On Cyclic $(C_{2m} + e)$-designs

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Abstract

An *almost-bipartite* graph is a non-bipartite graph with the property that the removal of a particular single edge renders the graph bipartite. A graph labeling of an almost-bipartite graph G with n edges that yields cyclic G-decompositions of the complete graph K_{2nt+1} (i.e., cyclic (K_{2nt+1}, G)-designs) was recently introduced by Blinco, El-Zanati, and Vanden Eynden. They called such a labeling a γ-labeling. Here we show that the class of almost-bipartite graphs obtained from C_{2m} by adding an edge joining distinct vertices in the same part in the bipartition of $V(C_{2m})$ has a γ-labeling if and only if $m \geq 3$. This, along with results of Blinco and of Froncek, shows that if G is a graph of size n consisting of a cycle with a chord, then there exists a cyclic (K_{2nt+1}, G)-design for every positive integer t.

1 Introduction

If a and b are integers we denote $\{a, a+1, \ldots, b\}$ by $[a, b]$ (if $a > b$, $[a, b] = \emptyset$). Let $\mathbb{N}$ denote the set of nonnegative integers and $\mathbb{Z}_n$ the group of integers modulo n. For a graph G, let $V(G)$ and $E(G)$ denote the vertex set of G and the edge set of G, respectively. The *order* and the *size* of a graph G are $|V(G)|$ and $|E(G)|$, respectively.

Let $V(K_k) = \mathbb{Z}_k$ and let G be a subgraph of K_k. By *clicking* G, we mean applying the isomorphism $i \to i + 1$ to $V(G)$. Let H and G be graphs such that G is a subgraph of H. A *G-decomposition* of H is a set $\Delta = \{G_1, G_2, \ldots, G_t\}$ of pairwise edge-disjoint subgraphs of H each

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of which is isomorphic to G and such that $E(H) = \bigcup_{i=1}^{t} E(G_i)$. A G-decomposition of K_k is also known as a (K_k, G)-*design* or a G-*design of order* k. A (K_k, G)-design Δ is *cyclic* if clicking is a permutation of Δ. For a comprehensive source on graph decompositions we refer the reader to [5]. For an excellent recent survey on G-designs, see [1].

Let $V(K_k) = \{0, 1, \ldots, k-1\}$. The *length* of an edge $\{i, j\}$ in K_k is $\min\{|i-j|, k-|i-j|\}$. Note that clicking an edge does not change its length. Also note that if k is odd, then K_k consists of k edges of length i for $i = 1, 2, \ldots, \frac{k-1}{2}$.

For any graph G, a one-to-one function $f : V(G) \to \mathbb{N}$ is called a *labeling* (or a *valuation*) of G. In [12], Rosa introduced a hierarchy of labelings. We add a few items to this hierarchy. Let G be a graph with n edges and no isolated vertices and let f be a labeling of G. Let $f(V(G)) = \{f(u) : u \in V(G)\}$. Define a function $\bar{f} : E(G) \to \mathbb{Z}^+$ by $\bar{f}(e) = |f(u) - f(v)|$, where $e = \{u, v\} \in E(G)$. We will refer to $\bar{f}(e)$ as the *label* of e. Let $\bar{E}(G) = \{\bar{f}(e) : e \in E(G)\}$. Consider the following conditions:

- $\ell 1$: $f(V(G)) \subseteq [0, 2n]$,
- $\ell 2$: $f(V(G)) \subseteq [0, n]$,
- $\ell 3$: $\bar{E}(G) = \{x_1, x_2, \ldots, x_n\}$, where for each $i \in [1, n]$ either $x_i = i$ or $x_i = 2n + 1 - i$,
- $\ell 4$: $\bar{E}(G) = [1, n]$.

If in addition G is bipartite with bipartition $\{A, B\}$ of $V(G)$ (with every edge in G having one endvertex in A and the other in B) consider also

- $\ell 5$: for each $\{a, b\} \in E(G)$ with $a \in A$ and $b \in B$, we have $f(a) < f(b)$,
- $\ell 6$: there exists an integer λ (called the *boundary value* of f) such that $f(a) \leq \lambda$ for all $a \in A$ and $f(b) > \lambda$ for all $b \in B$.

Then a labeling satisfying the conditions:

- $\ell 1, \ell 3$: is called a ρ-*labeling*;
- $\ell 1, \ell 4$: is called a σ-*labeling*;
- $\ell 2, \ell 4$: is called a β-*labeling*.

A β-labeling is necessarily a σ-labeling which in turn is a ρ-labeling. If G is bipartite and a ρ, σ or β-labeling of G also satisfies ($\ell 5$), then the labeling is *ordered* and is denoted by ρ^+, σ^+ or β^+, respectively. If in addition ($\ell 6$) is satisfied, the labeling is *uniformly-ordered* and is denoted by ρ^{++}, σ^{++} or β^{++}, respectively.

A β-labeling is better known as a *graceful* labeling and a uniformly-ordered β-labeling is an *α-labeling* as introduced in [12]. Labelings of the types above are called *Rosa-type* because of Rosa's original article [12] on the topic. For a survey of Rosa-type labelings and their graph decomposition applications, see [8]. A dynamic survey on general graph labelings is maintained by Gallian [11].

Labelings are critical to the study of cyclic graph decompositions as seen in the following two results from [12] and [9], respectively.

Theorem 1 *Let G be a graph with n edges. There exists a cyclic G-decomposition of K_{2n+1} if and only if G has a ρ-labeling.*

Theorem 2 *Let G be a graph with n edges that has a ρ^+-labeling. Then there exists a cyclic G-decomposition of K_{2nt+1} for all positive integers t.*

If G with n edges is not bipartite, then the best that could be obtained up until recently from a Rosa-type labeling was a cyclic G-decomposition of K_{2n+1}. A non-bipartite graph G is *almost-bipartite* if G contains an edge e whose removal renders the remaining graph bipartite (for example, odd cycles are almost-bipartite). In [4], Blinco et al. introduced a variation of a ρ-labeling of an almost-bipartite graph G of size n that yields cyclic G-decompositions of K_{2nt+1}. They called this labeling a γ-labeling. They showed that odd cycles (other than C_3) and certain other 2-regular almost-bipartite graphs admit γ-labelings. In [6], it is shown that every 2-regular almost-bipartite graph other than C_3 and $C_3 \cup C_4$ admits a γ-labeling.

In this article, we show that the class of almost-bipartite graphs obtained from C_{2m} by adding an edge joining distinct vertices in the same part in the bipartition of $V(C_{2m})$ has a γ-labeling if and only if $m \geq 3$. When combined with results from Blinco's Ph.D. thesis [3] and a recent result of Froncek [10], this shows that if G is a graph of size n consisting of a cycle with a chord, then there exists a cyclic (K_{2nt+1}, G)-design for every positive integer t.

2 Additional Definitions, Notation, and Some Known Results

Let G be a graph with n edges and h a labeling of the vertices of G. We call h a *γ-labeling* of G if the following conditions hold.

g1: The function h is a ρ-labeling of G.

g2: The graph G is tripartite with vertex tripartition A, B, C with $C = \{c\}$ and $\hat{b} \in B$ such that $\{\hat{b}, c\}$ is the unique edge joining an element of B to c.

g3: If $\{a, v\}$ is an edge of G with $a \in A$, then $h(a) < h(v)$.

g4: We have $h(c) - h(\hat{b}) = n$.

Note that if a non-bipartite graph G has a γ-labeling, then it is almost-bipartite as defined earlier. In this case, removing the edge $\{c, \hat{b}\}$ from G produces a bipartite graph. Figure 1 shows γ-labelings of two almost-bipartite graphs of orders 6 and 8, respectively.

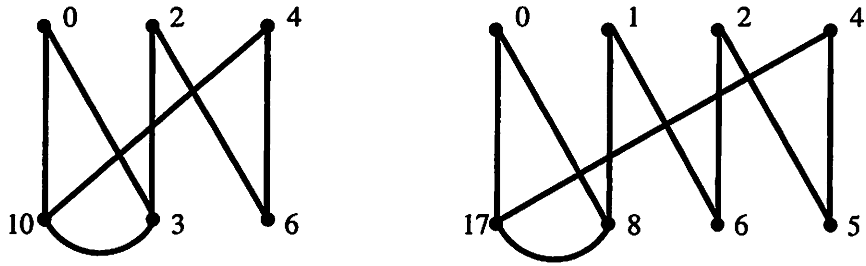

Figure 1: γ-labelings of $G(6, 2)$ and of $G(8, 2)$.

Let $G(2m, 2h)$ denote the graph formed by adding an edge between the endvertices of a path of length $2h$ in C_{2m}, where $m \geq 2$ and $h \geq 1$. As stated earlier, we will show that $G(2m, 2h)$ admits a γ-labeling for all $m \geq 3$. Note that $G(2m, 2h)$ is in a subclass of $C_k + e$, the class of graphs obtained by adding a chord e to the cycle C_k. In 1983, Delorme et al showed in [7] that $C_k + e$ is graceful. In 2003, Blinco re-looked in his Ph.D. thesis [3] at labelings of $C_k + e$ as part of his investigation of decompositions of complete graphs into θ-graphs. Blinco showed that if $C_k + e$ is bipartite, then it admits an α-labeling. He also showed that if k is odd, then $C_k + e$ admits a γ-labeling. He conjectured, but did not prove that if $k > 4$ is even and $C_k + e$ is almost-bipartite, then $C_k + e$ admits a γ-labeling. That is the case we cover in this paper. Although $C_4 + e$ cannot admit a γ-labeling (see Theorem 3), a recent (not yet published) result of Froncek [10] on cyclic $(K_{m,2} + e)$-designs shows that there exists a cyclic $(C_4 + e)$-decomposition of K_{10t+1} for all positive integers t.

To simplify our consideration of the labelings, we will henceforth consider graphs whose vertices are named by distinct nonnegative integers, which are also their labels. Recall that by the label of the edge $\{x, y\}$ in such a graph we mean $|x - y|$. If G is a graph with n edges and if m is the label of an edge e, let $m^* = \min\{m, 2n + 1 - m\}$ (thus m^* is the length of e). If S is a set of edge labels, let $S^* = \{m^* : m \in S\}$.

We denote the directed path with vertices $x_0, x_1, \ldots, x_k$, where x_i is adjacent to x_{i+1}, $0 \leq i \leq k - 1$, by $(x_0, x_1, \ldots, x_k)$. The *first vertex* of this path is x_0, the *second vertex* is x_1, and the *last vertex* is x_k. If $G_1 = (x_0, x_1, \ldots, x_j)$ and $G_2 = (y_0, y_1, \ldots, y_k)$ are directed paths with $x_j = y_0$, then by $G_1 + G_2$ we mean the path $(x_0, x_1, \ldots, x_j, y_1, y_2, \ldots, y_k)$.

Let $P(k)$ be the path with k edges and $k+1$ vertices $0, 1, \ldots, k$ given by $(0, k, 1, k-1, 2, k-2, \ldots, \lceil k/2 \rceil)$. Note that the set of vertices of this graph is $A \cup B$, where $A = [0, \lfloor k/2 \rfloor]$, $B = [\lfloor k/2 \rfloor + 1, k]$, and every edge joins a vertex from A to one from B. Furthermore the set of labels of the edges of $P(k)$ is $[1, k]$.

Now let a and b be nonnegative integers with $a \leq b$ and let us add a to all the vertices of A and b to all the vertices of B. We will denote the resulting graph by $P(a, b, k)$. Note that this graph has the following properties.

P1: $P(a, b, k)$ is a path with first vertex a and second vertex $b + k$. If k is even, its last vertex is $a + k/2$.

P2: Each edge of $P(a, b, k)$ joins a vertex from $A' = [a, \lfloor k/2 \rfloor + a]$ to a vertex with a larger label from $B' = [\lfloor k/2 \rfloor + 1 + b, k + b]$.

P3: The set of edge labels of $P(a, b, k)$ is $[b - a + 1, b - a + k]$.

Now let $R(a, b, k)$ be the path $P(a, b, k)$ with its orientation reversed. Note that this graph has the following properties.

R1: $R(a, b, k)$ is a path with last vertex a. If k is even, its first vertex is $a + k/2$.

R2: Each edge of $R(a, b, k)$ joins a vertex from $A' = [a, \lfloor k/2 \rfloor + a]$ to a vertex with a larger label from $B' = [\lfloor k/2 \rfloor + 1 + b, k + b]$.

R3: The set of edge labels of $R(a, b, k)$ is $[b - a + 1, b - a + k]$.

Figure 2 shows $P(6)$, $P(3, 5, 6)$, and $R(3, 5, 6)$.

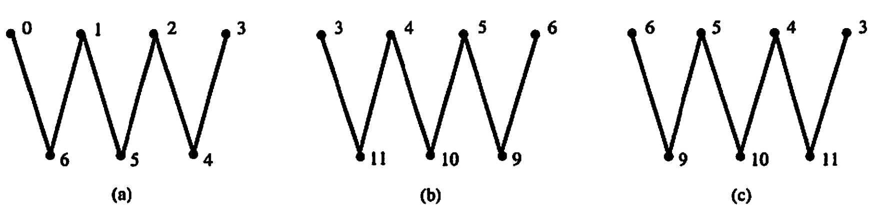

Figure 2: (a) $P(6)$, (b) $P(3, 5, 6)$, (c) $R(3, 5, 6)$.

3 Main Result

Theorem 3 *Let $G(2m, 2h)$ denote the graph formed by adding an edge between the endvertices of a path of length $2h$ in C_{2m}, where $m \geq 2$ and $1 \leq h \leq m$. Then $G(2m, 2h)$ has a γ-labeling if and only if $m \neq 2$.*

Proof. The graph $G(2m, 2h)$ is not bipartite, since it contains a cycle of length $2h+1$, but it is clearly almost-bipartite. Without loss of generality, we can assume that $h \leq m/2$. If $m = 2$, then $G(4, 2)$ is isomorphic to $K_4 \setminus K_2$, which consists of two K_3's sharing an edge e. In a γ-labeling of $G(4, 2)$, the edge e must have label 5. Let a be the vertex in A incident with the edge f of length 1. Note that $c - a = \hat{b} + 5 - a$. Thus if the label of f is 1, a must be adjacent to $\hat{b}$, while if the label on f is 10, then a must be adjacent to c. In the first case $c - a = 6$, while in the second case $\hat{b} - a = 5$. Either way we have two edges with length 5.

We divide the rest of the problem into five cases depending on the parities of m and h. At least one example is shown for each case.

Case 1 m and h are both even.
Let $m = 2x$ and $h = 2y$. Thus our graph is a C_{4x} with a chord resulting in a C_{4y+1}. We take the graph to be $G_1 + G_2 + G_3 + G_4 + (2x-1, 8x+2, 0)$ plus the edge $\{8x+2, 4x+1\}$, where

$$G_1 = P(0, 4x+2y+2, 2y),$$
$$G_2 = P(y, 4x-y, 2y),$$
$$G_3 = P(2y, 2x+2y, 2x-4y),$$
$$G_4 = P(x, x+1, 2x-2).$$

First, we show that $G_1 + G_2 + G_3 + G_4 + (2x-1, 8x+2, 0)$ is a cycle of

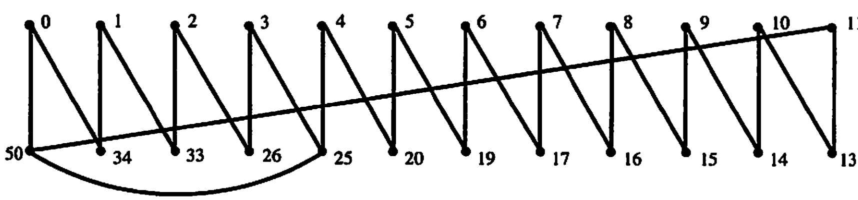

Figure 3: A γ-labeling of $G(24, 8)$.

length $4x$. Note that by P1, the first vertex of G_1 is 0 and the last is y, the first vertex of G_2 is y and the last is $2y$, the first vertex of G_3 is $2y$ and the last is x, the first vertex of G_4 is x and the last is $2x-1$. For $1 \leq i \leq 4$, let A_i and B_i denote the sets labeled A' or B' in P2, corresponding to the path G_i. Then using P2, we compute

$$\begin{aligned} A_1 &= [0, y], & B_1 &= [4x + 3y + 3, 4x + 4y + 2], \\ A_2 &= [y, 2y], & B_2 &= [4x + 1, 4x + y], \\ A_3 &= [2y, x], & B_3 &= [3x + 1, 4x - 2y], \\ A_4 &= [x, 2x - 1], & B_4 &= [2x + 1, 3x - 1]. \end{aligned}$$

Thus, $A_1 \leq A_2 \leq A_3 \leq A_4 < B_4 < B_3 < B_2 < B_1$. Also note that $V(G_1) \cap V(G_2) = \{y\}$, $V(G_2) \cap V(G_3) = \{2y\}$, $V(G_3) \cap V(G_4) = \{x\}$ and that, otherwise, G_i and G_j are vertex-disjoint. Therefore, $G_1+G_2+G_3+G_4$ is a path P of length $4x-2$ with first vertex 0 and last vertex $2x-1$. Since $V(P) \cap \{2x-1, 8x+2, 0\} = \{2x-1, 0\}$, the graph $G_1 + G_2 + G_3 + G_4 + (2x-1, 8x+2, 0)$ is a cycle of length $4x$.

With the additional edge $\{8x+2, 4x+1\}$, the resulting graph is $G(4x, 4y)$ and is tripartite with tripartition A, B, C, where $A = A_1 \cup A_2 \cup A_3 \cup A_4 \cup \{2x-1\} = [0, 2x-1]$, $B = B_4 \cup B_3 \cup B_2 \cup B_1$ and $C = \{8x+2\}$. With $\hat{b} = 4x+1$ and $c = 8x+2$, we satisfy condition **g2** for a γ-labeling. Since $c - \hat{b} = 4x+1$, condition **g4** is satisfied. Moreover, if $\{a, v\}$ is an edge in our graph with $a \in A$, then $a < v$. Thus **g3** is satisfied.

Therefore it remains to show that we have a ρ-labeling of $G(4x, 4y)$. Let E_i denote the set of edge labels in G_i for $1 \leq i \leq 4$. By P3, we have

$$\begin{aligned} E_1 &= [4x + 2y + 3, 4x + 4y + 2], \\ E_2 &= [4x - 2y + 1, 4x], \\ E_3 &= [2x + 1, 4x - 4y], \\ E_4 &= [2, 2x - 1]. \end{aligned}$$

Note that $E_1^* = [4x - 4y + 1, 4x - 2y]$ and $E_i^* = E_i$ for $2 \leq i \leq 4$. Moreover, the path $(2x-1, 8x+2, 0)$ consists of edges with labels $6x+3$ and $8x+2$. These labels correspond to edge lengths $(6x+3)^* = 2x$ and $(8x+2)^* = 1$, respectively. Thus the edges of $G(4x, 4y)$ have lengths $(\cup_{i=1}^4 E_i^*) \cup \{2x, 1\} \cup \{4x+1\} = [1, 4x+1]$. Hence the defined labeling is a ρ-labeling, and thus condition **g1** is satisfied. Therefore, we have a γ-labeling of $G(4x, 4y)$.

Case 2 m is even and h is odd.
Let $m = 2x$ and $h = 2y+1$. Thus our graph is a C_{4x} with a chord resulting in a C_{4y+3}. Note that we must have $x \geq 2y+1$ since we assumed $h \leq m/2$. We will consider the case $x = 2y+1$ separately.

If $x > 2y+1$, we take the graph to be $G_1+G_2+G_3+G_4+(2x-2, 2x+1, 2x, 8x+1, 0)$ plus the edge $\{8x+1, 4x\}$, where

$$G_1 = P(0, 4x+2y+4, 2y),$$
$$G_2 = P(y, 4x-y-2, 2y+2),$$
$$G_3 = P(2y+1, 2x+2y+3, 2x-4y-4),$$
$$G_4 = P(x-1, x+2, 2x-2).$$

First, we show that $G_1+G_2+G_3+G_4+(2x-2, 2x+1, 2x, 8x+1, 0)$ is a

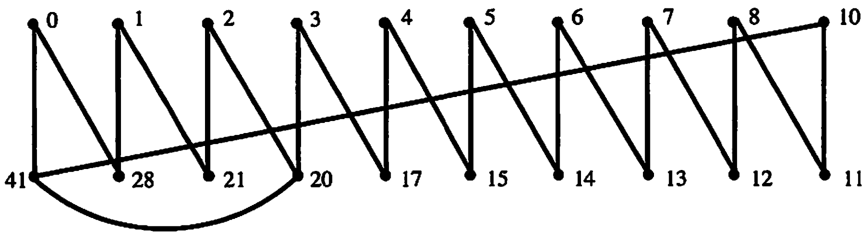

Figure 4: A γ-labeling of $G(20, 6)$.

cycle of length $4x$. Note that by P1, the first vertex of G_1 is 0 and the last is y, the first vertex of G_2 is y and the last is $2y+1$, the first vertex of G_3 is $2y+1$ and the last is $x-1$, the first vertex of G_4 is $x-1$ and the last is $2x-2$. For $1 \le i \le 4$, let A_i and B_i denote the sets labeled A' or B' in P2, corresponding to the path G_i. Then using P2, we compute

$$A_1 = [0, y], \qquad B_1 = [4x+3y+5, 4x+4y+4],$$
$$A_2 = [y, 2y+1], \qquad B_2 = [4x, 4x+y],$$
$$A_3 = [2y+1, x-1], \qquad B_3 = [3x+2, 4x-2y-1],$$
$$A_4 = [x-1, 2x-2], \qquad B_4 = [2x+2, 3x].$$

Thus, $A_1 \le A_2 \le A_3 \le A_4 < B_4 < B_3 < B_2 < B_1$. Also note that $V(G_1) \cap V(G_2) = \{y\}$, $V(G_2) \cap V(G_3) = \{2y+1\}$, $V(G_3) \cap V(G_4) = \{x-1\}$ and that, otherwise, G_i and G_j are vertex-disjoint. Therefore, $G_1+G_2+G_3+G_4$ is a path P of length $4x-4$ with first vertex 0 and last vertex $2x-2$. Since $V(P) \cap \{2x-2, 2x+1, 2x, 8x+1, 0\} = \{2x-2, 0\}$, the graph $G_1+G_2+G_3+G_4+(2x-2, 2x+1, 2x, 8x+1, 0)$ is a cycle of length $4x$.

With the additional edge $\{8x+1, 4x\}$, the resulting graph is $G(4x, 4y+2)$ and is tripartite with tripartition A, B, C, where $A = A_1 \cup A_2 \cup A_3 \cup A_4 \cup \{2x\} = [0, 2x-2] \cup \{2x\}$, $B = \{2x+1\} \cup B_4 \cup B_3 \cup B_2 \cup B_1$ and $C = \{8x+1\}$. With $\hat{b} = 4x$ and $c = 8x+1$, we satisfy condition **g2** for a γ-labeling. Since

$c - \hat{b} = 4x + 1$, condition **g4** is satisfied. Moreover, if $\{a, v\}$ is an edge in our graph with $a \in A$, then $a < v$. Thus **g3** is satisfied.

Therefore it remains to show that we have a ρ-labeling of $G(4x, 4y + 2)$. Let E_i denote the set of edge labels in G_i for $1 \leq i \leq 4$. By P3, we have

$$\begin{aligned} E_1 &= [4x + 2y + 5, 4x + 4y + 4], \\ E_2 &= [4x - 2y - 1, 4x], \\ E_3 &= [2x + 3, 4x - 4y - 2], \\ E_4 &= [4, 2x + 1]. \end{aligned}$$

Note that $E_1^* = [4x - 4y - 1, 4x - 2y - 2]$ and $E_i^* = E_i$ for $2 \leq i \leq 4$. Moreover, the path $(2x - 2, 2x + 1, 2x, 8x + 1, 0)$ consists of edges with labels $3, 1, 6x + 1$ and $8x + 1$. These labels correspond to edge lengths $3, 1, (6x + 1)^* = 2x + 2$ and $(8x + 1)^* = 2$, respectively. Thus the edges of $G(4x, 4y+2)$ have lengths $(\cup_{i=1}^{4} E_i^*) \cup \{1, 2, 3, 2x+2\} \cup \{4x+1\} = [1, 4x+1]$. Hence the defined labeling is a ρ-labeling, and thus condition **g1** is satisfied. Therefore, we have a γ-labeling of $G(4x, 4y + 2)$ when $x > 2y + 1$.

Now if $x = 2y + 1$, then our graph is $G(8y + 4, 4y + 2)$. Note that we must have $y > 0$ since $G(4, 2)$ does not admit a γ-labeling. We take our graph to be $G_1 + G_2 + G_3 + (4y, 4y + 3, 4y + 2, 16y + 9, 0, 12y + 9, 1)$ plus the edge $\{16y + 9, 8y + 4\}$, where

$$\begin{aligned} G_1 &= P(1, 10y + 9, 2y - 2), \\ G_2 &= P(y, 7y + 2, 2y + 2), \\ G_3 &= P(2y + 1, 2y + 4, 4y - 2). \end{aligned}$$

First, we show that $G_1 + G_2 + G_3 + (4y, 4y+3, 4y+2, 16y+9, 0, 12y+9, 1)$

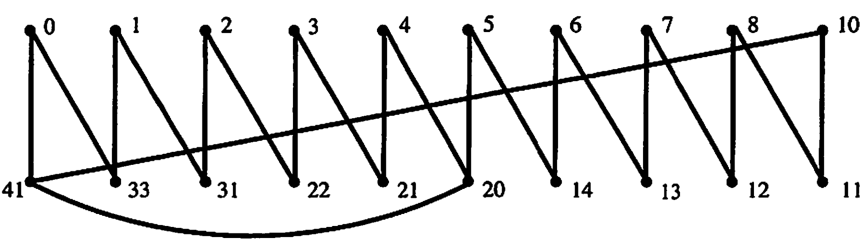

Figure 5: A γ-labeling of $G(20, 10)$.

is a cycle of length $8y + 4$. Note that by P1, the first vertex of G_1 is 1 and the last is y, the first vertex of G_2 is y and the last is $2y + 1$, the first vertex of G_3 is $2y + 1$ and the last is $4y$. For $1 \leq i \leq 3$, let A_i and B_i denote the sets labeled A' or B' in P2, corresponding to the path G_i. Then using P2, we compute

$$A_1 = [1, y], \qquad B_1 = [11y + 9, 12y + 7],$$
$$A_2 = [y, 2y + 1], \qquad B_2 = [8y + 4, 9y + 4],$$
$$A_3 = [2y + 1, 4y], \qquad B_3 = [4y + 4, 6y + 2].$$

Thus, $A_1 \leq A_2 \leq A_3 < B_3 < B_2 < B_1$. Also note that $V(G_1) \cap V(G_2) = \{y\}$, $V(G_2) \cap V(G_3) = \{2y+1\}$ and that, otherwise, G_i and G_j are vertex-disjoint. Therefore, $G_1+G_2+G_3$ is a path P of length $8y-2$ with first vertex 1 and last vertex $4y$. Since $V(P) \cap \{4y, 4y+3, 4y+2, 16y+9, 0, 12y+9, 1\} = \{4y, 1\}$, the graph $G_1 + G_2 + G_3 + (4y, 4y + 3, 4y + 2, 16y + 9, 0, 12y + 9, 1)$ is a cycle of length $8y + 4$.

With the additional edge $\{16y+9, 8y+4\}$, the resulting graph is $G(8y+4, 4y + 2)$ and is tripartite with tripartition A, B, C, where $A = A_1 \cup A_2 \cup A_3 \cup \{4y+2, 0\} = [0, 4y] \cup \{4y+2\}$, $B = \{4y+3\} \cup B_3 \cup B_2 \cup B_1 \cup \{12y+9\}$ and $C = \{16y+9\}$. With $\hat{b} = 8y+4$ and $c = 16y+9$, we satisfy condition **g2** for a γ-labeling. Since $c - \hat{b} = 8y + 5$, condition **g4** is satisfied. Moreover, if $\{a, v\}$ is an edge in our graph with $a \in A$, then $a < v$. Thus **g3** is satisfied.

Therefore it remains to show that we have a ρ-labeling of $G(8y+4, 4y+2)$. Let E_i denote the set of edge labels in G_i for $1 \leq i \leq 4$. By P3, we have

$$E_1 = [10y + 9, 12y + 6],$$
$$E_2 = [6y + 3, 8y + 4],$$
$$E_3 = [4, 4y + 1].$$

Note that $E_1^* = [4y + 5, 6y + 2]$ and $E_i^* = E_i$ for $2 \leq i \leq 3$. Moreover, the path $(4y, 4y + 3, 4y + 2, 16y + 9, 0, 12y + 9, 1)$ consists of edges with labels $3, 1, 12y + 7, 16y + 9, 12y + 9$ and $12y + 8$. These labels correspond to edge lengths $3, 1, (12y + 7)^* = 4y + 4, (16y + 9)^* = 2, (12y + 9)^* = 4y + 2$ and $(12y + 8)^* = 4y + 3$, respectively. Thus the edges of $G(8y + 4, 4y + 2)$ have lengths $(\cup_{i=1}^{3} E_i^*) \cup \{3, 1, 4y + 4, 2, 4y + 2, 4y + 3\} \cup \{8y + 5\} = [1, 8y + 5]$. Hence the defined labeling is a ρ-labeling, and thus condition **g1** is satisfied. Therefore, we have a γ-labeling of $G(8y + 4, 4y + 2)$. Thus we have a γ-labeling of $G(4x, 4y + 2)$.

Case 3 m is odd and h is even.
Let $m = 2x + 1$ and $h = 2y$. Thus our graph is a C_{4x+2} with a chord resulting in a C_{4y+1}.

We take the graph to be $(1, 8x+6, 0, 4x+2y+3, 2x) + G_1 + G_2 + G_3 + G_4 + G_5$ plus the edge $\{8x+6, 4x+3\}$, where

$$\begin{aligned} G_1 &= R(2x-2y+2, 2x-2y+4, 4y-4), \\ G_2 &= R(2x-2y+1, 4x+1, 2), \\ G_3 &= R(2x-3y+2, 6x-3y+6, 2y-2), \\ G_4 &= R(x-y+2, 5x+y+5, 2x-4y), \\ G_5 &= R(1, 6x-2y+7, 2x-2y+2). \end{aligned}$$

First, we show that $(1, 8x+6, 0, 4x+2y+3, 2x) + G_1 + G_2 + G_3 + G_4 + G_5$

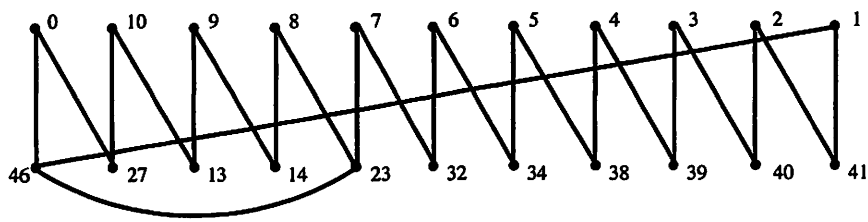

Figure 6: A γ-labeling of $G(22, 8)$.

is a cycle of length $4x+2$. Note that by R1, the first vertex of G_1 is $2x$ and the last is $2x-2y+2$, the first vertex of G_2 is $2x-2y+2$ and the last is $2x-2y+1$, the first vertex of G_3 is $2x-2y+1$ and the last is $2x-3y+2$, the first vertex of G_4 is $2x-3y+2$ and the last is $x-y+2$, the first vertex of G_5 is $x-y+2$ and the last is 1. For $1 \le i \le 5$, let A_i and B_i denote the sets labeled A' or B' in R2, corresponding to the path G_i. Then using R2, we compute

$$\begin{aligned} A_1 &= [2x-2y+2, 2x], & B_1 &= [2x+3, 2x+2y], \\ A_2 &= [2x-2y+1, 2x-2y+2], & B_2 &= \{4x+3\}, \\ A_3 &= [2x-3y+2, 2x-2y+1], & B_3 &= [6x-2y+6, 6x-y+4], \\ A_4 &= [x-y+2, 2x-3y+2], & B_4 &= [6x-y+6, 7x-3y+5], \\ A_5 &= [1, x-y+2], & B_5 &= [7x-3y+9, 8x-4y+9]. \end{aligned}$$

Thus, $A_5 \le A_4 \le A_3 \le A_2 \le A_1 < B_1 < B_2 < B_3 < B_4 < B_5$. Also note that $V(G_1) \cap V(G_2) = \{2x-2y+2\}$, $V(G_2) \cap V(G_3) = \{2x-2y+1\}$, $V(G_3) \cap V(G_4) = \{2x-3y+2\}$, $V(G_4) \cap V(G_5) = \{x-y+2\}$ and that, otherwise, G_i and G_j are vertex-disjoint. Therefore, $G_1+G_2+G_3+G_4+G_5$ is a path P of length $4x-2$ with first vertex $2x$ and last vertex 1. Since $V(P) \cap \{1, 8x+6, 0, 4x+2y+3, 2x\} = \{1, 2x\}$, the graph $(1, 8x+6, 0, 4x+2y+3, 2x) + G_1 + G_2 + G_3 + G_4 + G_5$ is a cycle of length $4x+2$.

With the additional edge $\{8x+6, 4x+3\}$, the resulting graph is $G(4x+2, 4y)$ and is tripartite with tripartition A, B, C, where $A = \{0\} \cup A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5 = [0, 2x]$, $B = B_1 \cup B_2 \cup B_3 \cup B_4 \cup B_5 \cup \{4x+2y+3\} \subset [2x+3, 8x-4y+9]$ and $C = \{8x+6\}$. With $\hat{b} = 4x+3$ and $c = 8x+6$, we satisfy condition **g2** for a γ-labeling. Since $c - \hat{b} = 4x+3$, condition **g4** is satisfied. Moreover, if $\{a, v\}$ is an edge in our graph with $a \in A$, then $a < v$. Thus **g3** is satisfied.

Therefore it remains to show that we have a ρ-labeling of $G(4x+2, 4y)$. Let E_i denote the set of edge labels in G_i for $1 \le i \le 5$. By R3, we have

$$\begin{aligned} E_1 &= [3, 4y-2], \\ E_2 &= [2x+2y+1, 2x+2y+2], \\ E_3 &= [4x+5, 4x+2y+2], \\ E_4 &= [4x+2y+4, 6x-2y+3], \\ E_5 &= [6x-2y+7, 8x-4y+8]. \end{aligned}$$

Note that $E_1^* = E_1$, $E_2^* = E_2$, $E_3^* = [4x-2y+5, 4x+2]$, $E_4^* = [2x+2y+4, 4x-2y+3]$, and $E_5^* = [4y-1, 2x+2y]$. Moreover, the path $(1, 8x+6, 0, 4x+2y+3, 2x)$ has edges with labels $8x+5, 8x+6, 4x+2y+3$ and $2x+2y+3$. These labels correspond to edge lengths $(8x+5)^* = 2, (8x+6)^* = 1, (4x+2y+3)^* = 4x-2y+4$, and $2x+2y+3$, respectively. Thus the edges of $G(4x+2, 4y)$ have lengths $(\cup_{i=1}^{5} E_i^*) \cup \{1, 2, 2x+2y+3, 4x-2y+4\} \cup \{4x+3\} = [1, 4x+3]$. Hence the defined labeling is a ρ-labeling, and thus condition **g1** is satisfied. Therefore, we have a γ-labeling of $G(4x+2, 4y)$.

Case 4 m is odd and $h = 1$.
Let $m = 2x+1$ and $h = 1$. Thus our graph is a C_{4x+2} with a chord resulting in a C_3. For $x = 1$, we give our labeling in Figure 1. For $x \ge 2$, we take our graph to be $G_1 + G_2 + (2x+3, 6x+9, 1, 2, 0, 4x+5, 5)$ plus the edge $\{2, 4x+5\}$, where

$$\begin{aligned} G_1 &= P(5, 2x+4, 2x), \\ G_2 &= P(x+5, x+7, 2x-4). \end{aligned}$$

First, we show that $G_1 + G_2$ is a path of length $4x-4$. Note that by P1, the first vertex of G_1 is 5 and the last is $x+5$, the first vertex of G_2 is $x+5$ and the last is $2x+3$. For $1 \le i \le 2$, let A_i and B_i denote the sets labeled A' or B' in P2, corresponding to the path G_i. Then using P2, we compute

$$\begin{aligned} A_1 &= [5, x+5], & B_1 &= [3x+5, 4x+4], \\ A_2 &= [x+5, 2x+3], & B_2 &= [2x+6, 3x+3]. \end{aligned}$$

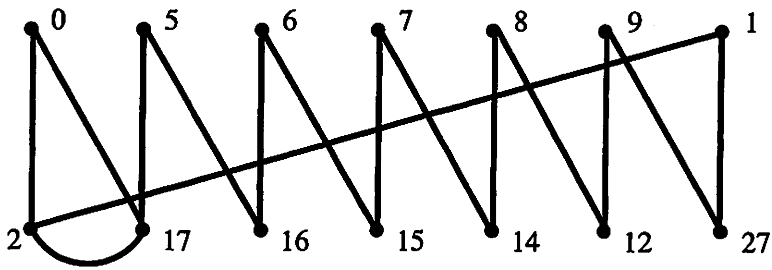

Figure 7: A γ-labeling of $G(14,2)$.

Thus, $A_1 \leq A_2 < B_2 < B_1$. Also note that $V(G_1) \cap V(G_2) = \{x+5\}$, and that, otherwise, G_1 and G_2 are vertex-disjoint. Therefore, G_1+G_2 is a path P of length $4x-4$. Since $V(P) \cap \{2x+3, 6x+9, 1, 2, 0, 4x+5, 5\} = \{2x+3, 5\}$, the graph $G_1 + G_2 + (2x+3, 6x+9, 1, 2, 0, 4x+5, 5)$ is a cycle of length $4x+2$.

With the additional edge $\{2, 4x+5\}$, the resulting graph is $G(4x+2, 2)$ and is tripartite with tripartition A, B, C, where $A = \{0\} \cup A_1 \cup A_2 \cup \{1\} \subset [0, 2x+3]$, $B = \{2\} \cup B_1 \cup B_2 \cup \{6x+9\}$ and $C = \{4x+5\}$. With $\hat{b} = 2$ and $c = 4x+5$, we satisfy condition **g2** for a γ-labeling. Since $c - \hat{b} = 4x+3$, condition **g4** is satisfied. Moreover, if $\{a, v\}$ is an edge in our graph with $a \in A$, then $a < v$. Thus **g3** is satisfied.

Therefore it remains to show that we have a ρ-labeling of $G(4x+2, 2)$. Let E_i denote the set of edge labels in G_i for $1 \leq i \leq 2$. By P3, we have

$$E_1 = [2x, 4x-1],$$
$$E_2 = [3, 2x-2].$$

Note that $E_1^* = E_1$ and $E_2^* = E_2$. Moreover, the path $(2x+3, 6x+9, 1, 2, 0, 4x+5, 5)$ has edges with labels $4x+6, 6x+8, 1, 2, 4x+5$ and $4x$. These labels correspond to edge lengths $(4x+6)^* = 4x+1$, $(6x+8)^* = 2x-1, 1, 2$, $(4x+5)^* = 4x+2$ and $4x$, respectively. Thus the edges of $G(4x+2, 2)$ have lengths $(E_1^* \cup E_2^* \cup \{1, 2, 2x-1, 4x, 4x+1, 4x+2\} \cup \{4x+3\} = [1, 4x+3]$. Hence the defined labeling is a ρ-labeling, and thus condition **g1** is satisfied. Therefore, we have a γ-labeling of $G(4x+2, 2)$.

Case 5 m is odd and $h \geq 3$ is odd.
Let $m = 2x+1$ and $h = 2y+1$ with $y \geq 1$. Thus our graph is a C_{4x+2} with a chord resulting in a C_{4y+3}. We take the graph to be $G_1 + (2y-1, 6x+5, 2y+1) + G_2 + G_3 + G_4 + (2x, 2x+2, 2x+1, 6x-2y+5, 0)$ plus the edge

$\{2x+2, 6x+5\}$, where

$$G_1 = P(0, 8x-4y+6, 4y-2),$$
$$G_2 = P(2y+1, 4x+5, 2y-2),$$
$$G_3 = P(3y, 2x+5y+3, 2x-4y),$$
$$G_4 = P(x+y, x+5y, 2x-2y).$$

First, we show that $G_1 + (2y-1, 6x+5, 2y+1) + G_2 + G_3 + G_4 + (2x, 2x+$

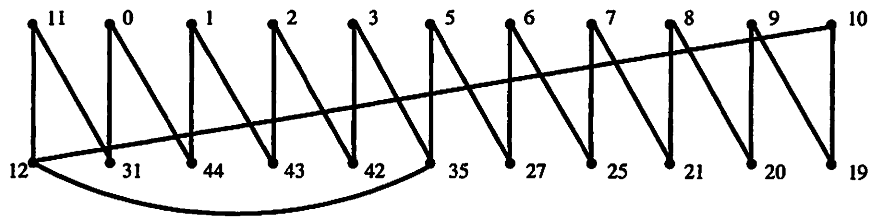

Figure 8: A γ-labeling of $G(22, 10)$.

$2, 2x+1, 6x-2y+5, 0)$ is a cycle of length $4x+2$. Note that by P1, the first vertex of G_1 is 0 and the last is $2y-1$, the first vertex of G_2 is $2y+1$ and the last is $3y$, the first vertex of G_3 is $3y$ and the last is $x+y$, the first vertex of G_4 is $x+y$ and the last is $2x$. Moreover, note that the path $(2y-1, 6x+5, 2y+1)$ starts at $2y-1$ (the last vertex of G_1) and ends at $2y+1$ (the first vertex of G_2). For $1 \le i \le 4$, let A_i and B_i denote the sets labeled A' or B' in P2, corresponding to the path G_i. Then using P2, we compute

$$A_1 = [0, 2y-1], \qquad B_1 = [8x-2y+6, 8x+4],$$
$$A_2 = [2y+1, 3y], \qquad B_2 = [4x+y+5, 4x+2y+3],$$
$$A_3 = [3y, x+y], \qquad B_3 = [3x+3y+4, 4x+y+3],$$
$$A_4 = [x+y, 2x], \qquad B_4 = [2x+4y+1, 3x+3y].$$

Thus, $A_1 \le \{2y-1, 2y+1\} \le A_2 \le A_3 \le A_4 < B_4 < B_3 < B_2 < \{6x+5\} < B_1$. Also note that $V(G_2) \cap V(G_3) = \{3y\}$, $V(G_3) \cap V(G_4) = \{x+y\}$, and that, otherwise, G_i and G_j are vertex-disjoint. Therefore, $G_1 + (2y-1, 6x+5, 2y+1) + G_2 + G_3 + G_4$ is a path P of length $4x-2$ with first vertex 0 and last vertex $2x$. Since $V(P) \cap \{0, 6x-2y+5, 2x+1, 2x+2, 2x\} = \{2x, 0\}$, the graph $G_1 + (2y-1, 6x+5, 2y+1) + G_2 + G_3 + G_4) + (2x, 2x+2, 2x+1, 6x-2y+5, 0)$ is a cycle of length $4x+2$.

With the additional edge $\{2x+2, 6x+5\}$, the resulting graph is $G(4x+2, 4y+2)$ and is tripartite with tripartition A, B, C, where $A = A_1 \cup A_2 \cup A_3 \cup A_4 \cup \{2x+1\} \subset [0, 2x+1]$, $B = \{2x+2\} \cup B_1 \cup B_2 \cup B_3 \cup B_4 \cup \{6x-2y+5\} \subset$

$[2x+4y+1, 8x+4]$ and $C = \{6x+5\}$. With $\hat{b} = 2x+2$ and $c = 6x+5$, we satisfy condition **g2** for a γ-labeling. Since $c - \hat{b} = 4x+3$, condition **g4** is satisfied. Moreover, if $\{a, v\}$ is an edge in our graph with $a \in A$, then $a < v$. Thus **g3** is satisfied.

Therefore it remains to show that we have a ρ-labeling of $G(4x+2, 4y+2)$. Let E_i denote the set of edge labels in G_i for $1 \le i \le 4$. By P3, we have

$$\begin{aligned} E_1 &= [8x-4y+7, 8x+4], \\ E_2 &= [4x-2y+5, 4x+2], \\ E_3 &= [2x+2y+4, 4x-2y+3], \\ E_4 &= [4y+1, 2x+2y]. \end{aligned}$$

Note that $E_1^* = [3, 4y]$ and $E_i^* = E_i$ for $2 \le i \le 4$. Also, the path $(2y-1, 6x+5, 2y+1)$ has edges with labels $6x-2y+6$ and $6x-2y+4$. These labels correspond to edge lengths $(6x-2y+6)^* = 2x+2y+1$ and $(6x-2y+4)^* = 2x+2y+3$, respectively. Moreover, the path $(0, 6x-2y+5, 2x+1, 2x+2, 2x)$ has edges with labels $6x-2y+5, 4x-2y+4, 1$ and 2. These labels correspond to edge lengths $(6x-2y+5)^* = 2x+2y+2, 4x-2y+4, 1$ and 2, respectively. Thus the edges of $G(4x+2, 4y+2)$ have lengths $(\cup_{i=1}^4 E_i^*) \cup \{2x+2y+1, 2x+2y+3\} \cup \{1, 2, 2x+2y+2, 4x-2y+4\} \cup \{4x+3\} = [1, 4x+3]$. Hence the defined labeling is a ρ-labeling, and thus condition **g1** is satisfied. Therefore, we have a γ-labeling of $G(4x+2, 4y+2)$. □

Because every cycle with a chord (other than $C_4 + e$) admits either an α-labeling (by Blinco's results [3]) or a γ-labeling (by Blinco's results [3] or Theorem 3 here) and in light of Froncek's result on cyclic $(K_{m,2}+e)$-decompositions [10], we have the following corollary.

Corollary 4 *Let G be a graph of size n consisting of a cycle C_{n-1} with a chord e. Then there exists a cyclic (K_{2nt+1}, G)-design for all positive integers t.*

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References

[1] P. Adams, D. Bryant, and M. Buchanan, A survey of the existence of G-designs, *J. Comb. Designs* **16** (2008), 1014–1037.

[2] A. Blinco, On diagonal cycle systems, *Australas. J. Combin.* **24** (2001), 221–230.

[3] A. Blinco, Graph decompositions, theta graphs and related labelling techniques, Ph.D. Thesis, Department of Mathematics, The University of Queensland, 2003.

[4] A. Blinco, S.I. El-Zanati, and C. Vanden Eynden, On the cyclic decomposition of complete graphs into almost-bipartite graphs, *Discrete Math.* **284** (2004), 71–81.

[5] J. Bosák, *Decompositions of Graphs*, Kluwer Academic Publishers Group, Dordrecht, 1990.

[6] R.C. Bunge, S.I. El-Zanati, and C. Vanden Eynden, On γ-labelings of 2-regular almost-bipartite graphs, in preparation.

[7] C. Delorme, M. Maheo, H. Thuillier, K.M. Koh, H.K. Teo, Cycles with a chord are graceful, *J. Graph Theory* 4 (1980), 409–415.

[8] S.I. El-Zanati and C. Vanden Eynden, On Rosa-type labelings and cyclic graph decompositions, *Math. Slovaca* **59** (2009), 1–18.

[9] S.I. El-Zanati, C. Vanden Eynden and N. Punnim, On the cyclic decomposition of complete graphs into bipartite graphs, *Australas. J. Combin.* **24** (2001), 209–219.

[10] D. Froncek, Private communication.

[11] J.A. Gallian, A dynamic survey of graph labeling, Dynamic Survey 6, *Electron. J. Combin.* (2009), 219 pp.

[12] A. Rosa, On certain valuations of the vertices of a graph, in: Théorie des graphes, journées internationales d'études, Rome 1966 (Dunod, Paris, 1967), 349–355.