ldots, d_n)(d_{d_1+1+(l+1)} \leq s)$. It is easy to compute that $\pi_2 = ((s-1)^s, (s+1)^l, s^{d_1-(s+1)}, d_{d_1+1+(l+1)}, \ldots, d_n)$. If $l \geq 2$, then $d_1 - (s+1) + l + s \geq s+3 \geq \frac{1}{s-1}\lfloor\frac{(s+1+s-1+1)^2}{4}\rfloor = \frac{1}{s-1}\lfloor(s-1)^2 + 3(s-1) + \frac{9}{4}\rfloor = s+2+\frac{2}{s-1}$ and π_2 is graphic by Theorem 2.3. If $l = 1$, then $\pi_2 = ((s-1)^s, s+1, s^{d_1-(s+1)}, d_{d_1+3}, \ldots, d_n)$. It is easy to check that the residual sequence π'_{21} obtained by laying off the largest term $s+1$ from π_2 is graphic by Theorem 2.8. Thus, π_2 is graphic by Theorem 2.1.

Subcase 2.2. $d_1 = s+1$. Then $\pi = ((s+1)^{s+2+l}, d_{s+2+l+1}, \ldots, d_n)$, where $(d_{s+2+l+1} \leq s)$. Thereby, $\pi_2 = ((s-1)^s, (s+1)^l, d_{s+2+l+1}, \ldots, d_n)$. If $d_{s+2+l+1} \geq s-1$ and $l \geq 2$, then $l+s+1 \geq s+3 \geq \frac{1}{s-1}[\frac{(s+1+s-1+1)^2}{4}] = s+2+\frac{2}{s-1}$ and π_2 is graphic by Theorem 2.3. If $d_{s+2+l+1} \geq s-1$ and $l = 1$, then $\pi_2 = ((s-1)^s, s+1, d_{s+2+l+1}, \ldots, d_n)$. According to Theorem 2.8, the residual sequence π'_{21} obtained by laying off the largest term $s+1$

from π_2 is graphic. Thus, π_2 is graphic according to Theorem 2.1. If $d_{s+2+l+1} \leq s-2$ and $l=1$, then $\pi_2 = ((s-1)^s, s+1, d_{s+2+l+1}, \ldots, d_n)$. It is easy to see that the residual sequence π'_{21} is also graphic by Theorem 2.8. If $d_{s+2+l+1} \leq s-2$ and $l=2$, then $\pi_2 = ((s-1)^s, (s+1)^2, d_{s+2+l+1}, \ldots, d_n)$. It is easy to check that the residual sequence π'_{211} obtained by laying off the largest term s from the residual sequence π'_{21} is graphic by Theorem 2.8. Thus, π_2 is graphic. If $d_{s+2+l+1} \leq s-2$ and $l \geq 3$, then $l+s \geq s+3 \geq \frac{1}{s-1}[\frac{(s+1+s-1+1)^2}{4}] = s+2+\frac{2}{s-1}$ and π_2 is graphic by Theorem 2.3. Hence, π is potentially $K_{1,1,s}$−graphic by Lemma 2.1. □

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